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Switched control of an electropneumatic clutch actuator using on/off valves

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Abstract— **In this paper we derive a controller for an electropneumatic clutch actuator system. On/off valves are being used as control valves, and this impose restrictions on the control design. We show that the system with the derived controller are asymptotically stable in the whole region of operation of the system, and we discuss robustness of the design. The presented simulations verify the results of the paper.**

I. INTRODUCTION

During the last years, the interest for automated manual transmission (AMT) systems has increased due to increases in the demand of driving comfort. Automated clutch actuation makes it easier for the driver, particularly in stop and go- traffic, and have especially seen a recent growth in the European automotive industry. An AMT system consist of a manual transmission through the clutch disc, and an automated actuated clutch during gear shifts. Some of the AMT's largest advantages are low cost, high efficiency, reduced clutch wear and improved fuel consumption.

The automatic control of the clutch engagement plays a crucial role in AMT vehicles, and in this paper we deal with the problem of controlling an electropneumatic clutch actuator for heavy-duty trucks using on/off valves. Hydraulic actuator systems are more common and have been the main focus in literature e.g [1]-[4], but as trucks already have pressurized air present it is desired to use this. Pneumatic systems are in general more difficult to model and control due to the compressible of air. The clutch actuators for heavy-duty trucks also differ from the ones for ordinary cars as a much higher level of torque has to be transmitted through the clutch plates. Thus, the dimension of the mechanical part of a truck clutch system has to be bigger, something which also influences the control task.

On/off valves are chosen as control valves over proportional valves. This because of cost, space and robustness advantages, but they also have a dynamic response that is harder to model accurately, especially for positions in between fully open and fully closed. On/off valves are often controlled by pulse width modulation as in [5]-[8], but to avoid the difficult modeling task, we choose to only allow the valves to be either fully open or fully closed. This is the same approach as we took in [9], and then we only need to know the size of the input voltage which guarantees fully opening/closing of the valves. While [9] investigated a modification of a backstepping design for switched control, here we consider a Lyapunov-design that yields a larger region of attraction.

The paper is organized as follows. In Section II we present the clutch system and its model. Section III gives the design of the controller and its stability proof. Simulation results are presented and robustness properties are discussed in Section IV, and conclusions are presented in Section V.

II. SYSTEM DESCRIPTION

Figure 1 shows a sketch of the system. The position of the piston in the clutch actuator valve decides the position of the clutch plates. These plates can either be fully engaged, fully disengaged or slipping. The on/off valves get electrical control signals from the electronic control unit (ECU), and control the flow of air to the clutch actuator valve based on these. A position sensor feeds back the position to the ECU, while the other states of the system have to be obtained from estimations.

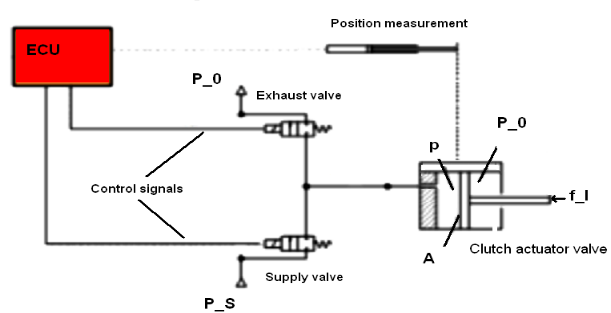

Fig. 1. Sketch of the clutch system

We use a rather simpler model of clutch actuator valve for the development of the controller

$$\begin{aligned} \dot{x}_{1e} &= x_2 \\ \dot{x}_2 &= f(x_{1e}, x_3) - \frac{D}{M}x_2 \\ \dot{x}_{3e} &= RT_0 w \end{aligned} \tag{1}$$

where $f(x_{1e}, x_3) = \frac{1}{M}(-K_l(1-e^{-L_l x_1}) + M_l x_1 + \frac{Ax_3}{V_0+Ax_1} - AP_0)$ and includes both the clutch load characteristic and the pressure forces. Parameter values and description can be found in Appendix A. The states, x_{ie}, are error in position, velocity and accumulated air, where accumulated air is proportional with the amount of air in the actuator valve. The errors can be written as $x_{ie} = x_i - x_i^*$, where x_i^* are the reference points,

