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Isolation of Process and Sensor Faults for a Class of Nonlinear Uncertain Systems

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Abstract— **This paper presents a unified fault isolation scheme for process faults and sensor faults in a class of nonlinear uncertain systems. The proposed fault diagnosis architecture consists of a fault detection estimator and a bank of isolation estimators, each corresponding to a particular fault type. Based on the class of nonlinear systems and fault types under consideration, adaptive thresholds are derived for the isolation estimators, and fault isolability conditions are rigorously investigated, characterizing the class of process faults and sensor faults that are isolable by the proposed scheme.**

I. INTRODUCTION

In recent years, there has been a lot of research activity in the design and analysis of fault diagnosis and accommodation schemes for different classes of dynamic systems (see, for example, [5], [1]). Considerable effort has been devoted to the development of fault diagnosis schemes for nonlinear systems in the framework of various kinds of assumptions and fault scenarios (see, for instance, [6], [7], [3], and the references cited therein).

In this paper, we focus on the *fault isolation* problem, which is a crucial step in the design of intelligent control and health management systems. Typically, fault isolation goes into effect after a fault is detected with the objective of determining the location/type of the fault. The fault isolation problem has been studied in the context of several different formulations including: (a) determining the particular type of the fault among a set of known (or partially known) possible fault types (for example, a bearing may exhibit abnormal behavior as a result of spalling, pitting, or overrolling of debris); (b) determining the particular faulty components among the set of all components under consideration (for example, sensor validation); (c) for spatially distributed systems, determining the physical location of the faulty subsystem (for example, locating faulty sensor clusters in a distributed wireless sensor networks monitoring a large physical region of interest).

In previous papers [11], [12], the authors investigated the nonlinear fault isolation problem in the context of type (a) and (b), respectively. However, for the process fault isolation problem we assume the sensors are healthy [11], and for the sensor fault isolation problem we assume there are no process faults [12]. In real-world applications, both the process (including plant and actuators) and the sensors are prone to faults. Fault isolation methods that only consider process faults could provide a wrong isolation decision in the presence of sensor faults, and vice versa. The objective of this paper is to develop a fault isolation method that deals with both process faults and sensor faults in a unified framework.

An architecture based on adaptive techniques has been presented with the aim of detecting and isolating process faults or sensor faults (see [6], [2], [10], [11], [12] for related results). In this connection, a design methodology for fault isolation of the considered class of process faults and sensor faults is provided, as well as a rigorous analytical framework aimed at characterizing the behaviors of the proposed scheme. The analysis of the fault isolation scheme focuses on: (i) determining adaptive thresholds for fault isolation; and (ii) deriving isolability conditions of the process faults and sensor faults under consideration on the basis of the so-called *fault mismatch function*, which provides a suitable measure of the mutual difference between faults.

The paper is organized as follows. Section II defines the classes of nonlinear systems and faults to be investigated. In Section III, the design of the proposed fault detection and isolation (FDI) scheme is described. In Section IV the design of adaptive thresholds is addressed, while in Section V the fault isolability analysis is carried out. Finally, the conclusions and future work are presented.

II. PROBLEM FORMULATION

Consider a class of nonlinear multi-input-multi-output (MIMO) dynamic systems described by

$$\begin{aligned} \dot{x} &= Ax + \gamma(y,u) + \eta(x,u,t) + \beta_x(t-T_x)\phi(y,u) \\ y &= Cx + d(x,u,t) + \beta_y(t-T_y)F\theta(t) \end{aligned} \tag{1}$$

where $x \in \Re^n$ is the system state vector, $u \in \Re^m$ is the input vector, $y \in \Re^l$ is the output vector, $\gamma : \Re^l \times \Re^m \mapsto \Re^n$, $\eta : \Re^n \times \Re^m \times \Re^+ \mapsto \Re^n$, $\phi : \Re^l \times \Re^m \mapsto \Re^n$, and $d : \Re^n \times \Re^m \times \Re^+ \mapsto \Re^l$ are smooth vector fields, and (A, C) is an observable pair. The state equations

$$\begin{aligned} \dot{x}_N &= Ax_N + \gamma(y_N,u) \\ y_N &= Cx_N \end{aligned}$$

represents the *known nominal* system dynamics, while the healthy system is described by

$$\begin{aligned} \dot{x}_H &= Ax_H + \gamma(y_H,u) + \eta(x_H,u,t) \\ y_H &= Cx_H + d(x_H,u,t). \end{aligned}$$

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The difference between the nominal model and the actual (healthy) system is due to the vector fields η and d, which represent the modeling uncertainties in the state equation and output equation, respectively. The changes in the system dynamics as a result of a process fault are characterized by the term $\beta_x(t-T_x)\phi(y,u)$ in (1). Specifically, $\beta_x(t-T_x)$ denotes the time profile of a process fault which occurs at some *unknown* time T_x, and $\phi(y,u)$ represents the nonlinear fault function. The changes in the system dynamics as a result of a sensor fault are characterized by $\beta_y(t-T_y)F\theta(t)$ in (1). Specifically, the vector $F\theta(t)$ represents a time–varying bias due to a sensor fault, and the function $\beta_y(t-T_y)$ characterizes the time profile of the sensor fault, where T_y is the *unknown* fault occurrence time.

