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Consensus over Networks with Dynamic Channels*

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Abstract— **In this paper, we investigate the consensus problem over communication networks where the communication channels are represented by stable LTI systems. We give sufficient conditions to reach consensus for our protocol and derive the consensus value which is function of the agents' initial state, the network topology and the channel dynamics. We show that if the $\mathcal{H}_\infty$ norm of each channel filter is equal to its DC gain, the proposed protocol will solve the consensus problem. This result is interpreted in terms of consensus robustness to channel uncertainty and compared with another robustness condition for a different uncertainty class.**

Keywords: Consensus, control over networks with dynamic channels, robust stability.

I. INTRODUCTION

Consensus problems have attracted much attention among researchers studying distributed and decentralized systems. This is due to the wide applications of multi-agent systems including autonomous formation flight [5], [9], cooperative search of unmanned air vehicles(UVAs)[7], swarms of autonomous vehicles or robots[11], [12], congestion control in the communication networks [15] and synchronization of oscillators [14]. Generally speaking, consensus means that all the agents of the network agree to a common value without recourse to a central coordinator or global communication. The consensus protocols are distributed feedback control laws based on local information that allow the coordination of multi-agent networks.

One interesting research area is to study consensus in the presence of propagation delays [1], [8], [13]. In particular, [1] provided a linear consensus protocol which can solve the consensus problem with the property that the consensus value is independent of the delays. However, this property is at the price of assuming reciprocal channels and equal delays, which may not be achievable in real network system. In [13] and [8], the authors extended the protocol in [1] to the case of nonhomogeneous delays and nonreciprocal channels. Although their protocol can guarantee the consensus, the agreement value is not known as a priori.

In this paper, we study the consensus problem in the networks where the channels are modeled by LTI systems. Thus, channels with delays is one special case in our development. We derive conditions for the protocols to reach consensus, and characterize the consensus value as a function of the agents' initial state, the network topology and the channel dynamics. In order to derive a more easily checkable condition, we show that the protocol solves consensus problem if the network topology is strongly connected and the H_∞ norm of each channel is equal to its DC gain. This result can be naturally interpreted in terms of consensus robustness to LTI uncertain channels. Along this line, we consider an alternative channel uncertainty class, and derive the corresponding robust consensus condition. We demonstrate that the network is robustly stable provided that the $\mathcal{H}_\infty$ norm of the channel filter vectors is bounded by a certain number which is fully characterized by the network topology and the protocol we use. We further compare the two robust consensus results, and extend the analysis to the discrete time case. In this case, We argue that to achieve consensus, the discretization step size should be less than the maximum in-degree of all the nodes. Finally, we use the example to demonstrate our results

II. BACKGROUND AND PROBLEM FORMULATION

A. Directed Graph

In this section, we review some notations and some relevant properties of directed graphs that will be used later. We consider a network of n agents, labeled by an index $i = 1, 2, \cdots, n$. The interaction among the agents is properly described by a weighted directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the set of nodes $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with nonnegtive adjacency elements a_{ij}. An edge of G is denoted by $e_{ij} = (v_i, v_j)$ which means that there exists a channel from v_j to v_i. The adjacency elements associated with the edges of the graph are positive, i.e. $e_{ij} \in \mathcal{E} \iff a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all i.

For any node $v_i \in \mathcal{V}$, we define the information neighbor of v_i as

$$N_i = \{v_j \in \mathcal{V} : e_{ij} = (v_i, v_j) \in \mathcal{E}\}. \tag{1}$$

The set N_i represents the set of nodes sending data to node i, and we use $|N_i|$ to denote the number of neighbors of node i. The in-degree and out-degree of the node $v_i \in \mathcal{V}$ are defined respectively as:

$$deg_{in} v_i \triangleq \sum_{j=1}^{n} a_{ij} \qquad deg_{out} v_i \triangleq \sum_{j=1}^{n} a_{ji} \tag{2}$$

We use d_i to represent the in-degree of the node v_i since this notation will be frequently used in this paper. For the digraph with $0-1$ adjacency elements, $d_i = |N_i|$.

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Definition 2.1: The degree matrix $\boldsymbol{\Delta}$ of a digraph $\mathcal{G}$ is a diagonal matrix with diagonal entries $[\boldsymbol{\Delta}]_{ii} = d_i$.