$$x^* = [x_1^*, 0, x_3^*]^T \tag{2}$$

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and the reference point x_1^* is given and x_3^* can be found from

$$0 = \frac{1}{M}(-K_l(1-e^{-L_l x_1^*}) + M_l x_1^* + A\frac{x_3^*}{V_0 + Ax_1^*} - AP_0). \quad (3)$$

or

$$x_3^* = \frac{V_0 + Ax_1^*}{A} (K_l(1-e^{-L_l x_1^*}) - M_l x_1^* + AP_0). \quad (4)$$

The region of operation for the clutch actuator valve considered in this paper is $O = \{x_1 \in [0, 0.0025], x_2 \in \mathbb{R}, x_3 \in [80, 504.68]\}$. The only available inputs due to our restrictions are $w \in \{-U_{\max}, 0, U_{\max}\}$ where $U_{\max} = \rho_0 C P_S$ is a simplified expression of the maximum flow capacity through the valves.

III. Controller design

The strict input restrictions complicate the control design. We choose a Lyapunov function based on knowledge of the system and use this to design a controller that will provide asymptotical stability results for the system.

The proposed Lyapunov function

$$V = \alpha \int_0^{x_{1e}} -f(y, x_3)dy + \frac{\beta}{2}x_2^2 + \frac{\lambda}{2}x_{3e}^2 \quad (5)$$

is chosen based on the function $V = \alpha \int_0^{x_{1e}} -f(y)dy + \frac{\beta}{2}x_2^2$ which can be used to show asymptotic stability of the open loop system, $w = 0$, with $x_3 = x_3^*$. (See Appendix B for proof of this). The proposed Lyapunov function is positive definite if the ratio $\frac{\alpha}{\lambda} \leq 3.839$, see Appendix C. The Lyapunov function time derivative is

$$\begin{aligned}\dot{V} &= -\alpha f(x_{1e}, x_3)x_2 - \frac{\alpha}{M}\ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right)RT_0 w \\ &\quad + \beta x_2(f(x_{1e}, x_3) - \frac{D}{M}x_2) + \lambda x_{3e} RT_0 w \qquad (6) \\ &= -\frac{\beta D}{M}x_2^2 + (\lambda x_{3e} - \frac{\alpha}{M}\ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right))RT_0 w \\ &= -\frac{\beta D}{M}x_2^2 + s(x_{1e}, x_{3e})RT_0 w\end{aligned}$$

where

$$\beta = \alpha \quad (7)$$

$$s(x_{1e}, x_{3e}) = \lambda x_{3e} - \frac{\alpha}{M}\ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right) \quad (8)$$

If we choose the following control law

$$w = \begin{cases} -U_{\max} sgn(s(x_{1e}, x_{3e})) & \text{if} \quad s(x_{1e}, x_{3e}) \neq 0 \\ 0 & \text{if} \quad s(x_{1e}, x_{3e}) = 0 \end{cases} \quad (9)$$

we get a negative semidefinite Lyapunov function time derivative,

$$\dot{V} = -\frac{\beta D}{M}x_2^2 - |\lambda x_{3e} - \frac{\alpha}{M}\ln(\frac{Ax_1 + V_0}{Ax_1^* + V_0})|RT_0 U_{\max} \leq 0 \quad (10)$$

and the variables α and λ can be tuned to weight the parts of $s(x_{1e}, x_{3e})$.

Proposition 1: The equilibrium point x^* of the system (1) with the switched controller given in (9) is asymptotical stable in the region of operation if $\frac{\alpha}{\lambda} < 3.839$.

