2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Peaking Free Output-Feedback Control of Uncertain Nonlinear Systems

Tiago Roux Oliveira, Alessandro Jacoud Peixoto and Liu Hsu

Abstract— **Output-feedback sliding mode controllers are designed for single-input-single-output (SISO) uncertain nonlinear plants with state dependent nonlinearities. The sliding surface is generated using the states of a high gain observer (HGO) and the control law is *peaking free*. The control signal amplitude is either obtained from a simple norm observer or from the HGO states in conjunction with a dwell-time strategy, depending on the nonlinearity growth condition. Global or semi-global exponential stability with respect to a small residual set is proved without requiring the control to be *a priori* globally bounded. The advantages are better transient behavior and improved domain of stability.**

Keywords: uncertain nonlinear systems, output-feedback, sliding mode control, high gain observer, peaking avoidance.

I. Introduction

The design of output-feedback control of uncertain systems with strong nonlinearities has been a challenging problem in the recent years. So far, globally stable closed-loop systems have been designed only for a restricted class of nonlinear plants and it is well known that some (quite simple) plants with polynomial nonlinearities cannot be globally stabilized by output-feedback [1].

In general, to solve the problem, some estimate of the plant state, or at least of the state norm, is necessary. In this respect, high-gain observers have been utilized owing to their robustness to plant uncertainties and arbitrarily small estimation error. However, the price to be paid is the generation of peaking which may potentially lead to either bad transient or even instability when the peaking signal is transmitted to the plant [2].

Oh and Khalil [3], [4] proposed a globally bounded control (GBC), which amounts essentially to saturating the control signal, in order to circumvent the deleterious effects of the peaking phenomena. In particular, this strategy was applied for output-feedback sliding mode control in which the switching (sliding) surface was given in terms of the HGO estimated states. However, the GBC does not guarantee global stability, for example, with open loop unstable plants. In order to increase the stability domain, the control saturation level has to be increased. This in turn can result in unacceptable transients since higher peaking signals are then transmitted to the plant.

In this paper, we propose two alternative control strategies which avoid the peaking effect according to the class of nonlinear systems to be controlled. In the simpler case of linear growth condition *w.r.t.* the unmeasured states, we obtain state norm estimates via non high-gain first order filters, similar to norm observers [5]. In the general case, such norm observers are not trivial to design since their existence is essentially tied to the knowledge of an Input-Output-to-State Stable (IOSS) Lyapunov function [5]. Thus, to include strong (e.g., polynomial) nonlinearities, the norm estimates require high gain. Then, peaking is avoided by introducing a dwell-time in the controller activation [6], [7].

Both peaking free strategies lead to quite satisfactory transient behavior. Furthermore, since GBC is not required as in [3], [4], larger domains of stability are expected. Semi-global practical stability is achieved with respect to a single parameter instead of two parameters, i.e., saturation level *and* observer high-gain as in [3], [4]. The residual set is smaller than in previous works *loc. cit.*. An intrinsic requirement of the GBC approach is the knowledge of some "domain of interest" or an estimate of the region of attraction to determine the controller parameters. In our schemes, no such explicit knowledge about domains or regions is necessary in order to stabilize the system and perform output tracking. Simulations illustrate their effectiveness.

Notation and Definitions: The Euclidean norm of a vector x and the corresponding induced norm of a matrix A are denoted by $|x|$ and $|A|$, respectively. The $\mathcal{L}_{\infty e}$ norm of signal $x(t) \in \mathbb{R}^n$, is defined as $\|x_{t,t_0}\| := \sup_{t_0 \le \tau \le t} |x(\tau)|$; for $t_0 = 0$, $\|x_t\|$ is adopted. The symbol "s" represents either the Laplace variable or the differential operator "d/dt", according to the context. The output of a linear system with transfer function $H(s)$ and input u is written $H(s)u$. Classes $\mathcal{K}$, $\mathcal{K}_\infty$ functions are defined as usual [8, pp. 144]. ISS and ISpS mean Input-to-State-Stable (or Stability) and Input-to-State-Practical-Stable (or Stability), respectively [9]. Filippov's definition for the solution of discontinuous differential equations [10] and the concept of *extended equivalent control* [11] are used throughout the paper.