Definition 2.2: The (weighted) Laplacian $L = [l_{ij}]$ of a digraph $\mathcal{G}$ is defined as

$$l_{ij} \triangleq \begin{cases} \sum_{k=1\neq i}^{n} a_{ik} & \text{if } j = i \\ -a_{ij} & \text{if } j \neq i \end{cases} \tag{3}$$

we can also equivalently define the Laplacian as $L \triangleq \boldsymbol{\Delta} - \mathcal{A}$.

According to the above definition of graph Laplacian, all the row-sums of L are zero, therefore L always has an eigenvalue of zero corresponding to the eigenvector $\mathbf{1} = (1, \cdots, 1)^T$, i.e. $L\mathbf{1} = 0$. Invoking the Gersgorin Disk Theorem [16], all the other eigenvalues of L have positive real parts.

Definition 2.3: The node v_i of a digraph $\mathcal{G}$ is called balanced if and only if its in-degree and out-degree are equal, A digraph $\mathcal{G}$ is said to be balanced if and only if all its nodes are balanced.

The Laplacian L of a balanced digraph G has the left eigenvector of $\mathbf{1}$ associated with its eigenvalue of zero, i.e.

$$\mathbf{1}^T L = 0 \tag{4}$$

Definition 2.4: A digraph is *strongly connected* (SC) if any two distinct nodes of the digraph can be connected through a path that follows the direction of the edges of the digraph.

The strongly connected digraph has the following property.

Lemma 2.5: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with Laplacian L. If $\mathcal{G}$ is strongly connected, then $\text{rank}(L) = n - 1$ and L has a positive left eigenvector associated to the eigenvalue of zero.

B. Problem Formulation

We consider the network with each agent has the following dynamics:

$$\dot{x}_i(t) = u_i(t) \tag{5}$$

$i = 1, 2, \ldots, n$, where u_i is the input to each node and x_i is the state of each network agent. we can also call u_i the protocol of network. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a continuous and differentiable function of n variables $x_1, x_2 \cdots, x_n$ and $x(0)$ denotes the initial value of x, we say the network asymptotically reach consensus on a group decision value $\chi(x(0))$ if $\|x_i - \chi(x(0))\| \to 0$ as $t \to \infty$, $\forall i$. The χ-consensus problem in a dynamic network is just a distributed way to calculate $\chi(x(0))$ by applying the input u_i which only depend on x_i and its neighbors, i.e.

$$\dot{x_i}(t) = u_i(x_i(t), x^{(i)}(t)) \tag{6}$$

where $x^{(i)}$ is the state vector of the agents in N_i.

In this paper, we investigate the consensus problem in the network whose channels are represented by LTI systems. Suppose there exists a channel from agent j to agent i, then the signal received by i from j is $\hat{x}_j$, where $\hat{x}_j(s) = h_{ij}(s)x_j(s)$. Here, $h_{ij}(s)$ represents the transfer function of the LTI channel e_{ij}. We can also interpret h_{ij} as the filter of the channel e_{ij}, which leads to the following definition.

Definition 2.6: A network is called LTI network if all of its channels are equipped with LTI filters.

In this paper, we always assume the channel filter transfer functions to be stable.

For a LTI network, we define its LTI topology as $\mathcal{G}_l = (\mathcal{V}, \mathcal{E}, \mathcal{A}(s))$, where $\mathcal{A}(s) = [a_{ij}(s)]$ with $a_{ij}(s) = a_{ij}h_{ij}(s)$ if $e_{ij} \in \mathcal{E}$ or $a_{ij}(s) = 0$ otherwise. The Laplacian $L(s)$ of $\mathcal{G}$ in s domain can be defined as

$$L(s) = [l_{ij}(s)] \triangleq \begin{cases} d_i & \text{if } j = i \\ -a_{ij}(s) & \text{if } j \neq i \end{cases} \tag{7}$$

or equivalently we can write $L(s) \triangleq \boldsymbol{\Delta} - \mathcal{A}(\mathbf{s})$, where $\mathcal{A}(s) = [a_{ij}(s)]$ and $\boldsymbol{\Delta}$ is defined as before.

