Leader-Follower strategies for a Multi-Plant differential game

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Abstract— In this paper the formulation of a concept for a type of robust Leader-Follower equilibrium for a Multi-Plant or multiple scenarios differential game is developed. The game dynamic is given by a family of N different possible differential equations (Multi-Model representation) with no information about the trajectory which is realized. The robust Leader-Follower strategy for each player must confront with all possible scenarios simultaneously. The problem of each player is the designing of min-max strategies for each player which guarantee an equilibrium for the worst case scenario. Based on the Robust Maximum Principle, the conditions for a game to be in Robust Leader-Follower Equilibrium are presented. As in the Nash equilibrium case the initial min-max differential game may be converted into a standard static game given in a multidimensional simplex. A numerical procedure for resolving the case of linear quadratic differential game is presented.

I. INTRODUCTION

Beginning from the seminal works of [1], [2], [3] the leader-follower or Stackelberg solution for an open loop information structure in a two person differential games has been well established (see also [4], [5]). This concept of a solution for a game, introduced in 1934 by the economist H. von Stackelberg [6], is suitable when one of the players is forced to subordinate to an authority that must announces his decision first, before play his own strategy, or when one of the players chooses to play the game passively, that is, composes just his optimal reaction (solve an one-player optimization problem) given that the other player has announced his strategy. As it is shown in the above publications this kind of problems are still tractable using results of optimal control theory. All of them tackle this problem when the model of the considered dynamics is exactly known.

In the case of one player optimization problems the robust version of the traditional Maximum Principle referred to as Robust Maximum Principle (RMP) (see [7], [8] and [9]) allows to design an "optimal policy" of a min-max type for a multi-plant or multi-scenario problem, where each possible scenario is seen as possible parametric realization of the dynamic equation. So, it appears that we have N possible linear state dynamic equations, each of them describing a model and there is no a priory information which will be the active one. The RMP is based on the concept of min-max control problem where the operation of the maximization is taken over a set of uncertainty (a parameter from a finite

set) and the operation of the minimization is taken over set of admissible control strategies. For a game with multi-participants, choosing Nash strategies, some similar concept of Multi-scenarios has been used to exemplify the discrepancies of the players in information sets, models, cost functions or even different amount of information that the players could hold of a large scale system, [10], [11], [12], and recently, applying the Robust Maximum Principle there was derived a type of Robust Nash equilibrium ([13]) for a multi-scenario game parametrized by a parameter belonging to a given finite parametric set and the problem was formulated as a min-max problem of the game.

The purpose of this paper is to develop the RMP for a family of two person Multi-Plant game when one player is considered as the leader and the second one as the follower. We formulate the problem for both leader and follower as min-max problem, that to the best of our knowledge has never been considered before, but presents a great interest because of its high spread applications. The focus is, as in [13], the designing of strategies that should provide a "Robust leader-follower equilibrium" being applied to all scenarios (or models) of the game simultaneously. It is shown that the resulting robust strategies for each player again appears as a mixture (with the weights fulfilling the leader-follower condition) of the controls which are the leader-follower for each fixed parameter value. This technique permits to transform the formulation of the game with a leader from a Banach space (where it was initially formulated) into a finite dimensional space where the strategies to be found are the preference weights of each scenario in the weighted sum of the individual optimal controls. The problem of finding such equilibrium weights is solve by the implementation of a special numerical procedure which extends the method given in [13]. A numerical example illustrate the effectiveness of this approach.

II. STANDARD LEADER-FOLLOWER EQUILIBRIUM

We begin with a brief ly review of the concept for leader-follower equilibrium. The basic idea of a leader-follower strategy for a static two-person game seems to be very simple. Consider two players Player 1 and Player. The cost function associated with the players are

$$
\begin{array}{ll}
J^1\left(u^1, u^2\right) & \text{for} \quad \text{Player 1} \\
J^1\left(u^1, u^2\right) & \text{for} \quad \text{Player 2}
\end{array}
$$

Both players want to minimize their criteria that naturally provokes a conf lict situation. Defining the sets U^1 and U^2 for the admissible strategies for the player 1 and player 2.

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The resolution of this problem is given by the following equilibrium concept. Choosing the player 1 as the leader and player 2 as the follower. The set of strategies are said to be in a *Leader-Follower equilibria with Player 1 as the leader and Player 2 as the follower* if:

$$J^1\left(u^{1*}, u^{2*}\left(u^{1*}\right)\right) \le J^1\left(u^1, u^{2^o}\left(u^1\right)\right) \qquad (1)$$

and

$$J^2\left(u^1, u^{2^o}\left(u^1\right)\right) = \min_{u^2 \in U^2} J^2\left(u^1, u^2\right) \qquad (2)$$

and $u^{2^o} = u^{2^o}\left(u^1\right)$ is the optimal policy of the player 2 for a given strategy of the leader, and as is usual $u^{2*}\left(u^{1*}\right) = u^{2^o}\left(u^{1*}\right)$. This means that being player 1 as the leader, he must advance his strategy to play first and because the player 2 want to minimize his functional J^2 then u^{2^o} is the "optimal reaction" (OR) of player 2 for the minimization of J^2 given u^1. If Player 1 chooses any other strategy u^1, then Player 2 will choose a strategy u^{2^o} that minimizes J^2, but the resulting cost for Player 1 may be greater than or equal to that when the Leader-Follower strategy with Player 1 as the leader is used.

III. MULTI-MODEL TWO PLAYERS GAME

Consider the following two players multi-model differential game:

$$\dot{x}^\alpha = f^\alpha\left(x^\alpha, u^1, u^2, t\right) \qquad (3)$$

where $x^\alpha \in \Re^n$ is the state vector of the game at time $t \in [t_0, T]$, $u^j \in \Re^{m_j}$ $(j = 1, 2)$ are the control strategies of each player at time t and α is the entire index from a finite set $\mathcal{A} := \{1, 2, .., M\}$ describing each possible α-model of the dynamics game (3), M is the number of possible model.

