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Averaging analysis of a sinusoidal disturbance rejection algorithm for unknown plants

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Abstract— **The paper discusses the dynamic behavior of an adaptive algorithm for the rejection of a sinusoidal disturbance acting on an unknown system. Averaging theory is used to approximate the nonlinear time-varying closed-loop system by a nonlinear time-invariant system. Then, it is shown that the four-dimensional averaged system has a two-dimensional equilibrium surface, which can be divided into stable and unstable subsets. Trajectories generally converge to a stable point of the equilibrium surface, implying that the disturbance is asymptotically cancelled even if the true parameters of the system may not be exactly determined. Simulations demonstrate the results of the analysis.**

I. Introduction

The paper considers an algorithm for the rejection of sinusoidal disturbances of known frequency acting on systems with unknown dynamics. The main contribution of the paper is an analysis of the dynamic properties of the algorithm using averaging theory. Few solutions have been proposed for the disturbance rejection problem under consideration, but even fewer have been either proved to work in practice or analyzed carefully. In the signal processing literature, algorithms have been presented that combine a gradient algorithm (*i.e.,* adaptive least-mean-squares or LMS algorithm) with an on-line identifier of the plant's impulse response [8][5][6]. For the identification, such methods require considerable excitation to be injected in the form of white noise added at the input of the system. An analysis of the stability of the closed-loop system has also not been provided, let alone any insight into the dynamics of the systems. Harmonic steady-state (HSS) methods have simplified the problem by approximating the plant by its steady-state sinusoidal response. In [2], Pratt and co-workers describe an HSS algorithm, known as higher harmonic control (HHC), for use in the reduction of vibrations in helicopters, and in [4], the algorithm is used for the cancellation of periodic noise in an acoustic drum. A proof of stability is provided in [4], although it assumes the injection of an excitation signal to ensure correct identification of the plant. In contrast, [11] proposes a clever algorithm that combines two gradient-type adaptation steps to obtain an algorithm with guaranteed stability properties without additional excitation. While successful experiments were reported, no data was shown on the transient properties of the algorithm or its ability to track variations in the parameters. On the other hand, such results were demonstrated in [9], which provided a remarkably simple algorithm inspired from [4]. Its advantage is that it eliminated the need to collect batches of data and updated control parameters continuously. The performance of the algorithm was verified through single-channel active noise control experiments and the ability to track abruptly or continuously time-varying system parameters was demonstrated. However, no formal proof of stability was obtained.

The objective of this paper is to provide a stability analysis for the adaptive system of [9]. Note that rigorous stability proofs have been the subject of research in adaptive control, but have turned out to be very complicated and have provided little insight in the dynamics of the systems. As an alternative, averaging methods have provided approximate results that were powerful in dealing with nonlinear time-varying systems [1][10][3]. Averaging theory shows how a set of nonlinear time-varying differential equations can be written as a perturbation about an averaged system, and how the much simpler averaged system can serve as an approximation of the original system. In [10] and other work, averaging theory was found to provide invaluable information on the dynamic properties of specific adaptive control systems. For periodic disturbance rejection problems, averaging is even more powerful, because the conditions for the existence of the averaged system are generally satisfied, due to the periodic nature of the signals. While averaging assumes low adaptation gains, experience shows that the approximation is useful for the typical adaptation gains used in practice, and that the loss of rigor due to the approximation is more than compensated for by the powerful insights that the approximation provides.

The paper is organized as follows. After formulating the system's equations, averaging theory [10] is reviewed. The averaged system associated with the problem is found and simulations are used to demonstrate the closeness of the responses. Next, the equilibrium points of the averaged system are determined and an eigenanalysis is used to understand the system's behavior around the equilibrium. This analysis enables one to understand how the algorithm handles uncertainty in the plant parameters in a way that a standard adaptive LMS cannot. Further simulations illustrate the results of the analysis of the averaged system.

II. System formulation

We consider the feedback system shown in Fig. 1. The output of the plant

$$y(t) = P(s)[u(t)] - p(t) \tag{1}$$

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is fed back in order to determine the control signal $u(t)$ needed to reject the sinusoidal disturbance $p(t)$. The notation $P(s)[(\cdot)]$ represents the time-domain output of the system with transfer function $P(s)$. $P(s)$ is assumed to be bounded-input bounded-output stable, but is otherwise unknown. C is a nonlinear and time-varying control law consisting of a parameter identification scheme and a disturbance cancellation algorithm.