In this paper, we consider the following consensus protocol

$$u_i(t) = \sum_{j \in N_i} a_{ij}(\hat{x}_j(t) - x_i(t)) \tag{8}$$

The primary objective in this paper is analysis of protocol (8). We give the sufficient conditions to achieve consensus and derive closed form formula for the consensus value. We point out that if the DC gain of the channel filter is equal to its $\mathcal{H}_\infty$ norm, we can solve the consensus problem by a modified protocol. we further focus on the robust analysis of protocol (8) and show the two robust conditions that we derived are not comparable. At the end we give the sufficient condition for the discretized version of protocol (8) to reach consensus and characterize consensus value.

III. Main Results

A. Consensus over LTI Networks

Given protocol (8), the dynamics of agent i can be written as

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}(\hat{x}_j(t) - x_i(t)) \tag{9}$$

$\forall i = 1, 2, \ldots, n$. Taking Laplace transform on both sides of (9), we get

$$sx_i(s) - x_i(0) = \sum_{j \in N_i} a_{ij}(h_{ij}(s)x_j(s) - x_i(s)) \tag{10}$$

where $x_i(s)$ denotes the Laplace transform of $x_i(t)$, $\forall i = 1, 2, \ldots, n$. The above n transformed equations can be rewritten in a more compact form as

$$X(s) = (sI_n + L(s))^{-1}X(0) \tag{11}$$

where I_n is the identity matrix with dimension n, $L(s)$ is defined in (7) and $X(s) = [x_1(s), x_2(s), \cdots, x_n(s)]$ is the transformed state vector. Now the convergence problem reduces to the analysis of the stability of a MIMO transfer function $G(s) = (sI_n + L(s))^{-1}$.

Theorem 3.1: Suppose $G(s)$ has only one pole at zero and all the other poles on the strict LHP, then protocol (8) can solve the consensus problem if $L(0)$ is a Laplacian of a strongly connected graph. Furthermore, the consensus value is given by

$$\alpha = \frac{\sum_{i=1}^{n} \gamma_i x_i(0)}{\sum_{i=1}^{n} \gamma_i + \gamma^T L_1 \mathbf{1}} \tag{12}$$

where γ is the left eigenvector associated with the zero eigenvalue of $L(0)$, i.e. $\gamma^T L(0) = 0$, $L_1 = \frac{dL(s)}{ds}|_{s=0}$ and $x_i(0)$ is the initial condition of the i-th agent for $i = 1, 2, \ldots, n$.

Proof. Since $G(s)$ has only one pole at zero and all the other poles on the strict LHP, we can apply the Final Value Theorem to (11) to get

$$X(\infty) = \lim_{s\to 0} s(sI + L(s))^{-1}X(0) \tag{13}$$

Define $\tilde{X}(s) = s(sI + L(s))^{-1}X(0)$, we have

$$X(0) = (I + \frac{L(s)}{s})\tilde{X}(s) \tag{14}$$

Let $\tilde{X}(s) = \alpha_1(s)\beta_1 + \alpha_2(s)\beta_2 + \cdots + \alpha_n(s)\beta_n$ be a basis decomposition of $\tilde{X}(s)$, where $\alpha_i(s)$ is some function of s for $i = 1, 2, \cdots, n$ and $\{\beta_i\}$ is one orthogonal basis of $\mathbb{R}^n$ with $\beta_1 = \mathbf{1}$.

We next show that $\lim_{s\to 0}\alpha_i(s) = 0$ for all $i > 1$. We take Taylor series expansion of $L(s)$ around $s = 0$ and expand (14) as

$$\begin{aligned} X(0) &= [I + \frac{L(0)}{s} + L_1 + \frac{L_2 s}{2} + \cdots]\cdot \\ &\quad [\alpha_1(s)\beta_1 + \cdots + \alpha_n(s)\beta_n] \end{aligned} \tag{15}$$

where $L_1 = \frac{dL(s)}{ds}|s = 0$, $L_2 = \frac{d^2L(s)}{ds^2}|s = 0, \ldots, etc.$ Taking limit as $s \to 0$ on both sides of (15), we get

$$\begin{aligned} X(0) = \lim_{s\to 0} \quad & [I + \frac{L(0)}{s} + L_1 + \frac{L_2 s}{2} + \cdots]\cdot \\ & [\alpha_1(s)\beta_1 + \cdots + \alpha_n(s)\beta_n] \end{aligned} \tag{16}$$