Proof: First we prove existence, uniqueness and continuity of the solution using Filippov solution theories in the same way as in [9] based on [10]. The discontinuity surface is described by

$$S := \{\mathbf{x} : s(x_{1e}, x_{3e}) = 0\} \quad (11)$$

which divides the solution domain Ω into two regions

$$\Omega^- := \{\mathbf{x} : s(x_{1e}, x_{3e}) < 0\} \quad (12)$$

$$\Omega^+ := \{\mathbf{x} : s(x_{1e}, x_{3e}) > 0\}. \quad (13)$$

The right hand side of (1) is defined everywhere in Ω and are measurable and bounded. This means that the system (1) satisfy condition B of Filippov's solution theory [11], and according to Theorems 4 and 5 in the same reference, we when have local existence and continuity of a solution. The right hand side of (1) is also continuous before and after the discontinuity surface, S, and this surface is smooth and independent of time. Hence, the conditions A, B and C of Filippov's solution [12] are satisfied. By following the procedure introduces in [11] the vector functions f^- and f^+ can be defined as the limiting values of the right-hand sides of the state space equations in Ω^- and Ω^+

$$f^- = \begin{bmatrix} x_2 \\ f(x_{1e}, x_3) - \frac{D}{M}x_2 \\ RT_0 U_{max} \end{bmatrix} \quad (14)$$

$$f^+ = \begin{bmatrix} x_2 \\ f(x_{1e}, x_3) - \frac{D}{M}x_2 \\ -RT_0 U_{max} \end{bmatrix}. \quad (15)$$

Vector $\mathbf{h}$, which is along the normal of the discontinuity surface,

$$\mathbf{N_s} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T \quad (16)$$

is defined as

$$\mathbf{h} = f^+ - f^- = \begin{bmatrix} 0 \\ 0 \\ -2RT_0 U_{max} \end{bmatrix} \quad (17)$$

for all points on the discontinuity surface. The scalar, h_N, defined as the projection of $\mathbf{h}$ on $\mathbf{N_s}$ is

$$h_N = \mathbf{N_s}\mathbf{h} = -\sqrt{2}RT_0 U_{max} < 0 \quad (18)$$

and will always be negative. According to Lemma 7 in [12], the uniqueness of the Filippov solution is then guaranteed.

Next we consider the stability properties of the solution. From (10) we have that $\dot{V} \leq 0$, and we use LaSalle's invariance principle to prove asymptotical stability. From $\dot{V} = 0$ we get

$$x_2 = 0 \quad \& \quad |s(x_{1e}, x_{3e})| = 0 \quad (19)$$

From this it follows that $x_2 = 0 \rightarrow \dot{x}_{1e} = 0 \rightarrow x_{1e} = c_1$, $x_2 = 0 \rightarrow \dot{x}_2 = 0 \rightarrow f(x_{1e}, x_3) = 0$ and $|s(x_{1e}, x_{3e})| = 0 \;\; \& \;\; x_{1e} = c_1 \rightarrow x_{3e} = c_2$. This gives the solution $(x_{1e}, x_2, x_{3e}) = (c_1, 0, c_2)$. It then remains to show that $c_1 = c_2 = 0$ is the only possible solution. The equation

$$|s(x_{1e}, x_{3e})| = 0 \tag{20}$$

have the two possible solutions

$$x_{3e} = 0 \quad \& \quad \frac{\alpha}{M} \ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right) = 0 \rightarrow x_{1e} = 0 \tag{21}$$

and

$$x_{3e} = \frac{\alpha}{\lambda M} \ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right) \tag{22}$$

The equation

$$f(x_{1e}, x_3) = f(x_{1e}) + \frac{Ax_{3e}}{M(V_0 + Ax_1)} = 0 \tag{23}$$

where $f(x_{1e}) = \frac{1}{M}(-K_l(1 - e^{-L_l x_1}) + M_l x_1 + \frac{Ax_3^*}{V_0 + Ax_1} - AP_0)$ and $f(0) = 0$, also have two possible solutions

$$\begin{aligned} f(x_{1e}) = 0 \quad & \& \quad \frac{Ax_{3e}}{M(Ax_1 + V_0)} = 0 \\ x_{1e} = 0 \quad & \& \quad x_{3e} = 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} f(x_{1e}) &= -\frac{Ax_{3e}}{M(V_0 + Ax_1)} \\ x_{3e} &= -\frac{M(V_0 + Ax_1)}{A} f(x_{1e}) \end{aligned} \tag{25}$$

