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Motion planning using navigation measure

U. Vaidya, and R. Bhattacharya,

Abstract— **Navigation measure is introduced as a new tool to solve the motion planning problem in the presence of static obstacles. Existence of the navigation measure guarantees collision free convergence to the final destination set. Navigation measure can be viewed as a dual to the navigation function, a popular tool used in the motion planning literature today. While navigation function has its minimum at the final destination set and peaks at the obstacle sets, the navigation measure on the other hand takes maximum value at the destination set and is zero on the obstacle set. A linear programming formalism is proposed for the construction of navigation measure. Set oriented numerical methods are used to obtain finite dimensional approximation of the navigation measure.**

I. INTRODUCTION

Motion planning problem has attracted the interest of different communities, including researcher in the area of nonlinear control, robotics, and artificial intelligence. The motion planning problem can be described as generation or execution of plan of moving from one location to another in space to accomplish a desired task while at the same time avoiding collision with obstacles or other undesirable behaviors [1]. Among various approaches to the motion planning problem three most popular approaches are cell decomposition method, roadmap method, and artificial potential field method.

The cell decomposition method rely on the partition of the configuration space into finite number of cells, in each of which the collision free path is found. Global path is then obtained by connecting the local collision free path between adjacent cells [2]. In roadmaps methods, a network of collision free connecting path is constructed which spans the free configuration space. The path planning problem then reduces to finding paths connecting the initial and final configuration to the roadmaps and then selecting the sequence of paths on the roadmaps [3], [4]. Potential field method was introduced to robotics research in the thesis work of Khatib [5]. In potential field method a collision free trajectory is generated by the robot moving locally according to forces defined by the negative gradient of a potential function. The function is designed to provide attractive forces towards the goal and repulsive forces which push the robot away from the obstacles. The control law using potential field methods is feedback in nature because the control action is computed at each instant of time depending upon the current state [6], [7]. The approaches using cell-decomposition and roadmaps are open loop in nature. The problem with the potential field methods is the possible existence of local minima in which robot might get trapped. Potential functions for motion planning were refined in the work of E. Rimon [8]. In particular the potential functions which do not have local minima are defined as navigation functions. Navigation functions have only one minimum at the desired goal configuration. The problem with the navigation functions is that although they exist, there is no systematic procedure for constructing one.

In this paper we introduce navigation measure as a new tool for motion planning problem. Navigation measure can be viewed as a dual to navigation function while navigation function has minimum at the final destination set and peaks at the obstacle set, navigation measure on the other hand takes maximum value on the final destination set and is zero on the obstacle sets. Navigation measure is inspired from *Lyapunov measure*, which is introduced in [9] for verifying weaker notion of almost everywhere stability in nonlinear systems. Lyapunov measure is shown to be dual to Lyapunov function. Application of Lyapunov measure for stabilization, called as *control Lyapunov measure*, is also studied in [10]. Motion planning can be viewed as the stabilization of final destination set using control Lyapunov measure under the additional constraints that the control Lyapunov measure is zero on the obstacle set to avoid collision with the obstacle sets. So navigation measure is defined as control Lyapunov measure that takes zero value on the obstacle sets. In this paper we propose a linear programming formalism for the construction of navigation measure. Set oriented numerical methods developed in [11] are used for the finite dimensional approximation of the navigation measure.

This paper is organized as follows. In section II, we review some of the key results from [9], [12], [10] on the applications of Lyapunov measure for stability analysis and stabilization problem. In section III, we present the main result of this paper on the use of navigation measure for the motion planning problem. Computation framework for the finite dimensional approximation of the navigation measure along with the simulation results for a model example problem are presented in section IV. Conclusion and discussion follows in section V.

II. LYAPUNOV MEASURE, STABILITY, AND STABILIZATION

In [9], [12], [10], *Lyapunov measure* is introduced for stability verification and for stabilizing controller design of an invariant set in nonlinear dynamical systems. Stability and stabilization problems for a nonlinear system $T : X \to X$, where $X \subset \mathbb{R}^n$ is compact, were studied using a weaker

U. Vaidya is with the Department of Electrical & Computer Engineering, Iowa State University, Ames, IA 50011 ugvaidya@iastate.edu

R. Bhattacharya is with the Department of Aerospace Engineering, Texas A & M University, raktim@aero.tamu.edu

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notion of almost everywhere stability. One such definition of almost everywhere stability is as follows.