Now we claim that

$$\lim_{s\to 0} L(0)\sum_{k=2}^{n}\alpha_k(s)\beta_k = 0 \tag{17}$$

Suppose not, then

$$\lim_{s\to 0}\frac{L(0)\sum_{k=2}^{n}\alpha_k(s)\beta_k}{s} \to \infty$$

From the Final Value Theorem, $X(\infty)$ is bounded, therefore $\lim_{s\to 0}\tilde{X}(s)$ is bounded. Noticing that β_1 is orthorgonal to the set $\{\beta_2, \beta_3, \ldots, \beta_n\}$, $\lim_{s\to 0}\alpha_1(s)\beta_1$ and $lim_{s\to 0}\sum_{k=2}^{n}\alpha_k(s)\beta_k$ must be bounded. Therefore from (16), $X(0) \to \infty$ and we get the contradiction. Let $B = [\beta_2, \beta_3, \ldots, \beta_n]$, then since $L(0)$ is the Laplacian of a strongly connected graph, rank$(L(0)) = n - 1$. Therefore rank$(L(0)B) = n - 1$ and the null space of $L(0)B$ is $\{0\}$. Thus, from (17), we must have $\lim_{s\to 0}\alpha_i(s) = 0$ for all $i > 1$.

Left multiplying γ on both sides of (15), since $\gamma^T L(0) = 0$, we get

$$\begin{aligned} \gamma^T X(0) &= \gamma^T(I + L_1 + \frac{L_2 s}{2} + \cdots) \\ &\quad [\alpha_1(s)\beta_1 + \cdots + \alpha_n(s)\beta_n] \end{aligned} \tag{18}$$

If we take the limit of (18) as $s \to 0$, since we have shown that $\lim_{s\to 0}\alpha_i(s) = 0$ for all $i > 1$

$$\gamma^T X(0) = \gamma^T(I + L_1)\alpha_1(0)\mathbf{1} \tag{19}$$

The result follows by dividing $\gamma^T(I + L_1)\mathbf{1}$ on both sides of (19) and $\alpha = \alpha_1(0)$. ∎

Note that the main difference with the results in [1] for networks without dynamics is the term $\gamma^T L_1\mathbf{1}$ [1]. There are several issues that are worth investigating in the future. First, it would be interesting to identify robust structures for which $\gamma^T L_1\mathbf{1} = \mathbf{0}$, so that the consensus value becomes independent from the channel dynamics. Note that when $\gamma^T L_1\mathbf{1} \neq \mathbf{0}$, the agents lose ability to compute the (weghted) average of their initial conditions. To recover it, a learning initial phase is needed, where the agents all starts with the same inital condition (say equal to 1) so that they know the resulting consensus value,

$$\alpha_{fixed} = \frac{\gamma^T\mathbf{1}}{\gamma^T\mathbf{1} + \gamma^T L_1\mathbf{1}}$$

Then, given any other $X(0)$, α is given by Equation (12). Therefore, each agent can compute the desired weighted average consensus value which is given by

$$\frac{\gamma^T X(0)}{\gamma^T\mathbf{1}} = \frac{\alpha}{\alpha_{fixed}}$$

Theorem 3.1 directly applies to the following consensus protocol which has been studied in [8], [13] .

$$u_i(t) = \sum_{j\in N_i} a_{ij}(x_j(t - \tau_{ij}) - x_i(t)) \tag{20}$$

where τ_{ij} denotes the communication delay in the channel e_{ij}. This protocol, $h_{ij}(s) = e^{-\tau_{ij}s}$ represents the channel filter and $\hat{x}_j(t) = x_j(t - \tau_{ij})$ is the filtered state of agent j. We provide the consensus value which is not given in [8], [13].