In this paper, we only consider the case of *abrupt* (sudden) faults; therefore, $\beta_x(\cdot)$ and $\beta_y(\cdot)$ take the form of a step function $\beta(\cdot)$ given by

$$\beta(t-T_0) = \begin{cases} 0 & \text{if} \quad t < T_0 \\ 1 & \text{if} \quad t \geq T_0 \end{cases},$$

where $T_0 = T_x$ for process faults, and $T_0 = T_y$ for sensor faults. Moreover, the analysis is based on the assumption that only *a single fault* occurs, which is either a process fault or a sensor fault.

The class of sensor faults under consideration is represented by $F\theta(t)$, where $F \in \Re^l$ is the fault distribution vector, and $\theta(t)$ is the magnitude of the time–varying sensor bias. Because of the single fault assumption, the sensor fault distribution vector F has only one non-zero entry, which represents the corresponding corrupted output measurement. Depending on the location of the fault, the distribution vector F belongs to a class of l possible vectors $\{F^1, F^2, \ldots, F^l\}$, where, for any $j = 1, \cdots l$, only the j–th component of vector F^j is different from zero. Accordingly, the scalar $\theta^j(t) \in \Re$ is the magnitude of the time–varying bias in the j–th sensor.

The process fault function ϕ under consideration is modeled as a nonlinear function of measurable quantities y and u. It is assumed that there are N types of possible process faults in the fault class; specifically, $\phi(y,u)$ belongs to a finite set of functions given by

$$\mathcal{F}_P \triangleq \left\{\phi^1(y,u), \ \ldots, \ \phi^N(y,u)\right\} . \tag{2}$$

Each fault function ϕ^p, $p = 1, \cdots, N$, is described by

$$\phi^p(y,u) \triangleq \left[\,(\theta_1^p)^\top g_1^p(y,u), \ \cdots, \ (\theta_n^p)^\top g_n^p(y,u)\,\right]^\top, \tag{3}$$

where θ_i^p, $i = 1, \cdots, n$, is an unknown parameter vector assumed to belong to a known compact and convex set Θ_i^p (i.e., $\theta_i^p \in \Theta_i^p \subset \Re^{\upsilon_i^p}$) and g_i^p: $\Re^l \times \Re^m \mapsto \Re^{\upsilon_i^p}$ is a known smooth vector field. As discussed in [11], the process fault model described by (2) and (3) characterizes a general class of nonlinear faults where the vector field g_i^p represents the functional structure of the p-th fault affecting the i-th state equation, while the unknown parameter vector θ_i^p characterizes the "magnitude" of the fault. The dimension υ_i^p of each parameter vector θ_i^p is determined both by the type of fault and by the specific state equation considered.

The main objective of this paper is to develop a robust fault isolation scheme for process faults and sensor faults in nonlinear dynamic systems modeled by (1). It is worth noting that in addition to nonlinear systems in the form of (1), the presented algorithm can also be applied to more general nonlinear systems which are transformable to (1) using a local diffeomorphism [9], [10].

Throughout the paper, the following assumptions are made:

Assumption 1. *The modeling uncertainties, represented by η and d in (1), are unstructured and unknown nonlinear functions of x, u, and t, but bounded by given known functionals, i.e.,*

$$|\eta(x,u,t)| \leq \bar{\eta}(y,u,t), \ \ |d(x,u,t)| \leq \bar{d}(y,u,t)\,, \qquad \forall (x,y,u) \in \mathcal{X}\times\mathcal{Y}\times\mathcal{U}\,, \forall t \geq 0\,, \tag{4}$$

where the bounding functions $\bar{\eta}(y,u,t)$ and $\bar{d}(y,u,t)$ are known and uniformly bounded in $\mathcal{Y}\times\mathcal{U}\times\Re^+$, $\mathcal{X} \subset \Re^n$ is some compact domain of interest, and $\mathcal{U} \subset \Re^m$ and $\mathcal{Y} \subset \Re^l$ are the compact sets of admissible inputs and outputs, respectively.

Assumption 2. *The system state vector x belongs to a possibly unknown compact set $\mathcal{X} \subset \Re^n$ before and after the occurrence of a fault, that is, $x(t) \in \mathcal{X}$ for all $t \geq 0$.*

Assumption 3. *The rates of change of the process fault parameter vector $\theta^p(t)$ ($p = 1, \cdots, N$), and the sensor bias $\theta^j(t)$ ($j = 1, \cdots, l$), are uniformly bounded, respectively, i.e.,*

- *for process faults, $|\dot{\theta}^p(t)| \leq \alpha_p$ for all $t \geq 0$, where*

$$\theta^p \triangleq \left[(\theta_1^p)^\top, \ \cdots, \ (\theta_n^p)^\top\right]^\top,$$

and α_p is a known positive constant;

- *for sensor bias, $|\dot{\theta}^j(t)| \leq \delta$ for all $t \geq 0$, where δ is a known positive constant.*

Assumption 1 characterizes the class of modeling uncertainties under consideration. The bounds on the *unstructured* modeling uncertainties are needed in order to be able to distinguish between the effects of faults and modeling uncertainty [11].