Definition 1 (Almost everywhere stable with geometric decay): A closed T invariant set $A \subset X$ i.e., $T(A) = A$ is said to be almost everywhere stable with geometric decay with respect to finite measure $m \in \mathscr{M}(A^c)$ if given $\delta > 0$, there exists $K(\delta) < \infty$ and $\beta < 1$ such that

$$m\{x \in A^c : T^n(x) \in B\} < K\beta^n \quad \forall n \geq 0$$

for all set $B \subset X \setminus U_\delta$, where U_δ is the open neighborhood of invariant set A.

This weaker notion of almost everywhere stability was studied using a linear transfer operator called as Perron-Frobenius (P-F) operator. P-F operator is used to study the evolution of the sets or the measure supported on the sets. For any given continuous mapping $T : X \to X$, linear P-F operator, denoted by $\mathbb{P}_T : \mathscr{M}(X) \to \mathscr{M}(X)$ is given by

$$\mathbb{P}_T[\mu](B) = \int_X \chi_B(T(x))d\mu(x) \tag{1}$$

where $\mathscr{M}(X)$ is the vector space of all measures supported on X, $\chi_B(x)$ is the indicator function supported on the set $B \subset \mathscr{B}(X)$, the Borel sigma-algebra of X. For more details on the P-F operator refer to [13]. Since the stability property of an invariant set in definition (1) is stated in terms of the transient behavior of the system on the complement of an invariant set A^c, we define sub-stochastic Markov operator as a restriction of the P-F operator on the complement of the invariant set as follows:

$$\mathbb{P}^1_T[\mu](B) := \int_{A^c} \chi_B(T(x))d\mu(x) \tag{2}$$

for any set $B \in \mathscr{B}(A^c)$ and $\mu \in \mathscr{M}(A^c)$. Necessary and sufficient condition for almost everywhere stability of an invariant set A with respect to finite non-negative measure m is obtained in the form of existence of the positive solution, *Lyapunov measure*, to the following *Lyapunov measure equation*:

$$\alpha\mathbb{P}^1_T\bar{\mu}(B) - \bar{\mu}(B) = -m(B) \tag{3}$$

where $\alpha \geq 1$ is a constant. The precise theorem for stability as proved in [12] is as follows:

Theorem 2: An invariant set A for the dynamical system $T : X \to X$ is almost everywhere stable with geometric stable with respect to some finite measure $m \in \mathscr{M}(A^c)$ if and only if there exists a non-negative measure $\bar{\mu}$ which is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\alpha\mathbb{P}^1_T\bar{\mu}(B) - \bar{\mu}(B) = -m(B)$$

for some $\alpha > 1$ and any set $B \subset X \setminus U_\delta$, where U_δ is the δ neighborhood of the invariant set A. Measure m is absolutely continuous with respect to measure $\bar{\mu}$.

Typically measure m in the Lyapunov measure equation (3) is taken to be Lebesgue measure. Stability of an invariant set with respect to Lebesgue almost every initial condition starting from a given set S can be studied by taking $m = m_S$ in the Lyapunov measure equation, where m_S is the Lebesgue measure supported on the set S.

Remark 3: In the subsequent section we use the notation m for the Lebesgue measure, m_S for the Lebesgue measure supported on set S and U_δ for the δ neighborhood of an invariant set A for a given $\delta > 0$.

Lyapunov measure as a solution of Lyapunov measure equation can also be used to characterize the unreachable sets in the phase space. The precise theorem in this direction is as follows.

Theorem 4: Let the invariant set A be almost everywhere stable with geometric decay with respect to measure m_S, where $S \subset X \setminus U_\delta$. Any set $D \subset X \setminus U_\delta$, s.t. $S \cap D = \emptyset$, is not reachable starting from almost every Lebesgue measure initial condition from set S i.e.,

$$m(D^n) = 0 \quad \forall n \geq 0 \quad \text{where} \quad D^n := \{x \in S : T^n(x) \in D\} \tag{4}$$

if and only if

$$\bar{\mu}_S(D) = 0$$

where $\bar{\mu}_S$ is the solution of following Lyapunov measure equation

$$\alpha\mathbb{P}_1\bar{\mu}_S(B) - \bar{\mu}_S(B) = -m_S(B)$$

for any set $B \subset X \setminus U_\delta$.