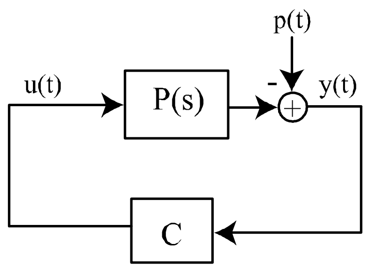

Fig. 1. Feedback control system.

The disturbance is assumed to be a sinusoidal signal given by

$$p(t) = p_c \cos(\omega_1 t) + p_s \sin(\omega_1 t) = w_m^T(t)\pi^* \tag{2}$$

where

$$\pi^* = \begin{pmatrix} p_c \\ p_s \end{pmatrix}, \quad w_m = \begin{pmatrix} \cos(\omega_1 t) \\ \sin(\omega_1 t) \end{pmatrix} \tag{3}$$

and ω_1 is the known frequency of the disturbance signal. Under these conditions, a control signal of the form

$$u(t) = \theta_c \cos(\omega_1 t) + \theta_s \sin(\omega_1 t) = w_m^T(t)\theta \tag{4}$$

is sufficient to cancel the disturbance in steady-state, provided that the controller parameter vector

$$\theta = \begin{pmatrix} \theta_c \\ \theta_s \end{pmatrix} \tag{5}$$

is chosen appropriately.

For the derivation of the algorithm, the response of the plant is approximated by the sinusoidal steady-state response

$$y(t) \simeq y_{ss}(t) = w_m^T(t) G^* \theta - p(t) \tag{6}$$

where

$$G^* = \begin{pmatrix} P_R & P_I \\ -P_I & P_R \end{pmatrix} \tag{7}$$

and P_R, P_I are the real and imaginary parts of the plant's frequency response

$$P(j\omega_1) \triangleq P_R + jP_I \tag{8}$$

In the problem considered here, there are four unknowns: two are associated with the plant (P_R and P_I) and two are associated with the disturbance (p_c and p_s). The parameters may be collected in a vector of parameters

$$x^* = \begin{pmatrix} P_R & P_I & p_c & p_s \end{pmatrix}^T. \tag{9}$$

Then, the steady-state output of the plant (6) can be written as

$$y_{ss}(t) = W^T(t,\theta)x^* \tag{10}$$

where $W(t,\theta)$ is a so-called *regressor matrix*

$$W(t,\theta) = \begin{pmatrix} \theta_c \cos(\omega_1 t) + \theta_s \sin(\omega_1 t) \\ \theta_s \cos(\omega_1 t) - \theta_c \sin(\omega_1 t) \\ -\cos(\omega_1 t) \\ -\sin(\omega_1 t) \end{pmatrix} \tag{11}$$

On the basis of the linear expression in (10), an estimate x of the unknown parameter vector x^* can be obtained using a gradient or a least-squares algorithm. For example, a gradient algorithm for the minimization of the error $e = W^T x - y$ that uses the approximation that $y(t) \simeq y_{ss}(t)$ is given by

$$\dot{x}(t) = -\epsilon W(t,\theta)\left(W^T(t,\theta)x(t) - y(t)\right) \tag{12}$$

The parameter $\epsilon > 0$ is the adaptation gain, which will be assumed to be small in the application of the averaging theory later in the paper.

Having derived an algorithm for the estimation of the unknown parameters, it remains to define the control law. Note that the disturbance is cancelled exactly in steady-state for a nominal control parameter

$$\theta^* = G^{*-1}\pi^* \tag{13}$$

Given an estimate of the unknown parameter vector x, a certainty equivalence control law will then select θ as a function of the estimate using

$$G(x) = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}, \ \pi(x) = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \tag{14}$$

and

$$\begin{aligned} \theta(x) &= G^{-1}(x)\pi(x) \\ &= \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_1 x_3 - x_2 x_4 \\ x_1 x_4 + x_2 x_3 \end{pmatrix} \end{aligned} \tag{15}$$

The nominal values satisfy

$$G^* = G(x^*), \ \pi^* = \pi(x^*), \text{ and } \theta^* = \theta(x^*) \tag{16}$$