Defining

$$g(x) = K_l(1 - e^{-Lx}) - M_l x + AP_0 \tag{26}$$

gives

$$f(x_{1e}) = \frac{1}{M}\left(-g(x_1) + \frac{Ax_1 + V_0}{Ax_1^* + V_0} g(x_1^*)\right) \tag{27}$$

By combining (22) and (25)

$$\begin{aligned} -\frac{Ax_1 + V_0}{A}\left(\frac{Ax_1^* + V_0}{Ax_1 + V_0} g(x_1^*) - g(x_1)\right) \\ = \frac{\alpha}{\lambda M} \ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right) \end{aligned} \tag{28}$$

and rearranging we get

$$\begin{aligned} (Ax_1 + V_0)g(x_1) - (Ax_1^* + V_0)g(x_1^*) \\ = \frac{\alpha A}{\lambda M} \ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right) \end{aligned} \tag{29}$$

From the proof of positive definiteness of the Lyapunov function (Appendix C) we have that

$$\begin{aligned} |(Ax_1 + V_0)g(x_1) - (Ax_1^* + V_0)g(x_1^*)| \\ > \frac{2A\alpha}{M\lambda}\left| ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right)\right| \end{aligned} \tag{30}$$

for $\frac{\alpha}{\lambda} \leq 3.839$. From this it follows that the only solution of (29) is $x_{1e} = 0$. This show that $x_{1e} = x_2 = x_{3e} = 0$ is the only solution and by the LaSalle's invariance principle we can show asymptotically stability; The only solution which can stay identically in the set $S = \left\{x \in O | \dot{V}(x) = 0\right\}$ is the reference point $(x_{1e}, x_2, x_{3e}) = 0$. ■

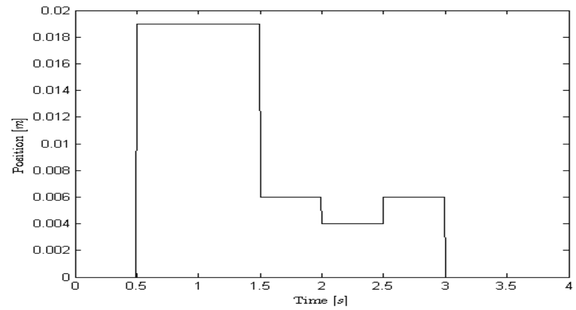

Fig. 2. The position reference

IV. Simulation

A. Practical considerations

The on/off valves cannot open/close instantaneously. The sampling time, and hence the minimum switching period, for the simulations is therefore set to 5 ms. This will guarantee that the valves have opened/closed before a new input signal is given.

In order to avoid unnecessary chattering, $w = 0$ has been chosen whenever V is sufficiently close to zero, which is the same as saying that x is close to x^*. All the simulations are done with the parameters $\alpha = 3.893$ and $\lambda = 1$.

The position reference used in the simulations is a typical clutch sequence, and is shown in Figure 2. It is desired that the controller makes the system reach the reference point within 0.1 s and with a steady state error of less than 0.2 mm in the area where the clutch engages/disengages. Outside this area, the requirements can be somewhat relaxed.

B. Results and robustness

Figure 3 shows the result of the simulation of the system with the derived controller. The requirements in precision and response are fulfilled, and this result verifies the stability results. The clutch load characteristic change during the life-time of the clutch, mostly due to wear. It is therefore natural to check how our design cope with changes in the load characteristics. A variation in the clutch load characteristic influence both the model of the system and the calculation of x_3^*. In the rest of the simulations we consider a different clutch load characteristic which is only encountered in the model, for robustness analysis. In Figure 4 both clutch loads are shown. As the ratio between α and λ is restricted by 3.893, it is the error in x_{3e} which is the the main factor in the calculation of the input. The requirements for the system is defined for position, and velocity and accumulated air are not important in this perspective. The results from simulation with the derived controller and the smaller load characteristics, see Figure 5, show as expected that the error in position is much greater than the specifications. To avoid