Corollary 3.2: Consider the network with its topology strongly connected, suppose each channel has different communication delays, i.e. if $e_{ij} \in \mathcal{E}$, it has communication delay τ_{ij} , then protocol (20) will solve the consensus problem and the consensus value is given by

$$\alpha = \frac{\sum_{i=1}^{n}\gamma_i x_i(0)}{\sum_{i=1}^{n}\gamma_i + \sum_{j\in N_i}\gamma_i a_{ij}\tau_{ij}}$$

To reach consensus, we need the stability of $G(s)$. the next result provides a sufficient condition to check the stability of $G(s)$ which also leads to consensus provided that extra condition is satisfied.

Theorem 3.3: If the channel filters satisfy $\sum_{j\in N_i}|a_{ij}h_{ij}(j\omega)| \leq d_i$, $\forall\omega \in \mathbb{R}$, $\forall i = 1, 2, \cdots, n$, then $G(s)$ will has all its poles on the LHP. If $G(s)$ has poles on the $j\omega$ axis, it can only has poles at the origin. Furthermore, if $L(0)$ is the Laplacian of a strongly connected graph, protocol (8) solves the consensus problem.

Proof. Let $Z(s) = G^{-1}(s) = (sI_n + L(s))$, then checking the poles of $G(s)$ is equivalent to checking the zeros of $Z(s)$.

[1]We are neglecting the difference in the γ. Here, γ depends not only on network topology but also on its dynamics.

we first show that for all s with Re$[s] > 0$, $\det[Z(s)] \neq 0$, which means $Z(s)$ has no zeros on the open RHP. From the Gersgorin Disk Theorem, all the eigenvalues of $Z(s) = [z_{ij}(s)]$ are located on the union of the following n disks:

$$D_i = \{z \in \mathbb{C} : |z - z_{ii}(s)| \leq \sum_{j \in N_i} |z_{ij}(s)|\} \tag{21}$$

where $i = 1, 2, \cdots, n$. We know that for the transfer matrix $Z(s)$, $z_{ii}(s) = s + d_i$, and $z_{ij}(s) = -a_{ij}h_{ij}(s)$ for $j \in N_i$. Since we assume $\sum_{j \in N_i} |a_{ij}h_{ij}(j\omega)| \leq d_i$, when Re$[s] > 0$, from the Maximum Modulus Principle, we have $\sum_{j \in N_i} |z_{ij}(s)| \leq d_i$. Now $|z_{ii}(s)| = |s + d_i| > d_i$, which means

$$|z_{ii}(s)| > \sum_{j \in N_i} |z_{ij}(s)| \quad \forall i = 1, 2, \ldots, n.$$

Therefore $\det[Z(s)] \neq 0$ and $Z(s)$ will have no zero on the open RHP.

When Re$[s] = 0$, $|z_{ii}(j\omega)| = |d_i + j\omega|$ and $\sum_{j \in N_i} |z_{ij}(j\omega)| \leq d_i$. If $\omega \neq 0$, since $|d_i + j\omega| > d_i$, all the eigenvalues of $Z(s)$ can not be zero, so $\det[Z(s)] \neq 0$ for any $s = j\omega$, $\omega \neq 0$. Thus $Z(s)$ has no zeros on the $j\omega$ axis except for the origin. When $\omega = 0$, we first show if $Z(0)v = 0$, i.e. Z(s) has zero at zero, $v = \mathbf{1}$. Suppose not, we can assume the largest number of v is v_i. Since $z_{ii}(0) \geq \sum_{j \in N_i} |z_{ij}(0)|$, $z_{ii}(0)v_i$ will be dominant in the product of the $i-th$ row of $Z(0)$ and v, i.e. the $i-$th element of $Z(0)v$ is not equal to zero. We now suppose for some node i, $\sum_{j \in N_i} a_{ij}h_{ij}(0) < d_i$. Apparently, $Z(0)\mathbf{1} \neq 0$ since $z_{ii}(0)$ is dominant on the $i-$th row of $Z(0)$. If $\sum_{j \in N_i} a_{ij}h_{ij}(0) = d_i$, $\forall i$, then $Z(0) = L(0)$ is a Laplacian. Therefore $Z(0)\mathbf{1} = 0$, which means $G(s)$ has poles at the origin. From the above analysis, $G(s)$ has poles at the origin iff $\forall i$, $\sum_{j \in N_i} a_{ij}h_{ij}(0) = d_i$.