II. Problem Formulation

Consider a SISO nonlinear uncertain plant described by:

$$\dot{x} = Ax + \phi(x,t) + Bu\,, \quad y = Cx\,, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output and $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a state dependent uncertain nonlinear disturbance, possibly unmatched. When no particular growth condition is imposed on ϕ, finite-time escape is not precluded *a priori* and for each solution of (1) a maximal time interval of definition is $[t_0, t_M)$, where t_0 is the initial time and t_M may be finite or infinite.

This work was supported in part by CNPq, FAPERJ and CAPES.

T. R. Oliveira, A. J. Peixoto and L. Hsu are with the Dept. of Electrical Eng./COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil. [tiagoroux,jacoud,liu]@coep.ufrj.br

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

A. Basic Assumptions

Without loss of generality, $t_0 = 0$ is the initial time. The triple (A, B, C) is assumed already in canonical controllable form with uncertain constant matrices A (lower companion form) and $C = [c_1 \ c_2 \ldots c_{(n-\rho+1)} \ 0 \ldots 0]$. Note that $c_{(n-\rho+1)} = CA^{\rho-1}B$ is the high frequency gain (HFG) of the subsystem (A, B, C) [8, p.512].

All uncertain parameters belong to some compact set Ω_p such that the necessary uncertainty bounds are available for design. In Ω_p we assume that: (i) ϕ is locally Lipschitz in x ($\forall x$), piecewise continuous in t ($\forall t$) and sufficiently smooth; (ii) (A, B, C) represents a linear plant which is minimum-phase, observable, of known order n, relative degree ρ and known HFG sign, as usual in Model Reference Adaptive Control (MRAC) [12]. Our main additional assumptions are:

(A1) There exists a global diffeomorphism $(\bar{x}, t) = T(x, t)$, $\bar{x}^T := [\eta^T \ \xi^T], \eta \in \mathbb{R}^{n-\rho}$, which transforms (1) to the *normal form* [8], with $\xi = [y \ \dot{y} \ \ldots y^{(\rho-1)}]^T$ and

$$\dot{\eta} = A_0\eta + \phi_0(x, t)\,, \qquad \dot{\xi} = A_r\xi + B_r k_p[u + d_\phi(x, t)]\,,$$

where $y = C_r\xi$, $k_p := CA^{\rho-1}B$ is the *constant* plant HFG and (A_r, B_r, C_r) is in the Brunovsky's controller form. In the η-dynamics: A_0 is Hurwitz and $|\phi_0| \leq \varphi_0(|\xi|, t)$, with φ_0 being a *known* non-negative function, piecewise continuous in t and $\mathcal{K}$ in $|\xi|$.

According to (A1), the nonlinear plant (1) is minimum phase and has strong relative degree ρ [13]. To obtain norm bounds for the matched disturbance $d_\phi(x, t)$, we further assume that:

(A2) There exist known locally Lipschitz functions $\varphi_{T1}, \varphi_{T2} \in \mathcal{K}_\infty$ and constants $k_{T1}, k_{T2} \geq 0$ such that $|\bar{x}| \leq \varphi_{T1}(|x|) + k_{T1}$ and $|x| \leq \varphi_{T2}(|\bar{x}|) + k_{T2}$.

(A3) There exists a known non-negative function $\varphi_d(|x|, t)$, piecewise continuous in t and $\mathcal{K}_\infty$ in $|x|$ such that $|d_\phi(x, t)| \leq \varphi_d(|x|, t)$.

Note that (A2)–(A3) are not restrictive since T, T^{-1} and d_ϕ are continuous in its arguments. Moreover, no particular growth condition is imposed on the bounding functions $\varphi_{T1}, \varphi_{T2}$ and φ_d.

B. Control Objective

The aim is, by output-feedback, to achieve global or semi-global stability properties in the sense of uniform signal boundedness and asymptotic output tracking, i.e., the *output tracking error*

$$e(t) = y(t) - y_m(t) \tag{2}$$

should tend to zero or to some small residual values.