Let the *individual aim performance* $h^{i,\alpha}$ of each player $(i = 1, 2)$ for each α-model (scenario) be given by

$$h^{i,\alpha} := h_0^i\left(x^\alpha\left(T\right)\right) + \int_{t=t_0}^T g_i^\alpha\left(x^\alpha, u^1, u^2, t\right) dt \qquad i = 1, 2 \qquad (4)$$

The *worst-case* (with respect to a possible scenario) cost functional F^i for each player under fixed admissible strategies $u^1 \in U^1$ and $u^2 \in U^2$ is defined by

$$F^i\left(u^1, u^2\right) := \max_{\alpha \in \mathcal{A}} h^{i,\alpha}\left(u^1, u^2\right)$$

A. Robust leader-follower Equilibrium

The set of strategies are said to be in a *Robust leader-follower equilibria with Player 1 as the leader and Player 2 as the follower* if:

- for any admissible strategy $(u^1, u^2) \in U^1 \times U^2$ the next inequalities hold

$$F^1\left(u^{1*}, u^{2*}\right) \le F^1\left(u^1, u^{2^o}\left(u^1\right)\right), \quad u^{2*} := u^{2^o}\left(u^{1*}\right) \qquad (5)$$

where:

$$F^2\left(u^1, u^{2^o}\left(u^1\right)\right) = \min_{u^2 \in U^2} F^2\left(u^1, u^2\right) \qquad (6)$$

Note here that the main difference with the Standard leader-follower equilibrium is the max operation taken over all possible scenarios.

B. Open-Loop Robust leader-follower Strategies for Multi-Model Differential Games

Proceeding first outlining the robust optimal reaction of the follower, let us represent the robust optimal control problem for the follower following the standard procedure in Optimal Control Theory. For each possible scenario $\alpha \in \mathcal{A}$ introduce the extended variables for the follower $\bar{x}^{\alpha,f} = \left(x_1^\alpha, ..., x_n^\alpha, x_{n+1}^{\alpha,f}\right)$ defined in $\Re^{n+1}$ and the last component $x_{n+1}^{\alpha,f}$ given by

$$x_{n+1}^{\alpha,f} = \int_{t=t_0}^T g_2^\alpha\left(x^\alpha, u^1, u^2, \tau\right) d\tau$$

or, in differential form,

$$\dot{x}_{n+1}^{\alpha,f} = g_2^\alpha\left(x^\alpha, u^1, u^2, t\right), \qquad x_{n+1}^{\alpha,f}\left(t_0\right) = 0 \qquad (7)$$

Now the initial individual aim performance for the follower (4) can be represented in the Mayer form (without an integral term):

$$h^{2,\alpha} = h_0^{2,\alpha}\left(x\left(T\right)\right) + x_{n+1}^{\alpha,f}\left(T\right) \qquad (8)$$

Notice that $h_0^{2,\alpha}\left(x^\alpha\right)$ does not depend on the last coordinate $x_{n+1}^{\alpha,f}$, that is, $\dfrac{\partial}{\partial x_{n+1}^{\alpha,f}} h_0^{2,\alpha}\left(x^\alpha\right) = 0$. Define also the new extended conjugate vector-variable by

$$\psi^\alpha := \left(\psi_1^\alpha, ..., \psi_n^\alpha\right) \in \Re^n$$
$$\bar{\psi}^\alpha := \left(\psi_1^\alpha, ..., \psi_n^\alpha, \psi_{n+1}^\alpha\right) \in \Re^{n+1}$$

satisfying

$$\dot{\psi}_j^\alpha = -\sum_{k=1}^n \frac{\partial f_k^\alpha\left(x^\alpha, u^1, u^2, t\right)}{\partial x_j^\alpha}\psi_k^\alpha - \frac{\partial g_2^\alpha\left(x^\alpha, u^1, u^2, t\right)}{\partial x_{n+1}^\alpha}\psi_{n+1}^\alpha$$
$$j = 1, ..., n+1 \qquad (9)$$

with some terminal condition

$$\psi_j^\alpha\left(T\right) = b_j^\alpha \qquad 0 \le t \le T, \qquad \alpha \in \mathcal{A} \qquad (j = 1, ..., n+1) \qquad (10)$$

where the vector b^α will be defined below. For the "*super-extended*" vectors for the follower defined by

$$\bar{x}^{\diamond,f} := \left(x_1^1, ..., x_{n+1}^1; ...; x_1^M, ..., x_{n+1}^M\right)^\mathsf{T}$$
$$\bar{\psi}^\diamond := \left(\psi_1^1, ..., \psi_{n+1}^1; ...; \psi_1^M, ..., \psi_{n+1}^M\right)^\mathsf{T}$$
$$\bar{f}^{\diamond,f} := \left(f_1^1, ..., f_n^1, g_2^1; ...; f_1^M, ..., f_n^M, g_2^M\right)^\mathsf{T}$$
$$\bar{f}^{\alpha,f} = \left(f_1^\alpha, ..., f_n^\alpha, g_2^\alpha\right) \in \Re^{n+1}$$

these vectors represent the complete family of trajectories including the $n+1$ state variables for the follower. Using the previous vectors, following [8], define the next "*generalized*" Hamiltonian function for the follower:

$$\mathcal{H}_2^\diamond\left(\bar{\psi}^\diamond, \bar{x}^{\diamond,f}, u^1, u^2, t\right) := \left\langle \bar{\psi}^\diamond, \bar{f}^{\diamond,f}\left(\bar{x}^{\diamond,f}, u^1, u^2, t\right)\right\rangle =$$
$$\sum_{\alpha \in \mathcal{A}} \mathcal{H}_2^\alpha\left(\bar{\psi}^\alpha, \bar{x}^{\alpha,f}, u^1, u^2, t\right) =$$
$$\sum_{\alpha \in \mathcal{A}} \left[\sum_{\alpha \in \mathcal{A}} \left(\psi_j^\alpha f_j^\alpha\left(x^\alpha, u^1, u^2, t\right)\right) + \psi_{n+1}^\alpha g_2^\alpha\left(x^{\alpha,f}, u^1, u^2, t\right)\right] \qquad (11)$$

the direct (3), (7) and conjugate (9) ODE equations may be represented shortly in the standard Hamiltonian form as

$$\frac{d}{dt}\bar{x}^{\diamond,f} = \frac{\partial \mathcal{H}_2^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{x}^{\diamond,f}, u^1, u^2, t\right)}{\partial \bar{\psi}^{\diamond}}$$
$$\frac{d}{dt}\bar{\psi}^{\diamond} = -\frac{\partial \mathcal{H}_2^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{x}^{\diamond,f}, u^1, u^2, t\right)}{\partial \bar{x}^{\diamond,f}} \tag{12}$$