Assumption 2 requires the boundedness of the state variables before and after the occurrence of a fault. Hence, it is assumed that the feedback control system is capable of retaining the boundedness of the state variables even in the presence of a fault. This is a technical assumption required for well posedness since the fault isolation design that we consider does not influence the closed-loop dynamics and stability. It is important to note that the proposed fault detection and isolation design does not depend on the structure of the controller.

Finally, in Assumption 3 known bounds on the rates of change of $\theta^p(t)$ and $\theta^j(t)$ are assumed. In practice, the rate bounds α_p and δ can be set by exploiting some *a priori* knowledge on the fault developing dynamics. In the case of a constant fault size or bias, we simply set $\alpha_p = 0$ or $\delta = 0$.

III. FAULT DETECTION AND ISOLATION ARCHITECTURE

The fault detection and isolation architecture is based on a bank of $N+l+1$ nonlinear adaptive estimators, where N is the number of different nonlinear process faults in the fault class $\mathcal{F}_P$, and l is the number of sensors or output variables under consideration. One of the nonlinear adaptive estimators is the *fault detection estimator* (FDE) used for detecting the occurrence of any faults, while the remaining $N+l$ nonlinear adaptive estimators are *fault isolation estimators* (FIEs) which are activated for the purpose of fault isolation only after a fault is detected.

A. Fault Detection Scheme

Based on the system model given by (1), the following FDE is chosen:

$$\begin{aligned} \dot{\hat{x}}^0 &= A\hat{x}^0 + L(y-\hat{y}^0) + \gamma(y,u), \qquad \hat{x}^0(0) = 0 \\ \hat{y}^0 &= C\hat{x}^0 , \end{aligned}$$

where $\hat{x}^0$ and $\hat{y}^0$ denote the estimated state and output vectors, respectively, and $L \in \Re^{n\times l}$ is design gain matrix. Moreover, let $\epsilon_x^0 \triangleq x-\hat{x}^0$ denote the state estimation error. Then

$$\dot{\epsilon}_x^0 = A_0\epsilon_x^0 - Ld(x,u,t) + \eta(x,u,t),\; t < min(T_x, T_y) \tag{5}$$

where the gain L is chosen such that the matrix $A_0 \triangleq (A-LC)$ is Hurwitz. Each component $\epsilon_{y_j}^0 \triangleq y_j - \hat{y}_j^0$, $j = 1,\ldots,l$ of the output estimation error is given by

$$\epsilon_{y_j}^0 = C_j\epsilon_x^0 + d_j(x,u,t)\,, \tag{6}$$

where C_j is the j-th row vector of matrix C, and d_j is the j-th component of the measurement uncertainty d. From (5) and (6), we have

$$\begin{aligned} |\epsilon_{y_j}^0(t)| \;\; &\leq \;\; \int_0^t k_j\, e^{-\lambda_j(t-\tau)}\,|\,\eta(x,u,\tau) - Ld(x,u,\tau)\,|d\tau \\ &\quad + |d_j(x(t),u(t),t)| + k_j\bar{x}e^{-\lambda_j t}\,, \end{aligned} \tag{7}$$

where k_j and λ_j are positive constants chosen such that $|C_je^{A_0t}| \leq k_je^{-\lambda_jt}$ (since A_0 is Hurwitz, constants k_j and λ_j satisfying the above inequality always exist [8]), and $\bar{x}$ is a (possibly conservative) bound for $|x|$, such that $|x| \leq \bar{x}$, $\forall x \in \mathcal{X}$ (by Assumption 2). It is worth noting that, since the effect of this bound decreases exponentially, the use of such a conservative bound will not weaken significantly the performance of the fault detection scheme. By using (4) and (7), the fault detection scheme is designed as follows. A fault is *detected* when at least one component of the modulus of the output estimation error $|\epsilon_{y_j}^0(t)|$ exceeds its corresponding threshold $\bar{\epsilon}_{y_j}^0(t)$, which is defined as

$$\begin{aligned} \bar{\epsilon}_{y_j}^0(t) \;\; &\triangleq \;\; \int_0^t k_j\, e^{-\lambda_j(t-\tau)}\left[\bar{\eta}(y,u,\tau) + ||L||\,\bar{d}(y,u,\tau)\right] \\ &\quad \cdot d\tau + k_j\bar{x}e^{-\lambda_jt} + \bar{d}_j(y(t),u(t),t)\,. \end{aligned} \tag{8}$$

More precisely, the *fault detection time* T_d is defined as $T_d \triangleq \inf \bigcup_{j=1}^{l} \left\{ t \geq 0 :\; \left|\epsilon_{y_j}^0(t)\right| > \bar{\epsilon}_{y_j}^0(t)\right\}$.