Proof: Refer to [12] for the proof. ■

Remark 5: In theorem (4), the condition of $D \cap S = \emptyset$ can be relaxed. This will correspond to the situation where the initial condition are allowed to start from the set D but no trajectory or almost every trajectories do not enter the set D. This situation can also be characterize in terms of Lyapunov measure [12].

In [9], set oriented numerical methods are used for the finite dimensional approximation of the Lyapunov measure $\bar{\mu}$. This finite dimensional approximation leads to further weaker notion of stability, which is referred to as coarse stability in [9]. Unlike almost everywhere stability, coarse stability of an invariant set allows for the existence of stable dynamics in the complement of an invariant set however the domain of attraction of the stable dynamics is strictly smaller than the size of the partition used in the finite dimensional approximation.

In [10], Lyaounov measure is used for the design of stabilizing feedback controller. For stabilization problem we consider the control dynamical system of the form

$$x_{n+1} = T(x_n, u_n)$$

where $x_n \in X$ and $u_n \in U$ is the state space and control space respectively. The objective is design feedback controller $u_n = K(x_n)$ to stabilize the invariant set A, which is assumed to be locally stable. The stabilization problem is solved using Lyapunov measure by extending the P-F operator formalism to the control dynamical system as follows. We define the feedback control mapping $C : X \to Y := X \times U$ as $C(x) = (x, K(x))$. Using the definition of feedback mapping C, we write the feedback control system as

$$x_{n+1} = T(x_n, K(x_n)) = T \circ C(x_n)$$

The system mapping $T : Y \to X$ and the control mapping $C : X \to Y$ can be associated with Perron-Frobenius operators

$\mathbb{P}_T : \mathscr{M}(Y) \to \mathscr{M}(X)$ and $\mathbb{P}_C : \mathscr{M}(X) \to \mathscr{M}(Y)$ respectively and are defined as follows

$$\mathbb{P}_T[\theta](B) = \int_Y \chi_B(T(y))d\theta(y)$$

$$\mathbb{P}_C[\mu](D) = \int_D f(a|x)dm(a)d\mu(x)$$

where $\theta \in \mathscr{M}(Y), \mu \in \mathscr{M}(X)$ and $B \subset X, D \subset Y$. $f(x|a)$ is the conditional probability density function and is introduced to incorporate the particular form of feedback controller mapping $C(x) = (x, K(x))$. The advantage of writing the feedback control dynamical system as the composition of two maps $T : Y \to X$ and $C : X \to Y$ is that the P-F operator for the composition $T \circ C : X \to X$ can be written as a product of $\mathbb{P}_T$ and $\mathbb{P}_C$ as follows:

$$\mathbb{P}_{T\circ C} = \mathbb{P}_T \cdot \mathbb{P}_C : \mathscr{M}(X) \to \mathscr{M}(X)$$

Refer to [10]. Just like Lyapunov measure is used for the stability analysis of an invariant set for the autonomous system. Control Lyapunov measure is introduced in [10] for the stabilization problem. Control Lyapunov measure is defined as any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}(B) < \beta \bar{\mu}(B) \tag{5}$$

for every set $B \subset X \setminus U_\delta$ and $\beta < 1$. $\mathbb{P}_T^1$ and $\mathbb{P}_C^1$ are the restriction of the P-F operator $\mathbb{P}_T$ and $\mathbb{P}_C$ to the complement of the invariant set A^c respectively and are defined similar to the restriction of the P-F operator in the autonomous case in equation (2). Stabilization of invariant set is posed as a co-design problem of jointly obtaining the control Lyapunov measure and the control P-F operator $\mathbb{P}_C$ or in particular the conditional probability density function $f(a|x)$. The co-design problem is formulated as an infinite dimensional linear program after suitable change of coordinates. Computational method based on set oriented numerical approach is proposed for the finite dimensional approximation of linear program in [10]. The finite dimensional approximation of the co-design problem reduces to solving finite number of linear inequalities.