A state-space representation of the overall system can be obtained as follows. With x_P denoting the states of $P(s) = C(sI - A)^{-1}B$, the plant has the following state-space representation

$$\begin{aligned} \dot{x}_P(t) &= Ax_P(t) + Bu(t) \\ &= Ax_P(t) + Bw_m^T(t)\theta(x) \end{aligned} \tag{17}$$

$$y(t) = Cx_P(t) - p(t) = Cx_P(t) - w_m^T(t)\pi^* \tag{18}$$

As is the case in most active noise and vibration control applications, the matrix A is assumed to be exponentially stable. Defining

$$E(x) = \begin{pmatrix} D(x) \\ -I_{2\times 2} \end{pmatrix}, \ D(x) = \begin{pmatrix} \theta_c(x) & \theta_s(x) \\ \theta_s(x) & -\theta_c(x) \end{pmatrix} \tag{19}$$

the matrix $W(t,\theta)$ is given by

$$W(t,\theta) = E(x)w_m(t) \tag{20}$$

and the overall system is described by a set of differential equations with two vectors x and x_P composing the total state vector with

$$\begin{aligned} \dot{x}_P &= Ax_P + Bw_m^T(t)\theta(x) && (21)\\ \dot{x} &= -\epsilon E(x)w_m(t)\left(w_m^T(t)E^T(x)x\right.\\ &\qquad \left.-Cx_P + w_m^T(t)\,\pi^*\right) && (22)\end{aligned}$$

Note that this set of differential equations is both time-varying and nonlinear, making direct analysis difficult. Fortunately, under the assumption of small gain ϵ, the application of averaging theory produces an approximate nonlinear time-invariant system whose dynamics can be analyzed, providing interesting insights in the behavior of the system.

III. AVERAGING ANALYSIS

A. Background - mixed time scale systems

Of particular interest to our problem is the continuous-time averaging method for mixed time scale systems as discussed in [10]. The theory applies to systems of the form

$$\begin{aligned} \dot{x} &= \epsilon f(t, x, x_P) && (23)\\ \dot{x}_P &= Ax_P + h(t, x) && (24)\end{aligned}$$

which includes the problem under consideration if one defines

$$\begin{aligned} f(t, x, x_P) &= -E(x)w_m(t)\left(w_m^T(t)E^T(x)x\right.\\ &\qquad \left.-Cx_P + w_m^T(t)\pi^*\right) && (25)\\ h(t, x) &= Bw_m^T(t)\theta(x) && (26)\end{aligned}$$

For ϵ small, x is a slow variable. x_P varies faster, except through its dependency on x. Averaging theory shows how the trajectories of (23)-(24) can be related to the trajectories of the so-called averaged system

$$\dot{x} = \epsilon f_{av}(x) \quad (27)$$

where

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, v(\tau, x))d\tau \quad (28)$$

and

$$v(t, x) := \int_0^t e^{A(t-\tau)} h(\tau, x)d\tau. \quad (29)$$

Central to the method of averaging is the assumption that the limit in (28) exist uniformly in t_0 and x. Then, there exists a strictly decreasing continuous function $\gamma(T)$, such that $\gamma(T) \to 0$ as $T \to \infty$ and

$$\left| \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, v(\tau, x))d\tau - f_{av}(x) \right| \leq \gamma(T). \quad (30)$$

The function $\gamma(T)$ is called the convergence function. If the limit exists, ϵ is sufficiently small, and certain technical conditions are satisfied, the response of (23)-(24) is close to the response of (27). Specifically, the theory is based on assumptions B1-B6 found in [10]. These assumptions establish certain continuity and boundedness conditions necessary for the successful application of averaging. After verification of assumptions B1-B6 the following result can be obtained.