If $L(0)$ is the Laplacian of a strongly connected graph, rank$(L(0)) = n - 1$. since $Z(0) = L(0)$, $Z(s)$ has only one zero on the origin. Thus $G(s)$ only has one pole at the origin. We can get the result from Theorem 3.1. ■

B. Consensus Robustness

Theorem 3.1 provides a general sufficient condition to reach consensus. However, testing the stability of $G(s)$ is not easy in some large network systems, especially when the network channels are not perfectly known. This leads us to search a more easily checkable condition.

Theorem 3.4: For the LTI network, suppose its topology is strongly connected, , then protocol (8) can solve the consensus problem if $\|h_{ij}(s)\|_\infty = h_{ij}(0) = 1$, $\forall i, j \in N_i$.

Remark 3.5: Theorem 3.4 characterize the robustness of protocol (8) in the presence of uncertain channels. Besides, the above condition is very useful for building the consensus network. Instead of concerning the incoming links of each node, we can concentrate on the network topology and the individual links to guarantee the conditions for reaching consensus. Moreover, if the DC gain of each channel is equal to its H_∞ norm, the network will reach consensus under the

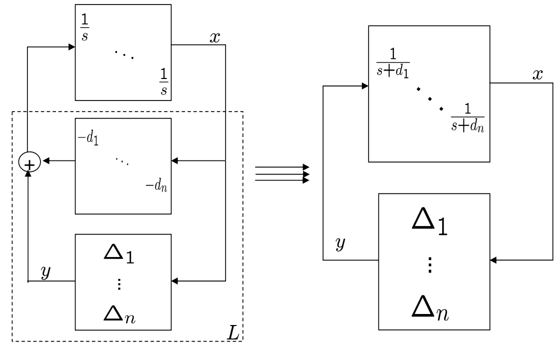

Fig. 1. The network system loop transformation

following modified consensus protocol.

$$\dot{u}_i(t) = \sum_{j \in N_i} a_{ij}(\frac{1}{g_{ij}}\hat{x}_j(t) - x_i(t))$$

provided we know the DC gain g_{ij} for each channel. This also gives the intuition that under the cases when the DC gain of the channel is not equal to its H_∞ norm, the agent should lowpass those channels to have the sufficient conditions satisfied.

Theorem 3.4 provides the robust condition for each channel. We next investigate another robust condition which gives more freedom to select the dynamics of the channel filters. The network system loop transformation is shown in figure 1, where $\Delta_i(s)$ is the i-th row of $\mathcal{A}(s)$ and y is the signal vector which can be represented as $y = [y_1, y_2, \cdots, y_n]^T$. Let $\phi_i(s) = \frac{1}{s+d_i}$ and define $\Delta_i(s)$ as the channel filter vector associated with the i-th agent. Here, we are ready to state our result concerning robust stability of the CNF.

Theorem 3.6: Consider the LTI network with its topology strongly connected, if the channel filter vectors satisfy $\|\Delta_i(s)\|_\infty < \sqrt{\frac{d_i^2}{|N_i|}}$, ∀i=1,2,$\cdots, n$, then $lim_{t\to\infty} x_i(t) = 0$, $\forall i = 1, 2, \ldots, n$.

Proof. We prove this theorem by contradiction. Suppose $\|x(t)\|_2 \neq 0$, from figure 1, $\|y_i\|_2^2 = \|\Delta_i(s)x\|_2^2$ for $i = 1, 2, \cdots, n$. Since $\|\Delta_i(s)\|_\infty < \sqrt{\frac{d_i^2}{|N_i|}}$, we have

$$\|y_i(t)\|_2^2 < \frac{d_i^2}{|N_i|} \sum_{j \in N_i} \|x_j(t)\|_2^2 \tag{22}$$

Since $\|x_i\|_2^2 = \|y_i\phi_i(s)\|_2^2$ and $\|\phi_i(s)\|_\infty \leq \frac{1}{d_i}$, we have

$$\|x_i(t)\|_2^2 \leq \frac{1}{d_i^2} \|y_i(t)\|_2^2 \tag{23}$$

From the above two equations, we get

$$\|y_i(t)\|_2^2 < \frac{d_i^2}{|N_i|} \sum_{j \in N_i} \frac{1}{d_j^2} \|y_j\|_2^2 \tag{24}$$