The *desired trajectory* $y_m(t)$ is generated by the following *reference model*:

$$y_m = M(s)r = \frac{k_m}{L(s)(s + a_m)}\, r\,, \quad k_m, a_m > 0\,, \tag{3}$$

where $r(t)$ is assumed piecewise continuous, uniformly bounded and the Hurwitz polynomial $L(s)$ is given by

$$L(s) := s^{\rho-1} + a_{\rho-2}s^{\rho-2} + \ldots + a_0\,. \tag{4}$$

C. Output Error Equation

Let the minimal realization of $M(s)$ in (3) be given by:

$$\dot{\xi}_m = A_m\xi_m + B_m k_m r\,, \quad y_m = C_m\xi_m\,, \tag{5}$$

where $\xi_m^T := [\, y_m \ \ \dot{y}_m \ \ \ldots \ \ y_m^{(\rho-1)}\,]$, $B_m := B_r$, $C_m := C_r$ and $A_m := A_r + B_r K_m$, with K_m obtained from the coefficients of the characteristic polynomial of $M(s)$.

Now, consider the ξ-dynamics of the plant in (A1). Replacing u by $u + K_m\xi/k_p - K_m\xi/k_p$, we obtain:

$$\dot{\xi} = A_m\xi + B_m k_p[u - K_m\xi/k_p + d_\phi]\,, \quad y = C_m\xi\,. \tag{6}$$

From (5) and (6), the state tracking error $x_e := \xi - \xi_m$ and the output tracking error e satisfy

$$\dot{x}_e = A_m x_e + k_p B_m[u + d]\,, \quad e = C_m x_e\,, \tag{7}$$

$$e = k^* M(s)[u + d]\,, \quad k^* = k_p/k_m\,, \tag{8}$$

where the *equivalent input disturbance* is defined by[1]

$$d(x, t) := -K_m\xi/k_p + d_\phi(x, t) - r/k^*\,. \tag{9}$$

III. Output-Feedback Sliding Mode Control

In this section, we describe the output-feedback sliding mode control approach, particulary the *sliding surface* and *modulation function* designs.

Sliding surface: when only y is available for feedback, the sliding surface is chosen as

$$\hat{\sigma} := S\hat{x}_e = 0\,, \quad S := [\ a_0 \ \ \ldots \ \ a_{\rho-2} \ \ 1\ ]\,, \tag{10}$$

with $a_0, \ldots, a_{\rho-2}$ defined in (4), $\hat{x}_e := \hat{\xi} - \xi_m$ and $\hat{\xi}$ being an estimate of ξ provided by an HGO due to its robustness to disturbances and parametric uncertainties.

Modulation (or control gain) function: following the state feedback design described in [14], the control law u can be defined by

$$u = -[\mathrm{sgn}(k_p)]\varrho(\chi, t)\mathrm{sgn}(\hat{\sigma}(t))\,, \tag{11}$$

where $\chi(t)$ is a scalar non-negative absolutely continuous function, obtained from available signals, which upper bounds the plant state norm $|x|$, *modulo* exponentially decaying terms. It would be desirable to obtain a *peaking free norm bound* χ such that the inequality (see details in [14])

$$\varrho(\chi, t) \geq |d(x, t)| + \delta\,, \tag{12}$$

holds, *modulo* exponentially decaying terms, where $\delta > 0$ is an arbitrarily small constant. This inequality, alone, is not sufficient to achieve global or semi-global tracking due to the disturbances caused by the inexact HGO estimation. Let the *estimation error* be defined by

$$\tilde{x}_e := x_e - \hat{x}_e = \xi - \hat{\xi}\,. \tag{13}$$

Now, setting $\sigma = Sx_e$, one can obtain from (4), (7), (10) and (13), the relationships $\hat{\sigma} = \sigma - S\tilde{x}_e$ and

$$\hat{\sigma} = k^* ML[u + d] - S\tilde{x}_e\,. \tag{14}$$

[1] To conclude that d is a function of x and t, we have considered the diffeomorphism $(\bar{x}, t) = T(x, t)$ introduced in (A1).