Again, as it follows from the definition (6), and as it is shown in [1], [2], the optimal reaction u^{2^o} of the follower to and fixed leader strategy u^1 of the leader, is solved as an standard optimal control problem, for the presented Multi-Model problem the solution is given as in [9], the follower should solve the following *robust optimal control problem*:

$$\max_{\alpha \in \mathcal{A}} h^{2,\alpha}\left(u^{1*}, u^2\left(u^1\right)\right) \to \min_{u^2 \in U_{adm}^2}$$
$$U_{adm}^i := \left\{u^i \mid \left(u^1, u^{2^o}\left(u^1\right)\right)\right\} \tag{13}$$

that provides the robust Stackelberg strategy for the follower.

Following the result of [9] the necessary condition for a robust optimality of a strategy for the follower must fulfill the next condition:

- (**the maximality condition**) The control strategies for the follower $u^{2^o}(t) \in U_{adm}^2$ $(t \in [0,T])$ satisfies

$$\mathcal{H}_2^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{x}^{\diamond,f}, u^1, u^{2^o}, t\right) \geq \mathcal{H}_2^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{x}^{\diamond,f}, u^1, u^2, t\right)$$

or, equivalently,

$$u^{2^o} \in \text{Arg} \max_{u^2 \in U_{adm}^2} \mathcal{H}_2^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{x}^{\diamond,f}, u^1, u^{2^o}, t\right) \tag{14}$$

- (**the complementary slackness condition**) For every $\alpha \in \mathcal{A}$ next conditions hold

$$\mu^2(\alpha)\left(h^{2,\alpha} - F^{2*}\right) = 0 \tag{15}$$

- (**the transversality condition (Follower)**) For every $\alpha \in \mathcal{A}$ and every $i = 1, ..., N$

$$b^{\alpha} + \mu^2(\alpha)\,\text{grad}h^{2,\alpha}\left(x^{\alpha*}(T)\right) = 0$$
$$\psi^{\alpha}(T) = b^{\alpha}, \qquad \psi_{n+1}^{\alpha}(T) + \mu^2(\alpha) = 0 \tag{16}$$

To derive the robust optimal condition for the leader, using the same Mayer representation as before, the functional for the leader is represented as:

$$h^{1,\alpha} = h_0^{1,\alpha}\left(x(T)\right) + x_{n+1}^{\alpha,l}(T)$$

where

$$x_{n+1}^{\alpha,l} = \int_{t=t_0}^{T} g_1^{\alpha}\left(x^{\alpha}, u^1, u^2, \tau\right) d\tau$$

Now consider the next hamiltonian representation for the leader:

$$\mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^1, u^2, t\right) :=$$
$$\left\langle \bar{\lambda}_1^{\diamond}, \bar{f}^{\diamond,l}\left(\bar{x}^{\diamond,l}, u^1, u^2, t\right)\right\rangle - \bar{\lambda}_2^{\diamond\mathsf{T}}\frac{\partial \mathcal{H}_2^{\diamond}}{\partial \bar{x}^{\diamond,f}} + \lambda_3^{\diamond\mathsf{T}}\frac{\partial \mathcal{H}_2^{\diamond}}{\partial u^2} =$$
$$\sum_{\alpha \in \mathcal{A}} \mathcal{H}_1^{\alpha}\left(\bar{\psi}^{\alpha}, \bar{\lambda}_1^{\alpha}, \bar{\lambda}_2^{\alpha}, \lambda_3^{\alpha}, \bar{x}^{\alpha,l}, u^1, u^2, t\right) =$$
$$\sum_{\alpha \in \mathcal{A}} \left[\sum_{\alpha \in \mathcal{A}} \left(\lambda_{1,j}^{\alpha} f_j^{\alpha}\left(x^{\alpha}, u^1, u^2, t\right)\right) + \right.$$
$$\left. \lambda_{1,n+1}^{\alpha} g_1^{\alpha}\left(x^{\alpha,f}, u^1, u^2, t\right) - \bar{\lambda}_2^{\alpha\mathsf{T}}\frac{\partial \mathcal{H}_2^{\diamond}}{\partial \bar{x}^{\diamond,f}} + \lambda_3^{\alpha\mathsf{T}}\frac{\partial \mathcal{H}_2^{\diamond}}{\partial u^2} \right.$$

where the extended vector are defined as:

$$\bar{x}^{\diamond,l} := \left(x_1^1, ..., x_{n+1}^{1,l}; ...; x_1^M, ..., x_{n+1}^{M,l}\right)^{\mathsf{T}}$$
$$\bar{\lambda}_1^{\diamond} := \left(\lambda_{1,1}^1, ..., \lambda_{1,n+1}^1; ...; \lambda_{1,1}^M, ..., \lambda_{1,n+1}^M\right)^{\mathsf{T}}$$
$$\bar{\lambda}_2^{\diamond} := \left(\lambda_{2,1}^1, ..., \lambda_{2,n+1}^1; ...; \lambda_{2,1}^M, ..., \lambda_{2,n+1}^M\right)^{\mathsf{T}}$$
$$\bar{f}^{\diamond,l} := \left(f_1^1, ..., f_n^1, g_1^1; ...; f_1^M, ..., f_n^M, g_1^M\right)^{\mathsf{T}}$$
$$\bar{f}^{\alpha,l} = \left(f_1^{\alpha}, ..., f_n^{\alpha}, g_1^{\alpha}\right) \in \Re^{n+1}$$

with the adjoint variables satisfying conditions that must be satisfy by the leader

$$\frac{d}{dt}\bar{\lambda}_1^{\diamond} = \frac{\partial \mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^1, u^2, t\right)}{\partial \bar{x}^{\diamond,l}}$$
$$\frac{d}{dt}\bar{\lambda}_2^{\diamond} = -\frac{\partial \mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^1, u^2, t\right)}{\partial \bar{\psi}^{\diamond}}$$
$$\bar{\lambda}_2^{\diamond}(t_0) = 0$$