Defining $\hat{y} = [\|y_1\|_2^2, \|y_2\|_2^2, \cdots, \|y_n\|_2^2]$, $D = diag\{\frac{1}{d_1^2}, \frac{1}{d_2^2}, \cdots, \frac{1}{d_n^2}\}$ and $N = diag\{\frac{1}{|N_1|}, \frac{1}{|N_2|}, \cdots, \frac{1}{|N_n|}\}$, we put (24) in a more compact form as

$$\hat{y} = D^{-1}NAD\hat{y} \tag{25}$$

where $A = [a_{ij}]$ with $a_{ij} = 1$ if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Furthermore, we left multiplying D on both sides of (25) and define $z = D\hat{y}$, then (25) can be transformed as

$$z < NAz \tag{26}$$

We now prove that (26) has no solution. Since $(I_n - NA)$ is an Laplacian with diagonal elements of 1, from the Gersgorin Disk Theorem, all the eigenvalues of the matrix $(I_n - NA)$ are located on the disk of $\{z \in \mathcal{C} : |z - 1| \leq 1\}$, therefore the spectral radius of NA satisfies $\rho\{NA\} \leq 1$ and the inequality (26) has no solution, which is a contradiction. Thus we got $\|X(t)\|_2 = 0$ is the only solution of the loop, which means all the states will converge to 0. ■

We now turn to the equal part of the above condition. Suppose for each agent i, $\|\Delta_i\|_\infty^2 = \frac{d_i^2}{|N_i|}$, which means that $\sup_{\omega\in\mathbb{R}}\{\sum_{j\in N_i} a_{ij}^2 |h_{ij}(j\omega)|^2\} = \frac{d_i^2}{|N_i|}$. We know for any vector $x \in \mathbb{R}^n$, $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$, so

$$[\sum_{j\in N_i} a_{ij}|h_{ij}(j\omega)|]^2 \leq |N_i| \sum_{j\in N_i} a_{ij}^2 \left|h_{ij}(j\omega)\right|^2 \leq d_i^2 \tag{27}$$

which means

$$\sum_{j\in N_i} a_{ij}\left|h_{ij}(j\omega)\right| \leq d_i \tag{28}$$

From the same analysis as in the proof of Theorem 3.3, we deduce that $G(s)$ has all its poles on the strict LHP or has a single pole at zero and $G(s)$ has a simple pole at zero if and only if for each node i, $a_{ij}h_{ij}(0)$ are equal for all $j \in N_i$, i.e. $a_{ij}h_{ij}(0) = \frac{d_i}{|N_i|}$. This is because the equal part of (27) holds iff $|a_{ij}h_{ij}(j\omega)|$ are equal for all $j \in N_i$ and we require this condition when $\omega = 0$.

This analysis together with Theorem 3.6 leads to the following corollary.

Corollary 3.7: Consider the LTI network with its topology strongly connected, if the channel filter vectors satisfy $\|\Delta_i(s)\|_\infty \leq \sqrt{\frac{d_i^2}{|N_i|}}$, ∀i=1,2,$\cdots, n$, then the network will be stable under protocol (8). Furthermore, under the above condition, the network will reach consensus if $\forall i = 1, 2, \cdots, n$, $a_{ij}h_{ij}(0) = \frac{d_i}{|N_i|}$ for all $j \in N_i$.

Remark 3.8: We should point out that although both Theorem 3.4 and Corollary 3.7 can lead to the same conditions required for Theorem 3.3, they are not comparable with each other. This is due to the fact that Corollary 3.7 allows individual channels have their H_∞ norm greater than 1 and on the other hand, some special cases satisfy Theorem 3.3 but violate Corollary 3.7, i.e. $L(s) = L =$ circul$[3, -1, -2]$, where circul denotes the circulant matrix.