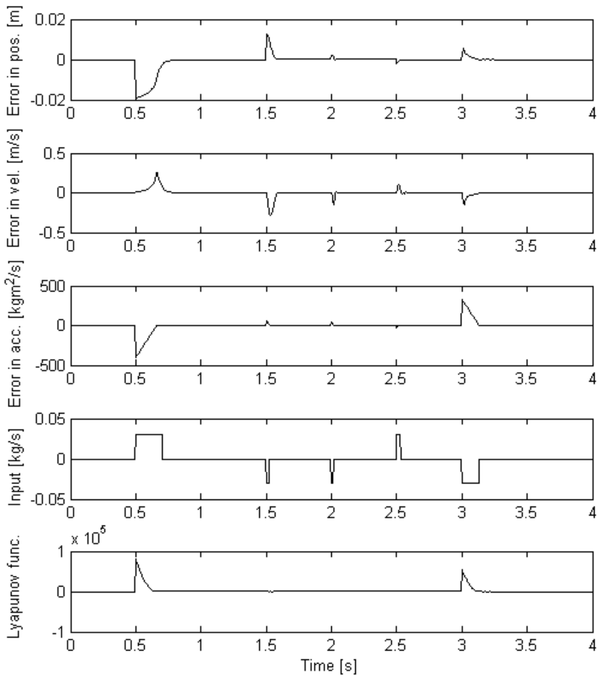

Fig. 3. Switched control

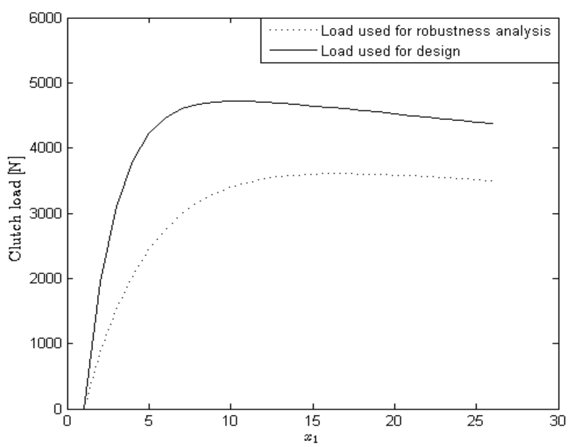

Fig. 4. Clutch load

this problem we propose a simple solution. Instead of the reference point x_3^* we use the reference $x_3^* - c$ where c is a correction term that we update by

$$c^+ = c + Ksgn(x_{1e}) \tag{31}$$

whenever $x_{3e} \cong 0$ and $|x_{1e}| > 0.2$ mm. The limit for position error is chosen as the upper permitted deviation in the requirements and K is the size of the correction step. Figure 6 show result from simulation where this correction term have been implemented. In this simulation, a sudden change in the clutch load is assumed before $t = 0$. This is not realistic, but show how the correction term works. We