The application of the result in [9], yields to the next conditions that must be satisfy by the leader

- (**the maximality condition**) The control strategies for the leader $u^{1*}(t) \in U_{adm}^1$ $(t \in [t_0, T])$, satisfies:

$$\mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^{1*}, u^2, t\right) \geq$$
$$\mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^1, u^2, t\right)$$

equivalently,

$$u^{1*} \in \text{Arg} \max_{u^1 \in U_{adm}^1} \mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^1, u^2, t\right) \tag{17}$$

- (**the complementary slackness condition**) For every $\alpha \in \mathcal{A}$ next conditions hold

$$\mu^1(\alpha)\left(h^{1,\alpha} - F^{1*}\right) = 0 \tag{18}$$

- (**the transversality condition (Follower)**) For every $\alpha \in \mathcal{A}$ and every $i = 1, ..., N$

$$c^{\alpha} + \mu^1(\alpha)\left[\text{grad}h^{1,\alpha}\left(x^{\alpha*}(T)\right) - \right.$$
$$\left. \frac{\partial^2}{(\partial x^{\alpha*})^2}h^{2,\alpha}\left(x^{\alpha*}(T)\right)\lambda_2^{\alpha}(T)\right] = 0; \quad \lambda_1^{\alpha}(T) = c^{\alpha}$$
$$\lambda_{n+1}^{\alpha}(T) + \mu^1(\alpha) = 0 \tag{19}$$

additionally the leader must satisfy the next relation:

$$\frac{\partial \mathcal{H}_1^{\diamond}\left(\bar{\psi}^{\diamond}, \bar{\lambda}_1^{\diamond}, \bar{\lambda}_2^{\diamond}, \lambda_3^{\diamond}, \bar{x}^{\diamond,l}, u^{1*}, u^2, t\right)}{\partial u^2} = 0 \tag{20}$$

C. Open-Loop Robust Leader-Follower Strategies in LQ Differential Games

Consider the next two-player linear quadratic multimodel game:

$$\dot{x}^{\alpha}(t) = A^{\alpha}(t)x^{\alpha}(t) + \sum_{j=1}^{2} B^{\alpha,j}(t)u^j(t)$$
$$x^{\alpha}(t) \in \Re^n, \quad x^{\alpha}(t_0) = x_0, \quad u^j(t) \in \Re^{m_j}, \quad j = 1, 2;$$
$$t \in [t_0, T], \quad T < \infty; \quad \alpha \in \mathcal{A} = \{1, 2, ..., M\} \tag{21}$$

Suppose that the *individual aim performance* $h^{i,\alpha}$ of each i-player $(i=1,2)$ for each α-model (scenario) is given by

$$h^{i,\alpha}\left(u^1,u^2\right) = \tfrac{1}{2}x^{\alpha\mathsf{T}}\left(T\right)Q_f^i x^\alpha\left(T\right)+$$
$$\tfrac{1}{2}\int\limits_{t=t_0}^{T}\left[x^{\alpha\mathsf{T}}Q^i x^\alpha + \sum_{j=1}^{2}u^{j\mathsf{T}}R^{ij}u^j\right]dt \quad (22)$$

As we mentioned before the robust optimal control problem for each player is formulated as (13). In what follows we illustrate how the construction given above is applied for the case of two players leader follower LQ multi-model differential games. The Hamiltonians for the leader and follower LQ are:

$$\mathcal{H}_2^\Diamond = \sum_{\alpha\in\mathcal{A}}\left[\tfrac{1}{2}\psi_{n+1}^{\alpha\mathsf{T}}\left(x^{\alpha\mathsf{T}}Q^2 x^\alpha + \sum_{j=1}^{2}u^{j\mathsf{T}}R^{2j}u^j\right)+\right.$$
$$\left.\psi^{\alpha\mathsf{T}}\left(A^\alpha x^\alpha + B^{\alpha,1}u^1 + B^{\alpha,2}u^2\right)\right]$$

$$\mathcal{H}_1^\Diamond = \sum_{\alpha\in\mathcal{A}}\left[\tfrac{1}{2}\lambda_{1,n+1}^{\alpha\mathsf{T}}\left(x^{\alpha\mathsf{T}}Q^1 x^\alpha + \sum_{j=1}^{2}u^{j\mathsf{T}}R^{1j}u^j\right)+\right.$$
$$\lambda_1^{\alpha\mathsf{T}}\left(A^\alpha x^\alpha + B^{\alpha,1}u^1 + B^{\alpha,2}u^2\right) - \lambda_2^{\alpha\mathsf{T}}\left(Q^2 x^\alpha + A^{\alpha,\mathsf{T}}\psi^\alpha\right)+$$
$$\left.\left(R^{22}u^2 + \psi^{\alpha\mathsf{T}}B^{\alpha,2}\right)\lambda_3^\alpha\right] \quad (23)$$