As shown in [14], (12) and (14) will imply only an ISS property from $\tilde{x}_e$ to x_e. Furthermore, the estimate $\hat{\xi}$ provided by an HGO has an ISpS property from x_e to $\tilde{x}_e$, with ISpS-gain given by the HGO small parameter μ. Thus, combining the above ISS properties, global or semi-global tracking can be proved through a small-gain analysis.

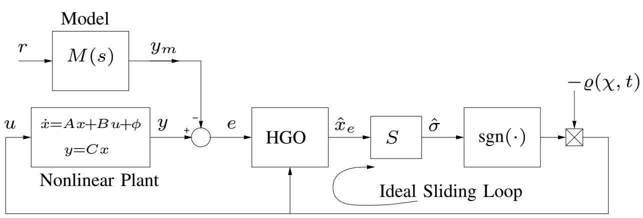

Fig. 1. Output-feedback sliding mode controller.

The proposed scheme is depicted in Fig. 1. An eventual peaking [2] in $\hat{\sigma}$ is blocked by the $\mathrm{sgn}(\cdot)$ function in (11) and the control signal u is peaking free since χ is implemented using only peaking free available signals.

In the following, we give a detailed description of the proposed controllers, stressing the HGO design and introducing two different strategies to obtain χ for classes of nonlinear systems with or without growth restriction *w.r.t.* the unmeasured states.

IV. High Gain Observer

An estimate $\hat{\xi}$ for ξ in (A1) is provided by the HGO:

$$\dot{\hat{\xi}} = A_r\hat{\xi} + k_p^{\mathrm{nom}} B_r u + H_\mu L_o C_r(\xi - \hat{\xi}) \,, \tag{15}$$

where L_o and H_μ are given by

$$L_o := [\; l_1 \quad \ldots \quad l_\rho \;]^T \quad , \quad H_\mu := \mathrm{diag}(\mu^{-1}, \ldots, \mu^{-\rho}) \tag{16}$$

and k_p^{nom} is a nominal value for k_p. The observer gain L_o is such that $N(s) = s^\rho + l_1 s^{\rho-1} + \ldots + l_\rho$ is Hurwitz.

Since it is desirable that the uncertainties and disturbances have negligible effects in $\hat{x}_e$ (10), the norm of H_μ should be large, which implies that μ should be small.

A. High Gain Observer Error Dynamics

As in [4], the following transformation is applied to (13)

$$\zeta := T_\mu \tilde{x}_e \,, \quad T_\mu := [\mu^\rho H_\mu]^{-1} \,, \tag{17}$$

which leads to: (i) $T_\mu(A_r - H_\mu L_o C_r)T_\mu^{-1} = \frac{1}{\mu}A_o$ and (ii) $T_\mu B_r = B_r$, where $A_o := A_r - L_o C_r$. Thus, from the ξ-dynamics in (A1), (13), (15) and (17), one has

$$\mu\dot{\zeta} = A_o\zeta + k_p B_\rho[\mu\nu] \,, \tag{18}$$

with

$$\nu := [\kappa u + d_\phi] \quad \text{and} \quad \kappa = (k_p - k_p^{\mathrm{nom}})/k_p \,. \tag{19}$$

B. Peaking Phenomenon

As is well known, HGO estimates may contain peaking [2]. Indeed, the estimation error $\tilde{x}_e$ will contain a transient term of the form $(a/\mu^b)e^{-ct/\mu}$, for some $a, b, c > 0$. Thus, these terms eventually exhibit an impulsive-like transient behavior, as $\mu \to 0$, where the transient peaks to $\mathcal{O}(1/\mu)$ values before it decays rapidly to zero. This behavior is known as the *peaking phenomenon* [8], [2].