Lemma 1: If the mixed time scale system (23)-(24) and the averaged system (27) satisfy assumptions B1-B6 of [10], then there is an $\epsilon_T > 0$ and a class K function $\Psi(\epsilon)$ such that

$$\|x(t) - x_{av}(t)\| \leq \Psi(\epsilon)b_T \quad (31)$$

for some $b_T > 0$ and for all $t \in [0, T/\epsilon]$ and $0 < \epsilon \leq \epsilon_T$. Further, if the function

$$d(t, x) = f(t, x, v(t, x)) - f_{av}(x) \quad (32)$$

has a bounded integral with respect to time, then $\gamma(T) \sim \frac{1}{T}$ and $\Psi(\epsilon)$ is on the order of ϵ.
A proof of *Lemma 1* can be found in [10]. This proof establishes a link between the convergence function $\gamma(T)$ and the order of the bound in (31). *Lemma 1* states that, for ϵ sufficiently small, the trajectories of (23) and (27) can be made arbitrarily close for all $t \in [0, T/\epsilon]$. This allows insight into the behavior of (23)-(24) by studying the behavior of (27).

B. Averaged system

We found earlier that the system under consideration fitted the averaging framework. It remains to determine what the averaged system is, whether the assumptions are satisfied, and what interesting properties the averaged system may have. In the computation of the averaged system, the parameter vector x is frozen. Further, all of the time variation in the functions is due to sinusoidal signals, and the systems to which they are applied are linear time invariant. The outcome is that the average of the function $f(t, x, x_P)$ is well-defined and can be computed exactly. Specifically, the function

$$\begin{aligned} v(t, x) &= \int_0^t e^{A(t-\tau)} Bw_m(\tau)d\tau \cdot \theta(x) && (33)\\ &= x_{P,ss}(t) + x_{P,tr}(t) && (34)\end{aligned}$$

where $x_{P,ss}(t)$ is the steady-state response of the state of the plant to the sinusoidal excitation $w_m(t)$ and, $x_{P,tr}$ is a transient response that decays to 0 exponentially, given that A is exponentially stable.

The averaged system is obtained by computing the average of

$$\begin{aligned} f_{av}(x) = -\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} & E(x)w_m(\tau)\left(w_m^T(\tau)E^T(x)x\right.\\ & \left.-Cv(\tau, x) + w_m^T(\tau)\pi^*\right) d\tau \quad (35)\end{aligned}$$

where

$$\begin{aligned} Cv(t, x) - w_m^T(t)\pi^* &= Cx_{P,ss}(t) + Cx_{P,tr}(t)\\ &\qquad - w_m^T(t)\pi^* && (36)\\ &= y_{ss}(t) + y_{tr}(t) && (37)\end{aligned}$$

and $y_{tr}(t) = Cx_{P,tr}(t)$. The derivation of the algorithm implied that

$$y_{ss}(t) = w_m^T(t)E^T(x)x^* \quad (38)$$

and since the transient response of the plant does not affect the average value of the function,.

$$f_{av}(x) = -\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} E(x)w_m(\tau)\left(w_m^T(\tau)E^T(x)x - w_m^T(\tau)E^T(x)\,x^*\right)d\tau \quad (39)$$

$$= -E(x)\left(\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} w_m(\tau)w_m^T(\tau)d\tau\right) \cdot E^T(x)(x-x^*) \quad (40)$$

$$= -\frac{1}{2}E(x)E^T(x)(x-x^*) \quad (41)$$

In other words, the averaged system is simply given by

$$\dot{x} = -\frac{\epsilon}{2}\begin{pmatrix} D(x) \\ -I_{2\times 2}\end{pmatrix}\begin{pmatrix} D(x) & -I_{2\times 2}\end{pmatrix}(x-x^*) \quad (42)$$

with

$$D(x) = \frac{1}{x_1^2+x_2^2}\begin{pmatrix} x_1x_3 - x_2x_4 & x_1x_4 + x_2x_3 \\ x_1x_4 + x_2x_3 & -x_1x_3 + x_2x_4\end{pmatrix} \quad (43)$$

Although (42) is nonlinear, the method of averaging has eliminated the time variation of the original system, providing an opportunity to understand much better the dynamics of the system.

C. Application of Averaging Theory

Application of the theory is relatively straightforward, and verification of the assumptions is left to the reader. A technical difficulty is related to the fact that both the adaptive and averaged systems have a singularity at $x_1^2 + x_2^2 = 0$ (see equations (15) and (43)). Such singularities are quite common in adaptive control, and occur when the estimate of the gain of the plant is zero. Here, the singularity occurs when the estimate of the plant's frequency response is zero. The problem is somewhat unlikely to occur as *two* parameters need to be small for the singularity to be encountered. Nevertheless, a cautious implementation of the algorithm would apply one of the standard adaptive control approaches to address singularities. For example, a simple practical fix of the algorithm consists in using in the control law the last value of the estimated parameter x such that $x_1^2+x_2^2 \geqslant \delta > 0$, where δ is a small parameter, when $x_1^2 + x_2^2 < \delta$. As far as the theory is concerned, we avoid the difficulty by adding the following assumption.