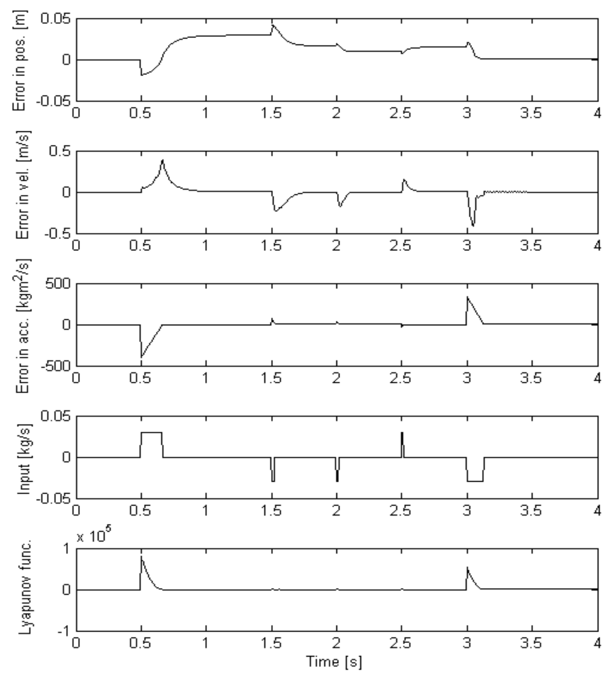

Fig. 5. Switched control with error in load characteristics

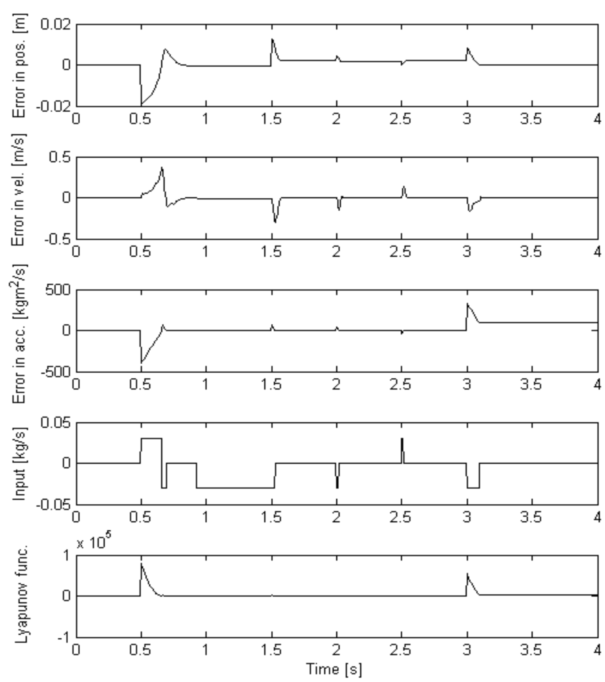

Fig. 6. Switched control with error in load characteristics and update of x_3^*

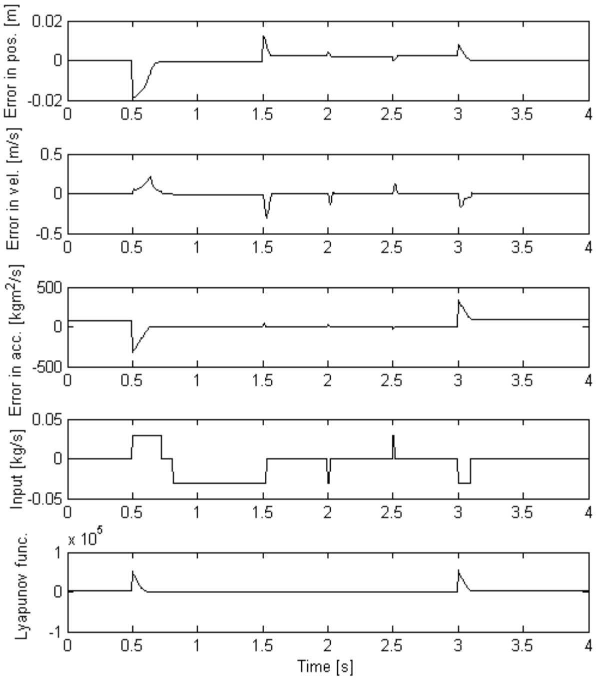

Fig. 7. Switched control which error in load characteristics and precalculated correction in x_3^*

see that the simple correction approach works satisfactory. We get a rather large deviation for a period after the first time the position reference change, at $t = 0.5$, but after this the control method seems to work satisfactory. Whenever the correction term are updated we would expect such periods where the position error is much larger than the requirements. But as the load characteristics are not likely to change rapid, the correction do not have to be used to often. One can make sure that the correction term is updated when the position reference is outside the area with the strictest restrictions.

In Figure 7 we see results from a more realistic simulation, where the correction term found from the previous simulation is used.

V. Concluding remarks

A stabilizing controller has been derived and verified by simulation results. The Lyapunov function used in the development is based on a Lyapunov function with which stability of the reduced second order system can be proven. To account for changes in the clutch load characteristics a correction term was implemented. This solves partly the problem that the input depend mostly on the error in accumulated air. In further work we will look at other and improved methods for adaption of x_3^*, and consider the option of estimation of the clutch load characteristic. It will also be considered to modify the Lyapunov function and to combine the control method in this paper and the one in [9] such that the position error influence more in the choice of input.