B. Fault Isolation Estimators and Decision Scheme

Assume now that a fault is detected at some time T_d; accordingly, at $t = T_d$ the fault isolation estimators are activated. Each FIE corresponds to one potential fault type. Specifically, among the $N+l$ FIEs employed in the fault isolation scheme, N FIEs are designed based on the functional structure of the process faults defined in (2) and (3), and the remaining l FIEs are designed based on the the functional structure of the potential sensor faults. The following N FIEs correspond to process faults: for $p = 1,\cdots,N$,

$$\begin{aligned} \dot{\hat{x}}^p &= A\hat{x}^p + \gamma(y,u) + L(y-\hat{y}^p) + \Omega^p\dot{\hat{\theta}}^p \\ &\quad + \hat{\phi}^p(y,u,\hat{\theta}^p)\,, \qquad \hat{x}^p(T_d) = 0\,, \\ \dot{\Omega}^p &= A_0\Omega^p + Z^p(y,u)\,, \qquad \Omega^p(T_d) = 0\,, \\ \hat{\phi}^p(y,u,\hat{\theta}^p) &= [(\hat{\theta}_1^p)^\top g_1^p(y,u),\; \cdots,\; (\hat{\theta}_n^p)^\top g_n^p(y,u)]^\top\,, \\ \hat{y}^p &= C\hat{x}^p\,, \end{aligned} \tag{9}$$

where $\hat{\theta}_i^p \in \Re^{\upsilon_i^p}$, for $i = 1,\cdots,n$, is the estimate of the fault parameter vector in the i-th state equation of the p-th isolation estimator. It is noted that according to (3) the fault approximation model $\hat{\phi}^p$ is linear in the adjustable weights $\hat{\theta}^p$, consequently, the gradient matrix $Z^p \triangleq \partial\hat{\phi}^p(y,u,\hat{\theta}^p)\,/\,\partial\hat{\theta}^p = \mathrm{diag}\,[(g_1^p)^\top,\cdots,(g_n^p)^\top]$ does not depend on $\hat{\theta}^p$.

The adaptation in the isolation estimators arises due to the unknown parameter vector $\theta^p \triangleq \left[(\theta_1^p)^\top,\cdots,(\theta_n^p)^\top\right]^\top$. The adaptive law for adjusting each $\hat{\theta}^p$ is derived using the Lyapunov synthesis approach (see for example [8]), with the projection operator restricting $\hat{\theta}^p$ to the corresponding *known* set Θ^p (in order to guarantee stability of the learning algorithm in the presence of modeling uncertainty [8], [4]). Specifically, the learning algorithm is chosen as follows

$$\dot{\hat{\theta}}^p = \mathcal{P}_{\Theta^p}\left\{\Gamma\Omega^{p\top}C^\top\epsilon_y^p\right\}, \tag{10}$$

where $\epsilon_y^p(t) \triangleq y(t) - \hat{y}^p(t)$ denotes the output estimation error of the p-th estimator, and $\Gamma > 0$ is a symmetric, positive-definite learning rate matrix.

Analogously, the following l FIEs correspond to sensor faults: for $q = N+1,\cdots,N+l$,

$$\begin{aligned} \dot{\hat{x}}^q &= A\hat{x}^q + \gamma(y,u) + L(y-\hat{y}^q) + \Omega^q\dot{\hat{\theta}}^q, \\ &\quad \hat{x}^q(T_d) = 0\,, \\ \dot{\Omega}^q &= A_0\Omega^q - LF^q\,, \qquad \Omega^q(T_d) = 0\,, \\ \hat{y}^q &= C\hat{x}^q + F^q\hat{\theta}^q\,, \\ \dot{\hat{\theta}}^q &= \mathcal{P}_{\Theta^q}\left\{\gamma^q\left(C\Omega^q + F^q\right)^\top\epsilon_y^q\right\}, \end{aligned} \tag{11}$$

where $\epsilon_y^q(t) \triangleq y^q(t) - \hat{y}^q(t)$ denotes the output estimation error, the *projection operator* $\mathcal{P}$ restricts the parameter estimation $\hat{\theta}^q$ to a predefined compact and convex region $\Theta^q \subset \Re$ [8], [4], and $\gamma^q \in \Re$ is the learning rate.

Remark 1. The parameter estimate $\hat{\theta}^p$ and $\hat{\theta}^q$ also provides useful information for fault isolation. However, it is important to stress that it cannot be guaranteed that, for the actual fault, the parameter estimate $\hat{\theta}^p$ and $\hat{\theta}^q$ will converge to the true value, unless we assume persistency of excitation [8], [4], a condition which, in general, is too restrictive (in this paper, we do *not* assume persistency of excitation.)

The stability and learning capability of the FIEs described by (9) and (11) have been rigorously investigated in [11] and [12].

Now, let us consider process fault p, where $p = 1, \cdots N$, and sensor fault q, where $q = N+1, \cdots N+l$, in a unified framework. Then we have $N + l$ faults in the augmented fault class. More specifically, for $s = 1, \cdots N+l$, fault s is a process fault, if $1 \le s \le N$, and fault s is a sensor fault, if $N+1 \le s \le N+l$. The fault isolation decision scheme is based on the following intuitive principle: for $s = 1, \cdots, N+l$, if fault s occurs at time T_0 and is detected at time T_d, then a set of adaptive threshold functions $\{\mu_j^s(t),\, j = 1, \cdots, l\}$ can be designed for the s-th isolation estimator, such that the j-th component of its output estimation error satisfies $|\epsilon_{y_j}^s(t)| \le \mu_j^s(t)$, for all $t > T_d$. Consequently, for each $s = 1, \cdots, N$, such a set of adaptive thresholds $\{\mu_j^s(t),\, j = 1, \cdots, l\}$ can be associated with the output estimation of the s-th isolation estimator. In the fault isolation procedure, if for a particular isolation estimator s, there exists some $j \in \{1, \cdots, l\}$, such that the j-th component of its output estimation error satisfies $|\epsilon_{y_j}^s(t)| > \mu_j^s(t)$ for some finite time $t > T_d$, then the possibility of the occurrence of fault s can be excluded. Based on this intuitive idea, the following fault isolation decision scheme is devised:

Fault Isolation Decision Scheme: *if, for each* $r \in \{1, \cdots, N+l\} \backslash \{s\}$*, there exist some finite time* $t^r > T_d$ *and some* $j \in \{1, \cdots, l\}$*, such that* $|\epsilon_{y_j}^r(t^r)| > \mu_j^r(t^r)$*, then the occurrence of fault* s *is concluded.*

In the next section, we will derive the threshold $\mu^s(t)$ associated with each isolation estimator.

IV. ADAPTIVE THRESHOLDS FOR FAULT ISOLATION

The threshold functions $\mu_j^s(t)$ play a significant role in the proposed fault isolation scheme. The following lemma provides a bounding function for the output estimation of the s–th isolation estimator in the case that fault s occurs.

Lemma 1. *Suppose that fault* s*, occurring at time* $t = T_0$ *is detected at time* $t = T_d$*, where* $s = 1, \cdots N+l$*. Then, for all* $t \ge T_d$*, the* j*-th component of the output estimation error of the* s*-th isolation estimator satisfies the following inequality:*

$$\begin{aligned} |\epsilon_{y_j}^s(t)| \;\; &\le \;\; |\Upsilon^s(t)|\,|\tilde{\theta}^s(t)| + k_j \bar{x} e^{-\lambda_j (t-T_d)} + \bar{d}_j(y,u,t) \\ &\quad + k_j \int_{T_d}^t e^{-\lambda_j (t-\tau)} \Big[\bar{\eta}(y(\tau),u(\tau),\tau) + \rho^s ||\Omega^s|| \\ &\quad + ||L||\, \bar{d}(y(\tau),u(\tau),\tau) \Big] d\tau \,, \end{aligned} \tag{12}$$

where

$$\Upsilon^s \triangleq \begin{cases} C_j \Omega^s & \text{if} \quad 1 \le s \le N \\ C_j \Omega^s + F_j^s & \text{if} \quad N < s \le N+l \end{cases} , \tag{13}$$

and

$$\rho^s \triangleq \begin{cases} \alpha_s & \text{if} \quad 1 \le s \le N \\ \delta & \text{if} \quad N < s \le N+l \end{cases} . \tag{14}$$

Proof: Denote the state estimation error of the s-th isolation estimator by $\epsilon_x^s(t) \triangleq x(t) - \hat{x}^s(t)$, for $s = 1, \cdots N+l$. The proof consists of two parts.

Part I: In the presence of a process fault p, $p = 1, \cdots, N$, (or equivalently, fault s with $1 \le s \le N$), by using (1), the system dynamics for $t > T_x$ are given by

$$\begin{aligned} \dot{x} \;\; &= \;\; Ax + \gamma(y,u) + \eta(x,u,t) + \phi^p(y,u) \\ y \;\; &= \;\; Cx + d(x,u,t) \,. \end{aligned} \tag{15}$$

Using (15), (9), and $Z^p = \dot{\Omega}^p - A_0\, \Omega^p$, after some simple algebraic manipulations, we obtain

$$\begin{aligned} \dot{\epsilon}_x^p(t) \;\; &= \;\; A_0 \left(\epsilon_x^p(t) + \Omega^p \tilde{\theta}^p \right) + \eta(x,u,t) - Ld(x,u,,t) \\ &\quad - \frac{d}{dt}(\Omega^p \tilde{\theta}^p) - \Omega^p \dot{\theta}^p \,. \end{aligned}$$

By letting $\bar{\epsilon}_x^p(t) \triangleq \epsilon_x^p(t) + \Omega^p \tilde{\theta}^p$, the above equation can be rewritten as

$$\dot{\bar{\epsilon}}_x^p(t) = A_0 \bar{\epsilon}_x^p(t) + \eta(x,u,t) - L\, d(x,u,t) - \Omega^p \dot{\theta}^p \,. \tag{16}$$

By defining $\epsilon_{y_j}^p(t) \triangleq y_j(t) - \hat{y}_j^p(t)$ and using (15) and (9), we have:

$$\epsilon_{y_j}^p(t) = C_j \left(\bar{\epsilon}_x^p(t) - \Omega^p \tilde{\theta}^p \right) + d_j(x,u,t) \,. \tag{17}$$

Now, based on (16), (17), (13), and (14), as well as assumption 1 and assumption 3, it can be easily shown that

$$\begin{aligned} |\epsilon_{y_j}^p(t)| \;\; &\le \;\; |C_j \Omega^p|\,|\tilde{\theta}^p(t)| + k_j \bar{x} e^{-\lambda_j (t-T_d)} + \bar{d}_j(y,u,t) \\ &\quad + k_j \int_{T_d}^t e^{-\lambda_j (t-\tau)} \Big[\bar{\eta}(y(\tau),u(\tau),\tau) + \alpha_p \\ &\quad \cdot ||\Omega^p|| + ||L||\, \bar{d}(y(\tau),u(\tau),\tau) \Big] d\tau \,, \end{aligned} \tag{18}$$