Assumption 1 Assume that trajectories of the original and averaged system are such that $x_1^2+x_2^2 \geqslant \delta$ for some $\delta > 0$.

Verification of assumptions B1-B6 found in [10] for the mixed time scale system given by (21)-(22) and the averaged system (42) is left to the reader. It can be shown that the system given by (21)-(22) satisfies the conditions of the theory. Thus, *Lemma 1* can be applied. Due to the periodic signals involved, it is easily shown that $d(t, x)$ has a bounded integral with respect to time, suggesting that $\Psi(\epsilon)$ in *Lemma 1* is on the order of ϵ. *Lemma 1* establishes that (42) can be used as an order of ϵ approximation of (21)-(22) for all $t \in [0, T/\epsilon]$. Note that *Lemma 1* only shows closeness of the original and averaged systems over finite time. Any stability properties of the averaged system would require a different theorem. The theorems of [10] do not apply because they assume a unique equilibrium point of the averaged system. As we will see, this is not the case here.

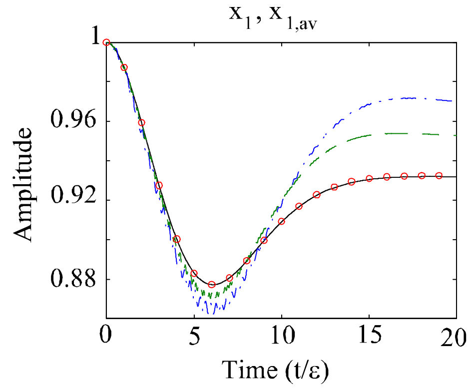

Fig. 2. The response of the first adapted parameter for the averaged system and three responses of the actual system.

D. Simulation example

To show the closeness of the responses (21)-(22) and (42), we let $\omega_1 = 330\pi$ and the plant is taken as a 250 coefficient FIR transfer function. The transfer function was identified from an active noise control system located at the University of Utah using a white noise input. The initial parameter estimate was $x(0) = x_{av}(0) = \begin{pmatrix} 1.0 & 1.0 & 0 & 0 \end{pmatrix}^T$. In Fig. 2, the response of the first adaptive parameter is shown. Four responses are shown: the averaged system with $\epsilon = 1$ (solid line), the actual system for $\epsilon = 100$ (dashed dot), the actual system for $\epsilon = 50$ (dashed), and the actual system for $\epsilon = 1$ (circles). As ϵ decreases, one finds that the trajectory of (21)-(22) approaches that of (42). Note that the parameter estimates do not converge to the nominal values. However, the control parameters θ_c and θ_s do converge to the nominal values, resulting in cancellation of the disturbance for all values of ϵ. The control parameters are shown in Fig. 3, along with θ^*, the nominal value that exactly cancels the disturbance (the constant line).

IV. Properties of the Averaged System

Several properties of the averaged system can be derived from the rather simple form that was obtained, enabling one to gain insight on the behavior of the closed-loop system.

A. Equilibrium surface

From the expression of the averaged system (42), we deduce that an equilibrium point of the averaged system must

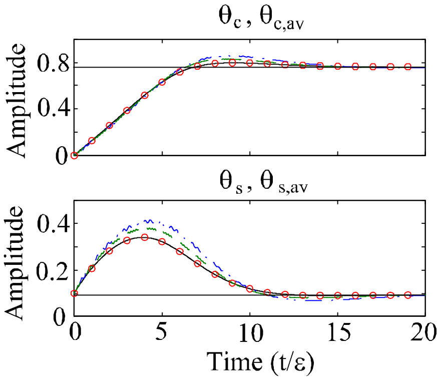

Fig. 3. Trajectories of control parameters for the actual and the averaged systems.

satisfy

$$E^T(x)(x-x^*) = \begin{pmatrix} D(x) & -I_{2\times 2} \end{pmatrix}(x-x^*) = 0 \quad (44)$$