III. Navigation measure for motion planning

In this section, we introduce *navigation measure* to solve the motion planning problem for a single vehicle in almost everywhere sense in the presence of static obstacles. We assume that the vehicle is modelled as a discrete time control dynamical system of the form

$$x_{n+1} = T(x_n, u_n) \tag{6}$$

where $x_n \in X$ represent the compact configuration space of the vehicle and $u_n \in U$ is the space of control input. The goal is to design the feedback control input $u_n = K(x_n)$ such that the final destination set A is asymptotically reached starting from the initial set S_i while avoiding the obstacle set S_o. Similar to the case of stabilization problem we define the control mapping $C : X \to Y := X \times U$ as $C(x, K(x))$. Using this definition of C, we define the feedback control system as

$$x_{n+1} = T \circ C(x_n)$$

Without loss of generality we assume that the final destination set A is invariant for the uncontrolled system i.e., $T(A, 0) = A$, moreover the control mapping C is designed such that $C(x) = (x, 0)$ for $x \in A$. Now we state the definition of almost everywhere motion planning.

Definition 6 (Almost everywhere motion planning): The feedback control system $x_{n+1} = T(x_n, K(x_n))$ is said to solve the motion planning problem in almost everywhere sense i.e., steer almost every with respect to Lebesgue measure initial condition from the given initial set $S_i \subset X \setminus U_\delta$ to the final destination set A, while avoiding the collision with the obstacle set S_o if there exists a feedback controller map $C : X \to Y$, where $C(x) = (x, K(x))$, such that following two conditions are satisfied

1) there exists an $K(\delta) < \infty$ and $\beta < 1$ such that
$$m\{x \in S_i : (T \circ C)^n(x) \in B\} < K\beta^n$$
for every set $B \subset X \setminus U_\delta$.
2) $m(S_o^n) = 0 \quad \forall n \geq 0$
where $S_o^n := \{x \in S_i : (T \circ C)^n(x) \in S_o\}$

Condition 1. of the theorem guarantee that the final destination set is almost everywhere stable with geometric decay with respect to measure m_{S_i} and the condition 2. of the definition ensures that almost every trajectory starting from the initial set S_i can be steered to the final destination set A while avoiding the collision with the obstacle set D. The framework that we use to solve the motion planning problem in almost everywhere sense (definition 6) is similar to the stabilization problem using Lyapunov measure discussed in previous section. We once again consider the restriction of the P-F operator $\mathbb{P}_T^1 : \mathscr{M}(A^c \times U) \to \mathscr{M}(X)$ corresponding to the dynamical system T and $\mathbb{P}_C^1 : \mathscr{M}(A^c) \to \mathscr{M}(A^c \times U)$ corresponding to the controller mapping C defined as follows:

$$[\mathbb{P}_T^1 \theta](B) = \int_{A^c \times U} \chi_B(T(y))d\theta(y) \tag{7}$$

$$[\mathbb{P}_C^1 \mu](D) = \int_D f(x|a)dm(a)d\mu(x) \tag{8}$$

where $\theta \in \mathscr{M}(A^c \times U), \mu \in \mathscr{M}(A^c)$, $B \subset A^c$, $D \subset A^c \times U$ and $f(x|a)$ is the conditional probability density function. The multiplication of $\mathbb{P}_T^1$ and $\mathbb{P}_C^1$ is a well defined operator corresponding to the feedback control dynamical system $T \circ C : A^c \times U \to X$ and is given by

$$\mathbb{P}_{T\circ C}^1 = \mathbb{P}_T^1 \cdot \mathbb{P}_C^1$$

For the proof of this refer to [10]. Note that for any set $B \subset A^c$

$$\begin{aligned}\mathbb{P}_{T\circ C}^1 m(B) = \textstyle\int_{A^c} \chi_B(T \circ C(x))dm(x) &= m((T \circ C)^{-1}(B) \cap A^c) \\ &= m\{x \in A^c : T \circ C \in B\}\end{aligned}$$

and similarly, we have

$$(\mathbb{P}^1_{T\circ C})^n m(B) = m\{x \in A^c : (T\circ C)^n(x) \in B\} \tag{9}$$

Solution to the almost everywhere motion planning problem depends upon the existence of navigation measure which is defined as follows

Definition 7 (Navigation measure): Let A be the final destination set and $S_i, S_o \subset X \setminus U_\delta$ be the initial and obstacle set respectively s.t. $S_i \cap S_o = \emptyset$. For a given controller mapping $C(x) = (x, K(x))$, navigation measure is defined as any non-negative measure μ_{S_i}, which is finite on $X \setminus U_\delta$ and satisfies following two equations.