Part II: In the presence of a sensor fault q, $q = 1, \cdots, l$, (or equivalently, fault s with $N < s \le N+l$), by using (1), the system dynamics for $t > T_y$ are given by

$$\begin{aligned} \dot{x} \;\; &= \;\; Ax + \gamma(y,u) + \eta(x,u,t) \\ y \;\; &= \;\; Cx + d(x,u,t) + F^q \theta^q(t) \,. \end{aligned} \tag{19}$$

The dynamics of the output estimation error of the q–th FIE described by (11) has been investigated in [12], the results are repeated as follows:

$$\begin{aligned} |\epsilon_{y_j}^q(t)| \;\; &\le \;\; |C_j \Omega^q + F_j^q|\,|\tilde{\theta}^q(t)| + k_j\, \bar{x}\, e^{-\lambda_j (t-T_d)} \\ &\quad + k_j \int_{T_d}^t e^{-\lambda_j (t-\tau)} \Big[\bar{\eta}(y(\tau),u(\tau),\tau) + \delta |\Omega^q| \\ &\quad + \|L\|\, \bar{d}(y(\tau),u(\tau),\tau) \Big] d\tau + \bar{d}_j(y,u,t) \,. \end{aligned} \tag{20}$$

Now the proof of (12) easily follows from (13), (14), (18), and (20). □

Although Lemma 1 provides an upper bound on the output estimation error of the s-th estimator, the right-hand side of (12) cannot be directly used as a threshold function for fault isolation because $\tilde{\theta}^s(t)$ is not available. However, as the estimate $\hat{\theta}^s$ belongs to the known compact set Θ^s, we have $\left|\theta^s - \hat{\theta}^s(t)\right| \leq \kappa^s(t)$ for a suitable $\kappa^s(t)$ depending on the geometric properties of set Θ^s (see [11]). Hence, based on the above discussions, the following threshold function is chosen:

$$\begin{aligned} |\mu_j^s(t)| \quad \leq \quad & |\Upsilon^s(t)|\,|\kappa^s(t)| + k_j \bar{x} e^{-\lambda_j(t-T_d)} + \bar{d}_j(y,u,t) \\ & + k_j \int_{T_d}^{t} e^{-\lambda_j(t-\tau)} \Big[\bar{\eta}(y(\tau),u(\tau),\tau) + \rho^s ||\Omega^s|| \\ & + ||L||\, \bar{d}(y(\tau),u(\tau),\tau) \Big] d\tau \,. \end{aligned} \tag{21}$$

Remark 2. The adaptive threshold function described by (21) is influenced by several sources of uncertainty entering the fault isolability problem, such as modeling uncertainty η, measurement errors d and parametric uncertainty κ^s. Intuitively, the smaller the uncertainty (resulting in a smaller threshold $\mu_j^s(t)$, the easier the task of isolating the faults. On the other hand, as clarified in the next section, the capability to isolate a fault depends not only on $\mu_j^s(t)$, but also on the degree that the faults are "different" from each other.

V. FAULT ISOLABILITY ANALYSIS

For our purpose, a fault is said to be *isolable* if the fault isolation scheme is able to reach a correct decision in finite time. Intuitively, faults are isolable if they are *mutually different* according to a certain measure quantifying the difference in the effects that different faults have on measurable outputs and on the estimated quantities in the isolation scheme. In this respect, we introduce the *fault mismatch function* between the s–th fault and the r–th fault:

$$h_j^{sr}(t) \stackrel{\triangle}{=} \Upsilon^s\theta^s - \Upsilon^r\hat{\theta}^r, \quad r,s = 1,\cdots,N+l,\, r \neq s \tag{22}$$

where Υ^s and Υ^r, defined in (13), represent respectively the functional structure of fault s and fault r related to the j–th output variable, y_j.

Remark 3. The fault mismatch function $h_j^{sr}(t)$ is related to both the structural difference between the faults (i.e., Υ^s and Υ^r) and the properties of the isolation scheme (i.e., the estimates $\hat{\theta}^r$). However, as described in Section III-B, the fault isolation estimators are designed based on the structure of the faults (i.e., Z^p and F^q). Moreover, the adaptive thresholds are designed in such a way that faults can not be isolated unless their structures are sufficiently different (see [11], [12]). Therefore, the structural difference between the faults has a predominant effect on the isolability properties than the actual behavior of the estimates $\hat{\theta}^r$ provided by the FIEs.

The following theorem characterizes (in a non-closed form) the class of isolable faults.