The conditions (14)-(20) for the LQ multimodel plant are as follows:

$$\dot\psi^\alpha = -\frac{\partial}{\partial\bar x^\alpha}\mathcal{H}_2^\alpha = -A^{\alpha\mathsf{T}}\psi^\alpha - \psi_{n+1}^{\alpha\mathsf{T}}Q^2 x^\alpha; \quad \dot\psi_{n+1}^{\alpha\mathsf{T}}(t)=0$$
$$\psi^\alpha\left(T\right) = -\mu^2\left(\alpha\right)\mathrm{grad}\left[x^{\alpha\mathsf{T}}\left(T\right)Q_f^2 x^\alpha\left(T\right)+x_{n+1}^{\alpha,f}\left(T\right)\right]=$$
$$-\mu^2\left(\alpha\right)Q^2 x^\alpha\left(T\right); \quad \psi_{n+1}^{\alpha\mathsf{T}}\left(T\right)=-\mu^2\left(\alpha\right)$$
$$\dot\lambda_1^\alpha = -\frac{\partial}{\partial\bar x^\alpha}\mathcal{H}_1^\alpha = -A^{\alpha\mathsf{T}}\lambda_1^\alpha - \lambda_{1,n+1}^\alpha Q^1 x^\alpha + Q^2\lambda_2^\alpha$$
$$\dot\lambda_{1,n+1}^\alpha\left(t\right)=0$$
$$\lambda_1^\alpha\left(T\right) = \mu^1\left(\alpha\right)\left[Q_f^1 x^\alpha\left(T\right) - Q_f^2\lambda_2^\alpha\left(T\right)\right]$$
$$\lambda_{1,n+1}^\alpha\left(T\right) = -\mu^1\left(\alpha\right)$$
$$\dot\lambda_2^\alpha = -\frac{\partial}{\partial\bar\psi^\alpha}\mathcal{H}_1^\alpha = -A^\alpha\lambda_2^\alpha + B^{\alpha,2}\lambda_3^\alpha; \quad \lambda_2^\alpha\left(t_0\right)=0$$
$$\quad (24)$$

the robust Stackelberg strategies for the leader satisfies:

$$u^{1*}(t) = \left(\sum_{\alpha\in\mathcal{A}}\mu^1\left(\alpha\right)\right)^{-1}\left(R^{11}\right)^{-1}\sum_{\alpha\in\mathcal{A}}B^{\alpha,1\mathsf{T}}\lambda_1^\alpha \quad (25)$$

and for the follower

$$u^{2*}(t) = \left(\sum_{\alpha\in\mathcal{A}}\mu^2\left(\alpha\right)\right)^{-1}\left(R^{22}\right)^{-1}\sum_{\alpha\in\mathcal{A}}B^{\alpha,2\mathsf{T}}\psi^\alpha \quad (26)$$

and

$$\frac{\partial\mathcal{H}_1^\Diamond}{\partial u^2} = \sum_{\alpha\in\mathcal{A}}\left[-\left(\mu^1\left(\alpha\right)\right)R^{12}u^{2*}\left(t\right)+\lambda_1^{\alpha\mathsf{T}}B^{\alpha,2}+R^{22}\lambda_3^\alpha\right]$$
$$= \sum_{\alpha\in\mathcal{A}}\left[-\mu^1\left(\alpha\right)\left(\mu^2\left(\alpha\right)\right)^{-1}R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\mathsf{T}}\psi^\alpha\right.$$
$$\left.+\lambda_1^{\alpha\mathsf{T}}B^{\alpha,2}+R^{22}\lambda_3^\alpha\right]=0 \quad (27)$$

finally

$$\sum_{\alpha\in\mathcal{A}}\lambda_3^\alpha = \sum_{\alpha\in\mathcal{A}}\left(R^{22}\right)^{-1}\times$$
$$\left[\mu^1\left(\alpha\right)\left(\mu^2\left(\alpha\right)\right)^{-1}R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\mathsf{T}}\psi^\alpha - B^{\alpha,2}\lambda_1^\alpha\right] \quad (28)$$

Since at least one index $\alpha\in\mathcal{A}$ is active we have: $\sum\limits_{\alpha\in\mathcal{A}}\mu^i\left(\alpha\right)>0$. Introducing the normalized adjoint variables with as

$$\tilde\psi_n^\alpha\left(t\right) = \begin{cases}\psi_n^\alpha\left(t\right)\left(\mu^2\left(\alpha\right)\right)^{-1} & \text{if } \mu^2\left(\alpha\right)>0 \\ 0 & \text{if } \mu^2\left(\alpha\right)=0\end{cases}$$
$$\tilde\lambda_{k,n}^\alpha\left(t\right) = \begin{cases}\lambda_n^\alpha\left(t\right)\left(\mu^1\left(\alpha\right)\right)^{-1} & \text{if } \mu^1\left(\alpha\right)>0 \\ 0 & \text{if } \mu^1\left(\alpha\right)=0\end{cases}$$
$$k=1,2,3$$

we get

$$\dot{\tilde\psi}^\alpha\left(t\right) = -A^{\alpha\mathsf{T}}\tilde\psi^\alpha - \tilde\psi_{n+1}^{\alpha\mathsf{T}}Q^2 x^\alpha$$
$$\dot{\tilde\psi}_{n+1}^\alpha\left(t\right) = 0;$$

with the corresponding transversality conditions given by

$$\tilde\psi^\alpha\left(T\right) = -Q^2 x^\alpha\left(T\right), \quad \tilde\psi_{n+1}^{\alpha\mathsf{T}}\left(T\right)=-1$$
$$\dot{\tilde\lambda}_1^\alpha\left(t\right) = -A^{\alpha\mathsf{T}}\tilde\lambda_1^\alpha - \tilde\lambda_{1,n+1}^{\alpha\mathsf{T}}Q^1 x^\alpha + Q^2\tilde\lambda_2^\alpha$$
$$\tilde\lambda_1^\alpha\left(T\right) = Q_f^1 x^\alpha\left(T\right) - Q_f^2\tilde\lambda_2^\alpha\left(T\right); \quad \tilde\lambda_{1,n+1}^\alpha\left(T\right)=-1$$
$$\dot{\tilde\lambda}_2^\alpha = -A^\alpha\tilde\lambda_2^\alpha + B^{\alpha,2}\tilde\lambda_3^\alpha; \quad \tilde\lambda_2^\alpha\left(t_0\right)=0$$