1)

$$\alpha \mathbb{P}^1_T \cdot \mathbb{P}^1_C \bar{\mu}_{S_i}(B) - \bar{\mu}_{S_i}(B) = -m_{S_i}(B) \tag{10}$$

for any set $B \subset X \setminus U_\delta$ and some $\alpha > 1$.

2)

$$\bar{\mu}_{S_i}(S_o) = 0 \tag{11}$$

Now we state the main theorem of this paper providing sufficient condition for the almost everywhere motion planning problem as defined in (6) in terms of existence of navigation measure.

Theorem 8: Almost everywhere motion planning problem (Definition 6) is solvable if there exists control mapping $C : X \to Y$, $C(x) = (x, K(x))$, and a navigation measure $\bar{\mu}_{S_i}$.

Proof: From the definition of navigation measure, we know that there exists the control mapping $C(x) = (x, K(x))$ and the navigation measure $\bar{\mu}_{S_i}$, such that

$$\bar{\mu}_{S_i}(B) = \alpha \mathbb{P}^1_T \cdot \mathbb{P}^1_C \bar{\mu}_{S_i}(B) + m_{S_i}(B) \tag{12}$$

Multiplying both the sides by $\alpha \mathbb{P}^1_T \cdot \mathbb{P}^1_C$, using (12)and after rearranging we get

$$\bar{\mu}_{S_i}(B) = \alpha^2 (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^2 \bar{\mu}_{S_i}(B) + \mathbb{P}^1_T \cdot \mathbb{P}^1_C m_{S_i}(B) + m_{S_i}(B)$$

Now multiplying both the sides by $\alpha^n (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k$ for $k = 1, 2, ..., n$ and using induction we get

$$\bar{\mu}_{S_i}(B) = \sum_{k=0}^{n} \alpha^k (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k m_{S_i}(B) + \alpha^n (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n \bar{\mu}_{S_i}(B) \tag{13}$$

Since the navigation measure $\bar{\mu}_{S_i}$ is finite on $X \setminus U_\delta$ and $\alpha^n (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n \bar{\mu}_{S_i}(B) \geq 0$ for all n and any set $B \subset X \setminus U_\delta$, we have

$$\begin{aligned}\sum_{k=0}^{n} \alpha^k (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k m_{S_i}(B) &\leq \sum_{k=0}^{n} \alpha^k (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k m_{S_i}(B) \\ &+ \alpha^n (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n \bar{\mu}_{S_i}(B) = \bar{\mu}_{S_i}(B)\end{aligned}$$

and hence

$$\sum_{k=0}^{n} \alpha^k (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k m_{S_i}(B) \leq \bar{\mu}_{S_i}(B) < K(\delta) \quad \forall n \geq 0$$

and

$$(\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n m_{S_i}(B) < \beta^n K(\delta)$$

where $\beta = \frac{1}{\alpha} < 1$.

Now using (9), we have

$$\begin{aligned}(\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n m_{S_i}(B) &= (\mathbb{P}^1_{T\circ C})^n m_{S_i}(B) \\ &= m_{S_i}\{x \in A^c : (T\circ C)^n(x) \in B\}\end{aligned}$$

and since $S_i \subset X \setminus U_\delta \subset A^c$, we have

$$m_{S_i}\{x \in A^c : (T\circ C)^n(x) \in B\} = m\{x \in S_i : (T\circ C)^n(x) \in B\}$$

so we get

$$m\{x \in S_i : (T\circ C)^n(x) \in B\} < \beta^n K$$

for any set $B \subset X \setminus U_\delta$ and for all $n \geq 0$. This proves the first condition of the motion planning definition. The second condition of the definition (6) follows from (13) and the property of navigation measure as follows:

$$\bar{\mu}_{S_i}(S_o) = \sum_{k=0}^{n} (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^k m_{S_i}(S_o) + (\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n \bar{\mu}_{S_i}(S_o) = 0$$