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Appendix

A. Parameters

Variable	**Value**	**Unit**	**Description**
A	$12.3e^{-3}$	m^2	Actuator area
P_0	$1e^5$	Pa	Ambient pressure
T_0	293	K	Temperature
R	288	$\frac{J}{kgK}$	Gas constant of air
M	10	kg	Mass of piston
V_0	$0.8e^{-3}$	m^3	Volume at y=0
D	2000	$\frac{Ns}{m}$	Viscous damping
K_l	5000/4000	$\frac{N}{m}$	Load char. term
L_l	500/250	-	Load char. term
M_l	25000/20000	$\frac{N}{m}$	Load char. term
C	$26.7e^{-9}$	$\frac{m^4 s}{kg}$	Capacity
ρ_0	1.185	$\frac{kg}{m^3}$	Density
P_S	$9.5e^5$	Pa	Supply pressure
α	3.839	$-$	Control parameter
β	1	$-$	Control parameter
λ	1	$-$	Control parameter

B. Open loop stability

Consider the input $w = 0$ and assume $x_3 = x_3^*$. We when can rewrite the system as a simple second order system

$$\begin{aligned} \dot{x}_{1e} &= x_2 \\ \dot{x}_2 &= f(x_{1e}) - \frac{D}{M}x_2 \end{aligned} \tag{32}$$

which also can be expressed as

$$\ddot{x}_{1e} + \frac{D}{M}\dot{x}_{1e} - \frac{1}{M}f(x_{1e}) = 0 \tag{33}$$

If

$$\begin{aligned} \frac{D}{M}x_{2e}^2 &> 0 \quad x_{2e} \neq 0 \\ -f(x_{1e})x_{1e} &> 0 \quad x_{1e} \neq 0 \end{aligned} \tag{34}$$

in the region $O' = \{x_1 \in [0, 0.025], x_2 \in \mathbb{R}\}$, we can choose Lyapunov function as the sum of potential and kinetic energy

$$V = \frac{1}{2}x_{2e}^2 + \int_0^{x_{1e}} -f(y)dy \tag{35}$$

and by differentiating it we obtain

$$\begin{aligned} \dot{V} &= x_{2e}\dot{x}_{2e} - f(x_{1e})\dot{x}_{1e} \\ &= -\frac{D}{M}x_{2e}^2 + f(x_{1e})x_{2e} - f(x_{1e})x_{2e} \\ &= -\frac{D}{M}x_{2e}^2 \leq 0 \end{aligned} \tag{36}$$

Consider $S = \left\{x \in O' | \dot{V}(x) = 0\right\}$, and the only solution that can stay identically in S is the reference point $(x_{1e}, x_{2e}) = 0$ as

$$x_{2e}(t) \equiv 0 \Longrightarrow \dot{x}_{2e} \equiv 0 \Longrightarrow f(x_{1e}) \equiv 0 \Longrightarrow x_{1e} \equiv 0. \tag{37}$$

As the second condition of (34) is satisfied, as shown below, when by LaSalle-Krasovski's theorem, the origin is proven asymptotically stable.

1) Proof of the second condition of (34): We have that

$$\begin{aligned} f(x_{1e}) &= \frac{1}{M}(-K_l(1 - e^{-L_l x_1}) + M_l x_1 \\ &\quad + \frac{Ax_3^*}{V_0 + Ax_1} - AP_0) \\ &= \frac{1}{M}(-K_l(1 - e^{-Lx_1}) + M_l x_1 \\ &\quad + \frac{A\frac{V_0 + Ax_1^*}{A}(K_l(1 - e^{-L_l x_1^*}) - M_l x_1^* + AP_0)}{V_0 + Ax_1} - AP_0) \\ &= \frac{1}{M}\left(-g(x_1) + \frac{V_0 + Ax_1^*}{V_0 + Ax_1}g(x_1^*)\right) \\ &= \frac{1}{M(V_0 + Ax_1)}(P(x_1^*) - P(x_1)) \end{aligned} \tag{38}$$