C. Discrete Time Case

To implement the consensus protocol in real time system, we need its discrete-time counterpart. We consider the following discrete-time consensus protocol which comes from the discretization of protocol (8)

$$x_i(k+1) = x_i(k) + \beta \cdot \sum_{j\in N_i} a_{ij}(\hat{x}_j(k) - x_i(k)) \tag{29}$$

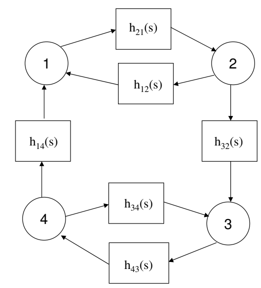

Fig. 2. Network topology: four agents with six channels.

where $\hat{x}_j(z) = h_{ij}(z)x_j(z)$ with $h_{ij}(z)$ being the transfer function of channel filter h_{ij}. Taking the z transformation of (29), we get

$$zx_i(z) - zx_i(0) = x_i(z) + \beta \sum_{j\in N_i} a_{ij}[h_{ij}(z)x_j(z) - x_i(z)]$$

We put the above equations into a compact form

$$x(z) = z[zI - I + \beta\bar{L}(z)]^{-1}x(0) \tag{30}$$

where $x(z) = [x_i(z), x_2(z), \ldots, x_n(z)]$ and $\bar{L}(z) = [l_{ij}(z)]_{n\times n}$ with $l_{ii}(z) = d_i$ and $l_{ij}(z) = -a_{ij}hij(z)$ $\forall j \in N_i$. We define $G(z) = z[zI - I + \beta\bar{L}(z)]^{-1}$. Almost all the results got from the continuous time system can be carried over to discrete time case except for the important role played by the discretizaiton step size. We state the following result without proof.

Theorem 3.9: Suppose $G(z)$ has one pole at 1 and all the other poles on the strictly unit disk, if $\bar{L}(1)$ is a graph Lapacian of a strongly connected graph and $\beta < 1/d$, where d is the maximum in-degree of all the nodes, then protocol (29) solves the consensus problem in LTI networks. Furthermore, the consensus value is given by

$$\alpha = \frac{\sum_{i=1}^n \gamma_i x_i(0)}{\sum_{i=1}^n \gamma_i + \beta \cdot \gamma^T L_1 \mathbf{1}}$$

where $L_1 = \frac{d\bar{L}(z)}{dz}|_{z=1}$.

IV. Simulation Results

In this section we use some example to demonstrate our results derived before. Consider a network of four agents, its topology is shown in figure 2 with all the channel weights equal to 1. Let all the channel filters to be the same and has the transfer function $\frac{1}{s+1}$. Therefore, we have the following parameters: $\gamma = [\frac{1}{2}, 1, \frac{1}{2}, 1]^T$

$$\mathbf{L} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \mathbf{L_1} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let the initial states of the four agents are 2, 4, 4 and 7. From figure 3, all the agents converge to the same value 2 which coincides with Theorem 3.1. Next, we change the transfer function of h_{14} to $\frac{s}{s+1}$. After this change, all the states converge to zero as shown in figure 3. Since $h_{14}(0) =$

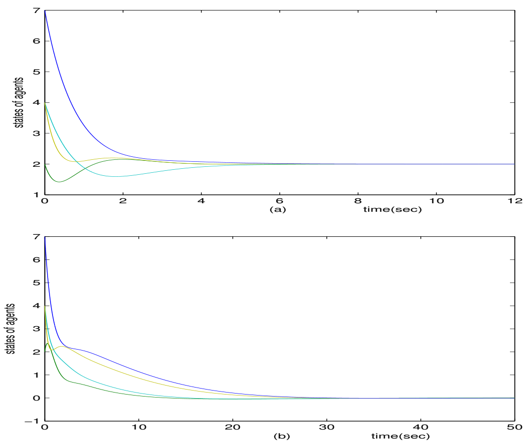

Fig. 3. (a) all the agents converge to 2. (b) all the agents converge to 0.

0, $L(0)$ is not a Laplacian graph and $G(s)$ does not have the pole at the origin. Actually, after the change, Theorem 3.3 is still satisfied, which means $G(s)$ must have all the poles on the strict LHP.

V. Conclusions

In this paper we have studied the consensus problem in the network with dynamic channels. We give the sufficient conditions to reach consensus and derive the formula for the consensus value. The robust condition to achieve consensus in our paper provides some insight in building dynamic consensus networks. We argue that in order to reach consensus, we can lowpass the channels at the agents to make Theorem 3.4 satisfied.

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