the robust Stackelberg strategies becomes:

$$u^{1*}(t) = \left(\sum_{\alpha\in\mathcal{A}}\mu^1\left(\alpha\right)\right)^{-1}\left(R^{11}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\left(\mu^1\right)B^{\alpha,1\mathsf{T}}\tilde\lambda_1^\alpha =$$
$$-\left(R^{11}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\nu^{\alpha,1}B^{\alpha,1\mathsf{T}}\tilde\lambda_1^\alpha \quad (29)$$

and for the follower

$$u^{2*}(t) = \left(\sum_{\alpha\in\mathcal{A}}\mu^2\left(\alpha\right)\right)^{-1}\left(R^{22}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\left(\mu^2\right)B^{\alpha,2\mathsf{T}}\tilde\psi^\alpha =$$
$$-\left(R^{22}\right)^{-1}\sum_{\alpha\in\mathcal{A}}\nu^{\alpha,2}B^{\alpha,2\mathsf{T}}\tilde\psi^\alpha \quad (30)$$

where the vectors $\boldsymbol\nu^i := \left(\nu_1^{\alpha,1},...,\nu_1^{\alpha,M}\right)^\mathsf{T}$ $(i=1,2)$ belongs to the simplex:

$$S^{i,M}:=\left\{\boldsymbol\nu^i\in R^{M=|\mathcal{A}|}:\nu^{\alpha,i}=\mu^i\left(\alpha\right)\left(\sum_{\alpha\in\mathcal{A}}\mu^i\left(\alpha\right)\right)^{-1}\geq 0,\right.$$
$$\left.\sum_{\alpha\in\mathcal{A}}\boldsymbol\nu^{\alpha,i}=1\right\} \quad (31)$$

Finally

$$\sum_{\alpha\in\mathcal{A}}\tilde\lambda_3^\alpha = \sum_{\alpha\in\mathcal{A}}\left(R^{22}\right)^{-1}\left[R^{12}\left(R^{22}\right)^{-1}B^{\alpha,2\mathsf{T}}\tilde\psi^\alpha - B^{\alpha,2}\tilde\lambda_1^\alpha\right] \quad (32)$$

As one can see from (29) and (30), the Robust Optimal Control for both leader an follower is a mixture of the control actions optimal for each independent index $\alpha\in\mathcal{A}$.

D. Extended form for the game

Now consider the next representation for the game

$$\mathbf{A}:=\begin{bmatrix} A^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^M \end{bmatrix}; \ \mathbf{Q}^i:=\begin{bmatrix} Q^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q^i \end{bmatrix}$$

$$\mathbf{Q}^i_f:=\begin{bmatrix} Q^i_f & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q^i_f \end{bmatrix}, \ \mathbf{\Gamma}^i:=\begin{bmatrix} \nu^{1,i}I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu^{M,i}I \end{bmatrix}$$

$$\mathbf{B}^{i\intercal} := \begin{bmatrix} B^{i,1\intercal} \cdots B^{i,M\intercal} \end{bmatrix}, \quad I \in \Re^{n \times n} \ \ i=1,2 \tag{33}$$

In the extended form we obtain the general dynamics given by (bold stand for extended vectors and matrices):

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x} + \mathbf{B}^1 u^1 + \mathbf{B}^2 u^2, \ \mathbf{x}^\intercal(t_0)=\left(x^{1\intercal}(0),...,x^{M\intercal}(0)\right)$$
$$\dot{\boldsymbol{\psi}}=-\mathbf{A}^\intercal\boldsymbol{\psi}+\mathbf{Q}^2\mathbf{x}; \ \boldsymbol{\psi}(T)=-\mathbf{Q}^2_f\mathbf{x}(T);$$
$$\dot{\boldsymbol{\lambda}}_1=-\mathbf{A}^\intercal\boldsymbol{\lambda}_1-\mathbf{Q}^1\mathbf{x}+\mathbf{Q}^2\boldsymbol{\lambda}_2; \ \boldsymbol{\lambda}_1(T)=\mathbf{Q}^1_f\mathbf{x}(T)-\mathbf{Q}^2_f\boldsymbol{\lambda}_2(T)$$
$$\dot{\boldsymbol{\lambda}}_2 = -\mathbf{A}\boldsymbol{\lambda}_2 + \mathbf{B}^2\boldsymbol{\lambda}_3; \ \boldsymbol{\lambda}_2(t_0)=0$$
$$\boldsymbol{\lambda}_3 = \left(R^{22}\right)^{-1}\left[R^{12}\left(R^{22}\right)^{-1}\mathbf{B}^{2\intercal}\boldsymbol{\psi} - \mathbf{B}^{2\intercal}\boldsymbol{\lambda}_1\right] \tag{34}$$

$$u^1 = -\left(R^{11}\right)^{-1}B^{1\intercal}\Gamma^1\boldsymbol{\lambda}_1$$
$$u^2 = -\left(R^{22}\right)^{-1}B^{2\intercal}\Gamma^2\boldsymbol{\psi} \tag{35}$$

where

$$\mathbf{x}^\intercal := \left(x_1^{1,\intercal},...,x_n^{1,\intercal};...;x_1^{M,\intercal},...,x_n^{M,\intercal}\right) \in \Re^{1 \times nM}$$
$$\boldsymbol{\psi}^\intercal := \left(\tilde{\psi}_1^{1,\intercal},...,\tilde{\psi}_n^{1,\intercal};...;\tilde{\psi}_1^{M,\intercal},...,\tilde{\psi}_n^{M,\intercal}\right) \in \Re^{1 \times nM}$$
$$\boldsymbol{\lambda}_k^\intercal := \left(\tilde{\lambda}_{k,1}^{1,\intercal},...,\tilde{\lambda}_{k,n}^{1,\intercal};...;\tilde{\lambda}_{k,1}^{M,\intercal},...,\tilde{\lambda}_{,n}^{M,\intercal}\right) \in \Re^{1 \times nM}$$
$$k=1,2,3$$