However, the peaking phenomenon can be circumvented by using the *peak extinction time* (t_e) concept, where t_e is defined as the solution of $(a/\mu^b)e^{-ct_e/\mu} = 1$, for each value of $\mu \in (0, 1]$. Note that t_e is a function of μ, which satisfies $t_e(\mu) \le \bar{t}_e(\mu)$, with $\bar{t}_e(\mu) \in \mathcal{K}$ (see [15] and *ref.* therein).

When no growth conditions are imposed on the unmeasured states (Section VI), this concept will be crucial in the design of a peaking free control signal.

V. Systems with Restricted Growth Condition

The class of nonlinear systems affinely bounded in the unmeasured states considered here allow us to implement χ and the modulation function $\varrho(\chi, t)$ using signals from *first order state filters* similar to *norm observers* [5]. An HGO is employed *only* to generate the switching law (10).

A. Linear Growth Condition

In this section, we assume that:

(A4) The term ϕ is norm bounded by $|\phi(x,t)| \le k_x|x| + \varphi(y,t)$, $\forall x, t$, where $k_x \ge 0$ is a *known* scalar and $\varphi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a known function piecewise continuous in t and continuous in y, and $\varphi(y,t) \le \Psi_\varphi(|y|) + k_\varphi$, where $\Psi_\varphi \in \mathcal{K}_\infty$ is locally Lipschitz and $k_\varphi > 0$ is a constant.

According to (A4), no particular growth condition such as linear growth or existence of a global Lipschitz constant is imposed on φ. For instance, (A1) and (A4) are satisfied by systems (1) with ϕ triangular in the unmeasured states.

B. State Filters (Norm Observers)

Following the usual MRAC approach [12] and the development described in [16], considering (A4) and using [16, Lemma 3], it is possible to find $k_x^* > 0$ such that, for $k_x \in [0, k_x^*]$, a norm bound for x can be obtained through *first order approximation filters* (FOAFs) (see details in [16]). Indeed, one can obtain $|x| \le \chi + \pi$ and

$$\chi(t) := \frac{1}{s + \lambda_x}[\bar{c}_1\varphi(|y(t)|, t) + \bar{c}_2|\omega(t)|] \,, \tag{20}$$

where the exponentially decaying term π accounts for initial conditions, $\bar{c}_1, \bar{c}_2, \lambda_x > 0$ are appropriate constants, $\omega = [\omega_1^T \; \omega_2^T \; y \; r]^T$ is the *regressor vector* [12] and ω_1, ω_2 are the states of the input/output filters:

$$\dot{\omega}_1 = \Lambda\omega_1 + gu \,, \qquad \dot{\omega}_2 = \Lambda\omega_2 + gy \,. \tag{21}$$

The matrix $\Lambda \in \mathbb{R}^{(n-1)\times(n-1)}$ is Hurwitz and g is a constant vector such that (Λ, g) is controllable. Such filters are needed due to the *lack of full state measurement* of the uncertain plant and *replace a state observer*.

To compute all the FOAF's parameters involved in (20), i.e., $\bar{c}_1, \bar{c}_2, \lambda_x$, one can use a simple technique based on Lyapunov quadratic forms, see [16].

From (A4) and (20), one has $|\phi(x,t)| \leq k_x \chi(t) + \varphi(y,t)$, *modulo* π term. Thus, from (A2)–(A3) and (9), one can write $|d| \leq \hat{d} + \hat{\pi}$, where $\hat{\pi}$ is a decaying term and

$$\hat{d}(t) := \hat{\varphi}(|\chi(t)|) + c_r \,, \tag{22}$$

with an appropriate constant $c_r > 0$ and $\hat{\varphi} \in \mathcal{K}$. Hence, one possible choice for ϱ satisfying (12) is

$$\varrho(\chi, t) := \hat{d}(t) + \delta \,, \tag{23}$$

with $\hat{d}(t)$ in (22) and $\delta > 0$ being an arbitrary small constant.

VI. General Nonlinear Systems

In this section, we consider a wider class of nonlinear systems without any type of growth condition imposed on ϕ. In this case, the foregoing approach can not be applied and the upper bound (20) is not valid. Thus, we will use HGO state not only to define the sliding surface but also use the HGO estimates (with peaking) to design χ and appropriate modulation function $\varrho(\chi, t)$ as in [3], [4].