Theorem 1. *Consider the fault isolation scheme described by (9)–(11) and (21). Suppose that Assumptions 1, 2, and 3 hold and that a fault s, $s = 1,\cdots N+l$, occurring at time $t = T_0$ is detected at time $t = T_d$. Then fault s is isolable if, for each $r \in \{1,\cdots,N+l\}\backslash\{s\}$, there exist some time $t^r > T_d$ and some $j \in \{1,\cdots l\}$, such that the fault mismatch function $h_j^{sr}(t^r)$ satisfies the following inequality:*

$$\begin{aligned} \left|h_j^{sr}(t^r)\right| \quad > \quad & |\Upsilon^r|\kappa^r(t^r) + 2\bar{d}_j(y(t^r),u(t^r),t^r) \\ & + \left| \int_{T_d}^{t^r} C_j e^{A_0(t^r-\tau)} \Big(\eta(x,u,\tau) - Ld(x,u,\tau) - \Omega^s\dot{\theta}^s \Big) \right| d\tau \\ & + k_j \int_{T_d}^{t^r} e^{-\lambda_j(t^r-\tau)} \Big[\bar{\eta}(y,u,\tau) + \|L\|\, \bar{d}(y,u,\tau) \\ & + \rho^r ||\Omega^r|| \Big] d\tau + \, 2k_j\bar{x}e^{-\lambda_j(t^r-T_d)} \,. \end{aligned} \tag{23}$$

Proof: The state estimation error and output estimation error associated with the r-th fault isolation estimator is $\epsilon_x^r(t) \stackrel{\triangle}{=} x(t) - \hat{x}^r(t)$, and $\epsilon_y^r(t) \stackrel{\triangle}{=} y(t) - \hat{y}^r(t)$, respectively. Based on the values of s and r, the proof consists of the following four parts.

Part I: the isolability of a process fault s from other process faults r, where $1 \leq s \leq N$, and $r \in \{1,\cdots,N\}\backslash\{s\}$. In this case, the dynamics of the system are given by (15), and the dynamics of FIE r are described by (9). Therefore, the output estimation error $\epsilon_y^r(t)$ is given by $\epsilon_y^r(t) = C\epsilon_x^r(t) + d(x,u,t)$. The dynamics of the state estimation error $\epsilon_x^r(t)$ is described by

$$\begin{aligned} \dot{\epsilon}_x^r(t) \quad = \quad & A_0\epsilon_x^r(t) - Ld(x,u,t) + \eta(x,u,t) + Z^s\theta^s \\ & - Z^r\hat{\theta}^r - \Omega^r\dot{\hat{\theta}}^r \,. \end{aligned} \tag{24}$$

By substituting $Z^s = \dot{\Omega}^s - A_0\Omega^s$ and $Z^r = \dot{\Omega}^r - A_0\Omega^r$ into (24), and by letting $\bar{\epsilon}_x^r(t) \stackrel{\triangle}{=} \epsilon_x^r(t) - \Omega^s\theta^s + \Omega^r\hat{\theta}^r$, we obtain:

$$\dot{\bar{\epsilon}}_x^r(t) = A_0\bar{\epsilon}_x^r(t) + \eta(x,u,t) - Ld(x,u,t) - \Omega^s\dot{\theta}^s \,. \tag{25}$$

By defining the j-th component of the output estimation error as

$$\epsilon_{y_j}^r(t) \stackrel{\triangle}{=} y_j(t) - \hat{y}_j^r(t)\,, \tag{26}$$

and by using (15), (9), (13), and (22), it can be shown that

$$\epsilon_{y_j}^r(t) = C_j\bar{\epsilon}_x^r(t) + h_j^{sr}(t) + d_j(x,u,t)\,. \tag{27}$$

Using (25) and the triangle inequality, we obtain

$$\begin{aligned} |\epsilon_{y_j}^r(t)| \quad \geq \quad & |h_j^{sr}(t)| - k_j\bar{x}e^{-\lambda_j(t-T_d)} \\ & - \left| \int_{T_d}^{t} C_j e^{A_0(t-\tau)} \Big(-Ld(x,u,\tau) - \Omega^s\dot{\theta}^s \right. \\ & \left. + \eta(x,u,\tau) \Big) \right| d\tau - |d_j(x,u,t)|\,. \end{aligned} \tag{28}$$

Now taking into account the corresponding adaptive threshold $\mu_j^r(t)$ given by (21), we can immediately conclude that, if condition (23) is satisfied at t^r, we obtain $|\epsilon_{y_j}^r(t^r)| > \mu_j^r(t^r)$, which implies that the possibility of the occurrence of fault r can be excluded at time $t = t^r$. Note that the special case of $\dot{\theta}^s = 0$ has been considered in [11].

Part II: the isolability of a process fault s from sensor faults r, where $1 \leq s \leq N$, and $r \in \{N+1, \cdots, N+l\}$. In this case, the dynamics of the system are given by (15), and the dynamics of FIE r are described by (11). Therefore, the output estimation error $\epsilon_y^r(t)$ is given by

$$\epsilon_y^r(t) = C\epsilon_x^r(t) + d(x,u,t) - F^r\hat{\theta}^r \,.$$

The dynamics of the state estimation error $\epsilon_x^r(t)$ is described by

$$\begin{aligned}\dot{\epsilon}_x^r(t) &= A_0\epsilon_x^r(t) + \eta(x,u,t) - Ld(x,u,t) + LF^r\hat{\theta}^r \\ &\quad + Z^s\theta^s - \Omega^r\dot{\hat{\theta}}^r \,. \qquad (29)\end{aligned}$$