Therefore, $x = x^*$ is an equilibrium point of the system. It is not the only one, however. Using the definition of the control law results in $E^T(x)x = 0$. Equilibrium points then satisfy

$$E^T(x)x^* = 0 \quad (45)$$

which can be rewritten as

$$\begin{pmatrix} \theta_c(x) \\ \theta_s(x) \end{pmatrix} = \begin{pmatrix} x_1^* & x_2^* \\ -x_2^* & x_1^* \end{pmatrix}^{-1} \begin{pmatrix} x_3^* \\ x_4^* \end{pmatrix} = \begin{pmatrix} \theta_c^* \\ \theta_s^* \end{pmatrix} \quad (46)$$

This last equation shows that any equilibrium state results in the cancellation of the disturbance. The equation also implies that

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \theta_c^* & \theta_s^* \\ \theta_s^* & -\theta_c^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (47)$$

In other words, the set of equilibrium points is a two-dimensional linear subspace of the four-dimensional state-space. Interestingly, for x constant,

$$f(t, x, x_{P,ss}) = -E(x)w_m(t)w_m^T(t)E^T(x)(x-x^*) \quad (48)$$

so that any equilibrium state of the averaged system is also an equilibrium state of the original system. Further, (10) and (20) yield

$$y_{ss}(t) = w_m^T(t)E^T(x)x^* \quad (49)$$

so that any equilibrium state corresponds to a perfect rejection of the disturbance.

B. Local stability

The local stability of the averaged system can be determined by linearizing (42) around an equilibrium state with x_1 and x_2. The following eigenvalues were computed

$$\lambda = \begin{pmatrix} 0 & 0 & \left(\frac{x_2^*+jx_1^*}{x_2+jx_1}\right)\beta & \left(\frac{x_2^*-jx_1^*}{x_2-jx_1}\right)\beta \end{pmatrix}^T \quad (50)$$

where $\beta = -\frac{g}{2}\left(\frac{x_1^{*2}+x_2^{*2}+x_3^{*2}+x_4^{*2}}{x_1^{*2}+x_2^{*2}}\right)$. The two eigenvalues at zero confirm the two-dimensional nature of the equilibrium surface. The nonzero eigenvalues are complex conjugates. An equilibrium point on the surface described by (45) is attracting if the following inequality is satisfied

$$x_1x_1^* + x_2x_2^* > 0 \quad (51)$$

or equivalently

$$x_3x_3^* + x_4x_4^* > 0. \quad (52)$$

The stability constraint can be interpreted in the $(x_1,\ x_2)$ plane, as shown in Fig.4. Specifically, the line going through the origin that is perpendicular to the line joining (0, 0) and (x_1^*, x_2^*) defines the boundary between the stable and unstable states. Interestingly, this is the same boundary that delineates the stable and unstable regions of a standard LMS algorithm that does not identify the plant parameters. In this case, however, the nonlinear dynamics ensure that all trajectories eventually converge to the stable subset of the equilibrium surface.

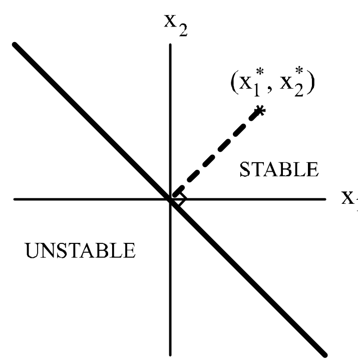

Fig. 4. Relationship between nominal parameters and stability of the equillibrium surface.

C. Lyapunov analysis

Lyapunov arguments can be used to establish further stability results. Specifically, the Lyapunov candidate function

$$V = \|x(t) - x^*\|^2 \quad (53)$$

evaluated along the trajectories of (42) gives

$$\dot{V} = -g\left\|E^T(x)\,(x-x^*)\right\|^2 \leq 0. \quad (54)$$

which implies that

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| \quad (55)$$

for all $t > 0$. Since x and $\dot{x}$ are bounded (using (42) and Assumption 1), one may also deduce that $E^T(x)\,(x-x^*) \to 0$ as $t \to \infty$. In turn, $E^T(x)x = 0$ and (49) imply that the disturbance is asymptotically cancelled.