for all n. Since each of the individual terms and the terms in the summation are non-negative, we have

$$(\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n m_{S_i}(S_o) = 0$$

for all $n \geq 0$ and since

$$(\mathbb{P}^1_T \cdot \mathbb{P}^1_C)^n m_{S_i}(S_o) = m\{x \in S_i : (T\circ C)^n(x) \in S_0\}$$

we have

$$m(S_o^n) = 0 \quad \forall n \geq 0$$

This proves the theorem. ■

So the solution to the motion planning problem lies in the co-design of the navigation measure $\bar{\mu}_{S_i}$ and the controller mapping $C : X \to Y$. In the next section we shown that the finite dimensional approximation of the co-design problem can be posed as a linear program and solved using set oriented numerical methods.

IV. Computation of Navigation measure

The procedure for the computation of navigation measure for the motion planning problem closely follows the computation of control Lyapunov measure for the stabilization problem as discussed in [10]. We consider a finite partition of $\mathscr{X}$ and $\mathscr{U}$ of the state space X and control space $\mathscr{U}$ respectively as follows

$$\mathscr{X} = \{D_1, ..., D_L\} \tag{14}$$

$$\mathscr{U} = \{U_1, ..., U_M\} \tag{15}$$

The partition of the product space $Y := X \times U$ is $\mathscr{X} \times \mathscr{U}$. These finite partition can be used to identify the measure space $\mathscr{M}(X)$ and $\mathscr{M}(Y)$ with the finite dimensional vector space $\mathbb{R}^L$ and $\mathbb{R}^{LM}$ respectively. We denote by $P_T : \mathbb{R}^{LM} \to \mathbb{R}^L$, and $P_C : \mathbb{R}^L \to \mathbb{R}^{LM}$ the finite dimensional approximation of the P-F operator $\mathbb{P}_T$ and $\mathbb{P}_C$ respectively. For more details on the finite dimensional approximation of P-F operators refer to [11]. Without loss of generality, we assume that

$$\mathscr{X}_0 := \{D_{N+1}, \cdots, D_L\} \tag{16}$$

$$\mathscr{X}_{11} := \{D_1, \cdots, D_K\} \tag{17}$$

$$\mathscr{X}_{12} := \{D_{K+1}, \cdots, D_N\} \tag{18}$$

such that $X_0 = \cup_{i=N+1}^{L} D_i \supset A$ contains the invariant set A and is assumed to be invariant. The assumption of X_0 being invariant can be satisfied by constructing fine enough partition around A if A is locally stable or by locally stabilizing the invariant set A if it is unstable to begin with. Obstacle set S_o is assumed to be contained in the partition $X_{12} = \cup_{i=K+1}^{N} D_i \supset S_o$, the initial set $S_i \subset X_{11} = \cup_{i=1}^{K} D_i$ and $X = X_0 \cup X_1$, with $X_1 = X_{11} \cup X_{12}$. Using the definition of finite partition and finite dimensional approximation of the P-F operators, the problem of constructing the finite dimensional approximation of the navigation measure can be posed as the co-design problem of jointly obtaining the Markov matrix P_C and the measure $\bar{\mu}$ such that

$$\alpha \bar{\mu} P_C \cdot P_T[1:N] - \bar{\mu} = -\bar{m} \tag{19}$$

$$\bar{\mu}_i = 0 \quad \text{for} \quad i = K+1, \ldots, N \tag{20}$$

for some $\alpha > 1$, where $\bar{m}$ is the row vector with support contained inside X_1 and is such that $\bar{m}_i = 0$ for $i = K+1, \ldots, N$. Equations (19) and (20) are the finite dimensional counterparts of equations (10) and (11) in the definition of navigation measure. The structure of P_C as it is too general since we know that control mapping C is of the form $C(x) = (x, K(x))$, this special structure of C can be incorporated as follows. For each fixed value of control $u_a \in \mathscr{U}$, we denote $P^a : \mathbb{R}^L \to \mathbb{R}^L$ to be the Markov matrix for the map $T(\cdot, u = u_a)$. In particular, P^a are sub-matrices of P_T. Next, define