By substituting $LF^r = -\dot{\Omega}^r + A_0\Omega^r$ and $Z^s = \dot{\Omega}^s - A_0\Omega^s$ into (29), and by means of some simple algebra (along the same reasoning that was reported in the proof of Part 1), we have

$$\dot{\bar{\epsilon}}_x^r(t) = A_0\bar{\epsilon}_x^r(t) + \eta(x(t),u(t),t) - Ld(x(t),u(t),t) - \Omega^s\dot{\theta}^s \qquad (30)$$

where $\bar{\epsilon}_x^r(t) \stackrel{\triangle}{=} \epsilon_x^r(t) - \Omega^s\theta^s + \Omega^r\hat{\theta}^r$. By using (15) and (11), we can see that the output estimation error $\epsilon_{y_j}^r(t)$, defined in (26), is given by

$$\begin{aligned}\epsilon_{y_j}^r(t) &= C_j\bar{\epsilon}_x^r(t) + \Big(C_j\Omega^s\theta^s - (C_j\Omega^r + F_j^r)\hat{\theta}^r\Big) \\ &\quad + d_j(x,u,t) \,. \qquad (31)\end{aligned}$$

By using (13) and (22), (31) can be rewritten in the same form as (27), where $\bar{\epsilon}_x^r(t)$ is given by (30).

Now the proof of Theorem 1 for this case easily follows by using the same reasoning method as reported in Part I of of the proof.

Part III: the isolability of a sensor fault s from process faults r, where $N+1 \leq s \leq N+l$, and $r \in \{1, \cdots, N\}$. In this case, the dynamics of the system are given by (19), and the dynamics of FIE r are described by (9). Therefore, the output estimation error $\epsilon_y^r(t)$ is given by

$$\epsilon_y^r(t) = C\epsilon_x^r(t) + d(x,u,t) + F^s\theta^s \,.$$

The dynamics of the state estimation error $\epsilon_x^r(t)$ is described by

$$\begin{aligned}\dot{\epsilon}_x^r(t) &= A_0\epsilon_x^r(t) + \eta(x,u,t) - Ld(x,u,t) - LF^s\theta^s \\ &\quad - Z^r\hat{\theta}^r - \Omega^r\dot{\hat{\theta}}^r \,. \qquad (32)\end{aligned}$$

By substituting $LF^s = -\dot{\Omega}^s + A_0\Omega^s$ and $Z^r = \dot{\Omega}^r - A_0\Omega^r$ into (32), and by means of some simple algebra (along the same reasoning reported in the proof of Part 1), we have

$$\dot{\bar{\epsilon}}_x^r(t) = A_0\bar{\epsilon}_x^r(t) + \eta(x(t),u(t),t) - Ld(x(t),u(t),t) - \Omega^s\dot{\theta}^s \qquad (33)$$

where $\bar{\epsilon}_x^r(t) \stackrel{\triangle}{=} \epsilon_x^r(t) - \Omega^s\theta^s + \Omega^r\hat{\theta}^r$. By using (19) and (9), we can see that the output estimation error $\epsilon_{y_j}^r(t)$, defined in (26), is given by

$$\begin{aligned}\epsilon_{y_j}^r(t) &= C_j\bar{\epsilon}_x^r(t) + \Big((C_j\Omega^s + F_j^s)\theta^s - C_j\Omega^r\hat{\theta}^r\Big) \\ &\quad + d_j(x,u,t) \,. \qquad (34)\end{aligned}$$

By using (13) and (22), (34) can be rewritten in the same form as (27), where $\bar{\epsilon}_x^r(t)$ is given by (33).

Now the proof of Theorem 1 for this case easily follows by using the same reasoning method as reported in Part I of of the proof.

Part IV: the isolability of a sensor fault s from other sensor faults r, where $N+1 \leq s \leq N+l$ and $r \in \{N+1, \cdots, N+l\}\backslash\{s\}$.

The dynamics of the output estimation error of this case have been analyzed in Theorem 1 of [12]. Specifically, the following results was proved:

$$\begin{aligned}|\epsilon_{y_j}^r(t)| &\geq \left|h_j^{sr}(t)\right| - \bar{d}_j(y(t),u(t),t) - |C_j e^{A_0(t-T_d)}|\bar{x} \\ &\quad - \bigg|\int_{T_d}^t C_j e^{A_0(t-\tau)}\big(\eta(x,u,\tau) - Ld(x,u,\tau) \\ &\quad - \Omega^s\dot{\theta}^s\big)d\tau\bigg|\end{aligned}$$

where $h_j^{sr}(t) = (C_j\Omega^s + F_j^s)\theta^s - (C_j\Omega^r + F_j^r)\hat{\theta}^r$. Clearly, the fault mismatch function $h_j^{sr}(t)$ follows the definition given by (13) and (22).

Now, the proof of this case easily follows by using the same reasoning method as reported in Part I of the proof.

Based on the analysis of the above four cases, it can be concluded that, if condition (23) is satisfied at t^r, for each $r \in \{1, \cdots, N+l\}\backslash\{s\}$, then fault s is isolable. □

VI. CONCLUDING REMARKS

In this paper, the design of a fault isolation method for process faults and sensor faults in a class of nonlinear uncertain systems is presented. Analytical results regarding adaptive thresholds for fault isolation and fault isolability conditions are established. Future research work will involve the consideration of a larger class of nonlinear systems.

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