Theorem: If for the two person linear quadratic differential game (34) with the following restrictions to the matrices: $R^{ii} > 0$, $R^{12} \geq 0$, $\mathbf{Q}^i \geq 0$ and $\mathbf{Q}^i_f \geq 0$ ($i = 1,2$) there exists a solution set of the following parametrized coupled differential equation [14]:

$$\dot{\mathbf{P}}_{\nu^{2*}}\left(\mathbf{A} - \mathbf{B}^1 R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{\Lambda}_{1,\nu^{1*}} - \mathbf{B}^2 R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right) +$$
$$\mathbf{P}_{\nu^{2*}} + \mathbf{A}^\intercal\mathbf{P}_{\nu^{2*}} + \Gamma^{2*}\mathbf{Q}^2 = 0;$$
$$\mathbf{P}_{\nu^{2*}}(T) = \Gamma^{2*}\mathbf{Q}^2_f; \ \ \mathbf{P}_{\nu^{2*}} = \mathbf{P}^\intercal_{\nu^{2*}} \in \Re^{nM \times nM}$$
$$\dot{\mathbf{\Lambda}}_{1,\nu^{1*}}\left(\mathbf{A} - \mathbf{B}^1 R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{\Lambda}_{1,\nu^{1*}} - \mathbf{B}^2 R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right) +$$
$$\mathbf{\Lambda}_{1,\nu^{1*}} + \mathbf{A}^\intercal\mathbf{\Lambda}_{1,\nu^{1*}} + \Gamma^{1*}\mathbf{Q}^1 - \Gamma^{1*}\mathbf{Q}^2\mathbf{\Lambda}_{2,\nu^{1*}} = 0;$$
$$\mathbf{\Lambda}_{1,\nu^{1*}}(T) = \Gamma^{1*}\left(\mathbf{Q}^1_f - \mathbf{Q}^2_f\mathbf{\Lambda}_{2,\nu^{1*}}(T)\right)$$
$$\dot{\mathbf{\Lambda}}_{2,\nu^{1*}}\left(\mathbf{A} - \mathbf{B}^1 R^{11^{-1}}\mathbf{B}^{1\intercal}\mathbf{\Lambda}_{1,\nu^{1*}} - \mathbf{B}^2 R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\right)$$
$$+\mathbf{\Lambda}_{2,\nu^{1*}}+\mathbf{A}\mathbf{\Lambda}_{2,\nu^{1*}} + \Gamma^{1*}\mathbf{B}^2 R^{22^{-1}}R^{12}R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}} -$$
$$\Gamma^{1*}\mathbf{B}^2 R^{22^{-1}}\mathbf{B}^{2\intercal}\mathbf{\Lambda}_{1,\nu^{1*}} = 0; \ \ \Gamma^{1*}\mathbf{\Lambda}_{2,\nu^{1*}}(t_0) = 0 \tag{36}$$

then the open-loop robust leader-follower equilibrium strategies with *Player 1* acting as the leader are:

$$u^1 = -\left(R^{11}\right)^{-1}\mathbf{B}^{1\intercal}\mathbf{\Lambda}_{1,\nu^{1*}}\mathbf{x}$$
$$u^2 = -\left(R^{22}\right)^{-1}\mathbf{B}^{2\intercal}\mathbf{P}_{\nu^{2*}}\mathbf{x} \tag{37}$$

define the open-loop robust leader-follower equilibrium solution if the matrix $\Gamma^{i*}(i=1,2)$ in (33) contains the vectors

ν^{i*} which satisfy the leader-follower equilibrium condition:

$$J^1\left(\nu^{1*}, \nu^{2*}\left(\nu^1\right)\right) \leq J^1\left(\nu^1, \nu^{2o}\left(\nu^1\right)\right)$$
$$\nu^{2o}\left(\nu^1\right) := \arg\min_{\nu^2 \in S^N} J^2\left(\nu^1, \nu^2\right) \tag{38}$$
$$\text{for any } \nu^i \in S^{i,M}$$

where

$$J^i\left(\nu^{1*}, \nu^{2*}\right) := \max_{\alpha \in \mathcal{A}} h^{i,\alpha}\left(u^1, u^2\right)$$
$$i = 1,2 \tag{39}$$

with given by (37) parametrized by ν^{1*} and ν^{2*} ($\nu^i \in S^{i,M}$) through (36).

Proof: Let us try to represent $\Gamma^2\boldsymbol{\psi}$ as $\Gamma^2\boldsymbol{\psi} = -\mathbf{P}_{\nu^2}\mathbf{x}$, $\Gamma^1\boldsymbol{\lambda}_1 = -\mathbf{\Lambda}_{1,\nu^1}\mathbf{x}$, and $\Gamma^1\boldsymbol{\lambda}_2 = -\mathbf{\Lambda}_{2,\nu^1}\mathbf{x}$. By (36) and by the commutation of the operators $\Gamma^i\mathbf{A}^\intercal = \mathbf{A}^\intercal\Gamma^i$ and $\Gamma^i\mathbf{Q}^i = \mathbf{Q}^i\Gamma^i$ the result follows.∎