$$Q_{ia} = \text{Prob}(u_n = u_a | x_n \in D_i) \text{ for } a \in [1, \ldots, M]. \tag{21}$$

to be the probability of choosing the a^{th} control value conditioned on state being in cell D_j. Q is the discrete counterpart of conditional distribution $f(a|x)$ in Eq. (8). We note that Q_{ia} describe all of the non-zeros entries of P_C for control maps of the form $C(x) = (x, K(x))$. Using this definition of Q equation (19) can be written as

$$\alpha \sum_{a=1}^{M} \sum_{i=1}^{N} \mu_i Q_{ia} P_{ij}^a - \mu_j = \bar{m}_j \text{ for } j = 1, \ldots, N. \tag{22}$$

Finite dimensional solution of the motion planning problem then involves co-designing the Markov matrix $Q : \mathbb{R}^N \to \mathbb{R}^M$ along with the navigation measure $\bar{\mu}$ such that equations (22) and (20) are satisfied. This problem is solved by introducing the change of coordinates as follows

$$R_{ia} := \mu_i Q_{ia} \quad \text{for} \quad i \in [1, \ldots, N], \quad a \in [1, \ldots, M]. \tag{23}$$

By virtue of the fact that Q is a Markov matrix, we have

$$\mu_j = \sum_a R_{ja}. \tag{24}$$

The equations in the new coordinates can be written

$$\alpha \textstyle\sum_{i,a} R_{ia} P_{ij}^a - \sum_a R_{ja} = -\bar{m}_j, \quad \text{for} \quad j \in [1, \ldots, N]. \tag{25}$$

$$R_{ia} = 0 \quad \text{for} \quad i = K+1, \ldots, N \quad \text{for } \forall a. \tag{26}$$

$$R_{ia} \geq 0. \tag{27}$$

The equations (25)-(26) thus represent a system of linear equations with linear inequality constraint (27) in unknowns R_{ia}. A feasible solution for this can be obtained using linear programming. From any admissible solution to the linear program, the navigation measure and control is easily obtained as

$$\mu_i = \sum_a R_{ia}, \tag{28}$$

$$Q_{ia} = \frac{R_{ia}}{\mu_i} \quad i = 1, \ldots, K. \tag{29}$$

Note that since $\mu_i = 0$ corresponding to the obstacle set i.e., $\mu_i = 0$ for $i = K+1, \ldots, N$, Q_{ia} can take any values for $i = K+1, \ldots, N$ as long as $Q_{ia} \geq 0$ and is row stochastic $\sum_a Q_{ia} = 1$. This arbitrariness in the entries of Q is because of our assumption that $S_i \cap S_o = \emptyset$ or in other words $\bar{m}_i = 0$ for $i = K+1, \ldots, N$ i.e., the system never starts with initial condition in the obstacle set. We note that the control Markov matrix Q is stochastic in general. In particular, solution to Eq. (29) in general leads to $Q_{ia} \in [0,1]$. To obtain deterministic controller i.e., the case when $Q_{ia} \in \{0,1\}$ impose integer constraints on the entries of Q, which requires solving a mixed integer linear programming problem; details of which are discussed in [10]. Existence of the stochastic matrix Q and the finite dimensional navigation measure μ to solve the motion planning problem depends upon the feasibility of linear program (25)-(26)-(27). Establishing conditions for the feasibility of the linear program will be addressed in our future publication. In the next section we present simulation result for the motion planning problem.