Further conditions may be obtained by noting that

$$\begin{pmatrix} I_{2\times 2} & D(x) \end{pmatrix} E(x) = 0 \quad (56)$$

so that

$$\begin{pmatrix} I_{2\times 2} & D(x) \end{pmatrix} \dot{x} = 0 \quad (57)$$

Given that (43) can be rewritten

$$D(x) = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} x_3 & x_4 \\ x_4 & -x_3 \end{pmatrix} \quad (58)$$

implies that

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix} \dot{x} = 0 \tag{59}$$

From the first equation, one has that

$$\|x(t)\| = \|x(0)\| . \tag{60}$$

for all $t > 0$. In other words, while the norm of the parameter error is monotonically decreasing, the norm of the parameter vector is constant. In particular, the state is bounded, regardless of the local instability near one side of the equilibrium surface. (60) along with (15) indicate that any change in the magnitude of the first two estimated parameters $\sqrt{x_{1,av}^2 + x_{2,av}^2}$ must be inversely proportional to change in the magnitude of the last two estimated parameters $\sqrt{x_{3,av}^2 + x_{4,av}^2}$. Note that, if the two magnitudes changed proportionally in the same direction, there would be no change in control parameter and no impact on the output error. The second equation in (59) yields a further constraint on the state vector but is not as easily integrated.

D. Simulation results

In this section, we discuss an example that illustrates the properties of the averaged system. Consider the nominal parameter

$$x^* = \begin{pmatrix} 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}^T , \tag{61}$$

with the initial vector $x(0) = \begin{pmatrix} 1.1 & -2.0 & -2.0 & 1.0 \end{pmatrix}^T$ and the gain $\epsilon = 2.0$. The eigenvalues of (42) are given in (50). The eigenvalues may be complex, suggesting that a phase plot of the system might exhibit some spiralling behavior. This property is indeed found in the simulation result of Figure 5. $x(0)$ was chosen in a neighborhood of an unstable equilibrium point whose eigenvalues have relatively large imaginary part. The trajectories of the parameter estimates were projected into the $x_{1,av} - x_{2,av}$ plane for visualization.

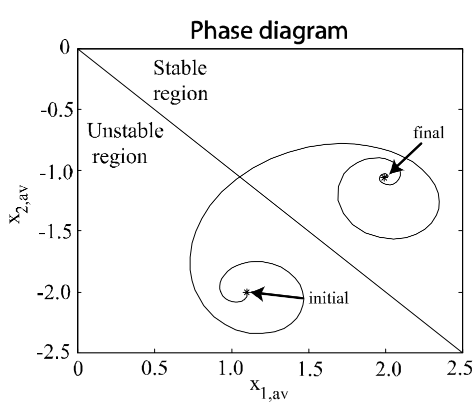

Fig. 5. Phase plot of identified parameters

While the initial conditions were chosen very close to the unstable region of the equilibrium surface, we see that the trajectory spirals as predicted and crosses over into the stable region. Also, note that the phase of the nominal plant is given by

$$\measuredangle P(j\omega_o) = 45^o \tag{62}$$

while the phase of the initial plant estimate is

$$\measuredangle \hat{P}(j\omega_o) = -61.2^o. \tag{63}$$

This is a phase difference of $\measuredangle P(j\omega_o) - \measuredangle \hat{P}(j\omega_o) = 106^o$, beyond the 90^o angle condition that applies to a gradient algorithm without plant identification. Although not shown, it was observed that the norm of trajectories remained constant at $\|x(0)\| = 3.20$.

V. Conclusions

The algorithm of [9] for the rejection of periodic disturbances of known frequency affecting unknown plants was considered. Since the overall closed-loop system is nonlinear and time-varying, exact analysis would be difficult and averaging theory was applied to simplify the analysis. By averaging over time, a much simpler time invariant system was obtained, whose dynamics closely approximate the dynamics of the actual system. It was shown that the averaged system for the algorithm under consideration was a nonlinear system with a two-dimensional equilibrium surface. Half of the surface was locally stable and the other half was unstable. Generally, trajectories converge to the stable subset of the equilibrium surface. Further properties of the trajectories of the systems were obtained from an analysis for their dynamics. Simulations illustrated the results of the analysis.

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