IV. NUMERICAL PROCEDURE

For the problem of finding the leader-follower equilibrium weights $\left(\nu^1, \nu^2\left(\nu^1\right)\right)$ we propose the use of the next minimizing numerical procedure. To make this analytically is not a simple task. Assuming that $J^i\left(\nu^1, \nu^2\left(\nu^1\right)\right) > 0$ for all $\nu^i \in S^{i,M}$ ($i = 1,2$) (if not, one can add to the cost functions any necessary positive constant that, evidently does not change anything) , define the series of the vectors iterations $\{\nu^{i,k}\}$ (for any fixed n) as

$$\nu^{i,k+1} = \pi_{S^{i,M}}\left\{\nu^{i,k} + \frac{\gamma^{i,k}}{\tilde{J}^i\left(\nu^{1,k}, \nu^{2,k}\right)}\tilde{F}^i\left(\nu^{1,k}, \nu^{2,k}\right)\right\}$$
$$\nu^{i,0} \in S^{i,M}, \ \ k = 1,2,...$$
$$\tilde{F}^i\left(\nu^{1,k}, \nu^{2,k}\right) = \left[\tilde{h}^{1,i}\left(\nu^{1,k}, \nu^{2,k}\right), \tilde{h}^{M,i}\left(\nu^{1,k}, \nu^{2,k}\right)\right]$$
$$\tilde{J}^i\left(\nu^{1,k}, \nu^{2,k}\right) = \max_{\alpha \in \overline{1,N}}\tilde{h}^{\alpha,i}\left(\nu^{1,k}, \nu^{2,k}\right) \tag{40}$$

where $\pi_{S^{i,M}}\{\cdot\}$ is the projector of an argument to the simplex $S^{i,M}$ and the new functional $\tilde{h}^{\alpha,i}$ is defined as:

$$\tilde{h}^{\alpha,i}\left(\nu^{1,k}, \nu^{2,k}\right) := \frac{\delta}{2}\left\|\nu^{(i)}\right\|^2 + h^{\alpha,i}\left(\nu^{1,k}, \nu^{2,k}\right) \tag{41}$$

whether this algorithm converges to a unique point or not we let this discussion for a large version of this work.

Algorithm. The algorithm for finding a solution for the robust leader-follower strategies is summarized as follows:

- Step 1) Select an initial condition for the leader's weights Γ^1.
- Step 2) Applying the control action equal to the weighted combination of standard stackelberg strategies, calculate all possible (for each scenario) dynamics and the corresponding cost functionals.
- Step 3) Using the corresponding cost functionals perform iteratively the minimizing procedure (40) to find the optimal response of the follower Γ^2 for a given fixed strategy of the leader.
- Step 4) Given the optimal response of the follower Γ^2 keep fixed this

values and use again (40) to minimize the leader's weights Γ^1.

- Step 5) Continue form the Step 1 making the initial conditions with values found in 3) until convergence.

A. Solving Coupled equations

For the solution of the set of coupled Riccati equations (36), we follow the work of [2], which is based on the solution of an auxiliary system. For the lack of space we omit the details of this method.

V. NUMERICAL EXAMPLE

Consider the next two-scenarios two-players LQ differential game given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}^1\mathbf{u}^1 + \mathbf{B}^2\mathbf{u}^2,$$

$$\mathbf{A} = \left[\begin{array}{cc} A^1 & 0 \\ 0 & A^2 \end{array}\right]; x_0 = \left(\begin{array}{c} 3 \\ -1 \end{array}\right)$$

$$A^1 = \left(\begin{array}{cc} 0.25 & 0.2 \\ 0.7 & 0.7 \end{array}\right); A^2 = \left(\begin{array}{cc} 0.3 & 0.15 \\ 0.7 & -0.5 \end{array}\right);$$

$$B^{1,1} = B^{1,2} = \left(\begin{array}{c} 2 \\ 0 \end{array}\right); B^{2,1} = B^{2,2} = \left(\begin{array}{c} 0 \\ 1 \end{array}\right)$$

$$Q^{1,1} = Q^{1,2} = 1 \times I_{2\times 2}; Q^{2,1} = Q^{2,2} = 1.5 \times I_{2\times 2}$$

$$Q_f^{1,1} = 3I_{2\times 2}; Q_f^{1,2} = 2I_{2\times 2}; Q_f^{2,1} = 3I_{2\times 2}; Q_f^{2,2} = 2I_{2\times 2}$$

$$R^{ii} = R^{ij} = I_{2\times 2} \qquad i,j = 1,2$$

The tables below show the convergence to dependence of the cost functionals on the weights $\nu^1 = \left(\nu_1^1, 1 - \nu_1^1\right)$, $\nu^2 = \left(\nu_1^2, 1 - \nu_1^2\right)$, $\left(\nu_1^1 \geq 0, \nu_1^2 \geq 0\right)$.

k	ν_1^1	ν_1^2	$h^{1,1}$	$h^{1,2}$
1	0.5000	0.5000	6.3146	6.3181
⋮	⋮	⋮	⋮	⋮
7	0.4972	0.5028	6.3203	6.3201
8	0.4972	0.5028	6.3202	6.3202
⋮	⋮	⋮	⋮	⋮
36	0.4969	0.5031	6.3269	6.3269
37	0.4969	0.5031	6.3269	6.3269

Table 1 Cost Function Leader

k	ν_1^1	ν_1^2	$h^{2,1}$	$h^{2,2}$
1	0.5000	0.5000	2.2697	2.2697
⋮	⋮	⋮	⋮	⋮
6	0.3742	0.6258	2.2717	2.2690
7	0.3742	0.6258	2.2677	2.2727
⋮	⋮	⋮	⋮	⋮
36	0.2659	0.7341	2.2609	2.2709
37	0.2659	0.7341	2.2609	2.2709

Table 1 Cost Function Follower

As one can see in the tables 1 and 2 the numerical procedure works efficiently finding the robust strategies, in this case the algorithm practically finish after one cycle of performing 15 iterations for the follower and 5 iterations for the leader. At the end of the process the cost functionals arrives to the practically same values as we expected from the complementary slackness condition (15) and (18).

VI. CONCLUSIONS

In this paper the formulation of a concept for a type of robust Leader-Follower equilibrium for a Multi-Plant differential game was presented. The dynamic of the game for this kind of problems is given by a set of N different possible differential equations (Multi-Plant problem) with no information about the trajectory which is realized. The problem is solved designing of min-max strategies for each player which guarantee an equilibrium for the worst case scenario. The suggested approach was based on the Robust Maximum Principle, the conditions for a game to be in Robust Leader-Follower Equilibrium was given. As in the Nash equilibrium case the initial min-max differential game is converted into a standard static game given in a multidimensional simplex. The realization of the numerical procedure confirms the effectiveness of the suggested approach.

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