where $P(x) = (V_0 + Ax)g(x)$ is a positive function strictly increasing in x for $x \in [0, 0.025]$ and

$$g(x) = K_l(1 - e^{-Lx)}) - M_l x + AP_0 > 0 \tag{39}$$

For $\forall\ x_{1e} > 0$ we have

$$x_1 > x_1^* \Leftrightarrow P(x_1) > P(x_1^*) \Leftrightarrow f(x_{1e}) < 0 \tag{40}$$

and

$$-f(x_{1e})x_{1e} > 0 \tag{41}$$

For $\forall\ x_{1e} < 0$ we have

$$x_1 < x_1^* \Leftrightarrow P(x_1) < P(x_1^*) \Leftrightarrow f(x_{1e}) > 0 \tag{42}$$

and

$$-f(x_{1e})x_{1e} > 0 \tag{43}$$

When we have $-f(x_{1e})x_{1e} > 0\ \forall\ x_{1e} \neq 0$ and the second condition of (34) is satisfied.

C. Positive definiteness of the Lyapunov function

We rewrite the Lyapunov function (5) as

$$\begin{aligned} V &= \alpha \int_0^{x_{1e}} -f(y, x_3)dy + \frac{\beta}{2}x_2^2 + \frac{\lambda}{2}x_{3e}^2 \\ &= \alpha \int_0^{x_{1e}} -f(y)dy + \frac{\beta}{2}x_2^2 + \frac{\lambda}{2}x_{3e}^2 \\ &\quad -\frac{\alpha}{M}\int_0^{x_{1e}} \frac{Ax_{3e}}{V_0 + A(x_1^* + y)}dy \end{aligned} \tag{44}$$

where we have that$-\int_0^{x_{1e}} f(y)dy$ is positive from the calculations above. When if

$$\begin{aligned} &\frac{\alpha}{M}\int_0^{x_{1e}} -\frac{Ax_{3e}}{V_0 + A(x_1^* + y)}dy + \frac{\lambda}{2}x_{3e}^2 \\ &\qquad = x_{3e}(-\frac{\alpha}{M}\ln(\frac{V_0 + Ax_1}{V_0 + Ax_1^*}) + \frac{\lambda}{2}x_{3e}) > 0 \end{aligned} \tag{45}$$

we have $V > 0$, as the other parts of (44) are positive. Equation (45) is satisfied if

$$|\frac{\lambda}{2}x_{3e}| > |\frac{\alpha}{M}\ln(\frac{V_0 + Ax_1}{V_0 + Ax_1^*})| \tag{46}$$

This is true except for small x_{3e}. For small x_{3e}, $|x_{3e}| < |\frac{2\alpha}{\lambda M}\ln(\frac{V_0 + Ax_1}{V_0 + Ax_1^*})|$, we need to show that

$$-f(x_{1e}, x_3)x_{1e} > 0 \tag{47}$$

is fulfilled. Remember that $f(x_{1e}, x_3)$ can be written as

$$\begin{aligned} f(x_{1e}, x_3) &= f(x_{1e}) + \frac{Ax_{3e}}{M(V_0 + Ax_1)} \\ &= \frac{P(x_1^*) - P(x_1)}{M(Ax_1 + V_0)} + \frac{Ax_{3e}}{M(Ax_1 + V_0)} \end{aligned} \tag{48}$$

As $-f(x_{1e})x_{1e} > 0$ the condition (47) is satisfied if $f(x_{1e})$ dominates the term $\frac{Ax_{3e}}{M(Ax_1 + V_0)}$ i.e:

$$|P(x_1^*) - P(x_1)| > |Ax_{3e}| \tag{49}$$

This is the case if

$$|P(x_1^*) - P(x_1)| > \frac{2\alpha A}{\lambda M}|\ln\left(\frac{Ax_1 + V_0}{Ax_1^* + V_0}\right)| \tag{50}$$

and this inequality is satisfied if $\frac{\alpha}{\lambda} \leq 3.839$. The proposed Lyapunov function (5) is positive definite when $\frac{\alpha}{\lambda} \leq 3.839$.