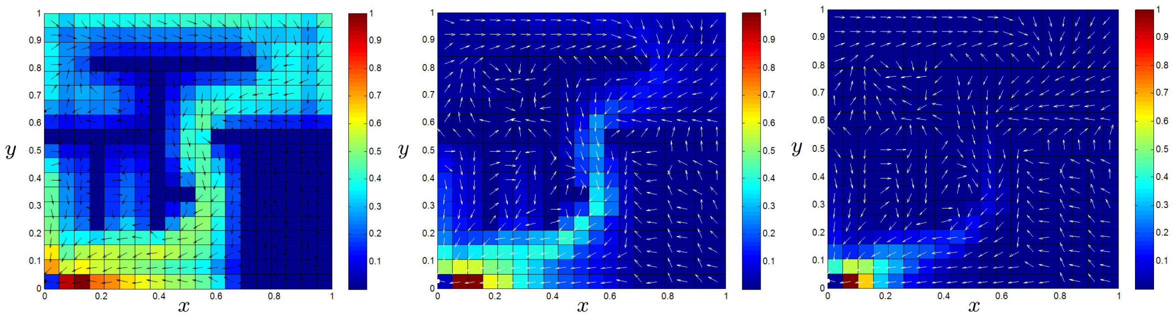
A. Simulation results

In this section we apply the computation framework developed in the previous section for the motion planning problems to a simplified Dubin's car like model where the motion of a vehicle in (x,y) plane is described by the following kinematics,

$$\begin{aligned} \dot{x} &= V_0 \cos(\theta) \\ \dot{y} &= V_0 \sin(\theta). \end{aligned} \tag{30}$$

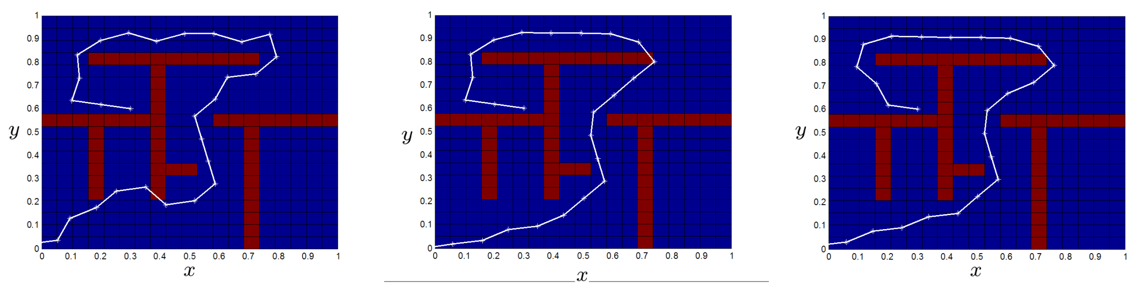
In the simulation results that follow, we have assumed $V_0 = 1$ and $\theta(t)$ is the control variable, which defines the direction of the velocity vector. The control space has been discretized as $\theta_i = \frac{2\pi i}{20}, i = 0, \cdots, 19$.. The continuous time dynamics in eqn.(30) was discretized with time step $dt = 0.1$, for every θ_i.

Figures 1(a), 1(b) and 1(c) illustrates the navigation measure for $\alpha = 1.02, 1.11, 1.42$ respectively. Role of α is to control the rate of convergence, larger the value of α faster the convergence to the final destination set. The color map represents the magnitude of the navigation measure on each cell. Notice that the navigation measure is maximum at or near the final destination set and is zero on the obstacle sets. The trajectories of the corresponding closed-loop systems, from initial condition $x_0 = 0.3$ and $y_0 = 0.6$, are shown in figures 2(a), 2(b) and 2(c); with the time history of the discrete (x,y) points are highlighted with '$*$'. Since the control is stochastic in this framework, the control that was applied to the system at every time step is the θ_i with



(a) $\alpha = 1.02$ (b) $\alpha = 1.11$ (c) $\alpha = 1.42$

Fig. 1. Plot of navigation measure for different values of α



(a) $\alpha = 1.02$ (b) $\alpha = 1.11$ (c) $\alpha = 1.42$

Fig. 2. Trajectories of the closed loop system.

maximum probability. The objective of the control is to drive the system to (0,0). The state feedback control law $\theta(x,y)$ is the direction of the velocity vector at each cell as shown in fig.1. In this framework, the control law is identical for all points in a given cell. Obstacles in fig.2 are shown in red. Observe that the navigation measure is zero on the obstacle sets fig.1, implying that the obstacle sets are avoided.

V. Conclusion

In this paper we have introduced navigation measure as a new tool to solve the motion planning problem in the presence of static obstacles. Motion planning problem is cast as a co-design problem of jointly obtaining navigation measure and the feedback controller. This co-design problem is posed as a linear program. Computational framework based on set oriented numerical methods is proposed for the finite dimensional approximation of the linear program. Future research efforts will focus on application of this framework to motion planning in higher dimension configuration space and extension of navigation measure to the motion planning problems with moving obstacles.

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