In [3], [4], a globally bounded sliding mode control law is applied by using saturation functions in order to avoid the *peaking phenomena*. As a consequence, the region of interest of the control effort must be first estimated in order to tune the saturation level and guarantee semi-global stability. *One major problem in this approach is that,* to enlarge the domain of stability, it is necessary to increase the level of the saturation function. Then, considerable peaking energy is transmitted to the plant, which leads to large transients particularly for smaller initial conditions since the control signal would still preserve a large peaking, only bounded by saturation level. This can be observed from the simulations discussed in Section VIII.

A. High Gain Observer plus Dwell-Time

Inspired by the recent developments in *supervisory control* and *logic-based switching* schemes [6], [7], a novel strategy is proposed based on the *peak extinction time* and *dwell-time* concepts to cope with the problems induced by peaking, particularly the *shrinking* of the region of attraction. The new scheme is developed trying to retain the desirable qualities of the state-feedback based sliding mode controller such as good transient performance and global or semi-global stability. Our *key idea* consists in combining the high gain estimates from HGO with an appropriate dwell-time strategy to obtain a peaking free norm bound χ. In this respect, we only apply the HGO estimates after a certain *dwell-time* τ_D, which is chosen just large enough to allow the peaking transients of the HGO to settle down, and small enough to ensure that the trajectories do not leave a prescribed compact set, thus avoiding finite time escape. It will be shown that this choice is possible for μ sufficiently small and

$$\tau_D := \bar{t}_e(\mu) \,, \tag{24}$$

where $\bar{t}_e(\mu)$ is the known upper bound for the peaking extinction time t_e, given in Section IV-B.

B. Norm Bound from HGO

Due to the high gain properties of the HGO, it can be shown that: while the plant state x remains within any given compact ball, the observer error $\tilde{x}_e$ (13) can be made arbitrary small by reducing the parameter μ. Indeed, the following proposition can be demonstrated.

Proposition 1: Consider (1) under the assumptions (A1)–(A3) and τ_D defined in (24). Let $t^* \in [0, t_M)$ be the first time instant such that $|x|$ exits a given ball $\mathcal{B} := \{x : |x| \leq R\}$ of radius $R > |x(0)|$ and $\hat{\xi}$ (15) be the estimate for the state ξ given in (A1). Then, if the HGO parameter μ is sufficiently small such that $\tau_D(\mu) \in [0, t^*)$, one has

$$|\xi - \hat{\xi}| \leq \tilde{k}^R \mu \,, \quad \tilde{k}^R > 0 \,, \quad \forall t \in [\tau_D, t^*) \,, \tag{25}$$

where $\tilde{k}^R$ is a constant possibly depending on R. Moreover,

$$|x(t)| \leq \varphi_{T2} \left(c_0 |\hat{\xi}(t)| + \frac{c_1}{s + \lambda_1} \varphi_0(c_2 |\hat{\xi}(t)|, t) \right) + \Delta := \chi(t) \,, \tag{26}$$

$\forall t \in [\tau_D, t^*)$, *modulo* exponentially decaying terms, where $c_0, c_1, c_2, \lambda_1, \Delta > 0$ are appropriate constants and φ_0, φ_{T2} are given in (A1) and (A2), respectively.

Proof: See Appendix. ■

Thus, using (A2)–(A3) and (9), a peaking free modulation function $\varrho(\chi, t)$ can be obtained:

$$\varrho(\chi, t) = \begin{cases} 0 & , \ \forall t \in [0, \tau_D) \\ \hat{d}(t) + \delta & , \ \text{otherwise} \,, \end{cases} \tag{27}$$

such that (12) holds $\forall t \in [\tau_D, t_M)$ with (22) redefined for χ given by (26). The stability results are summarized in the next section.

VII. Stability Analysis

In order to fully account for the initial conditions of the *error system* (7) and (18), let:

$$z^T(t) := [z^0(t), x_e^T(t), \zeta^T(t)] \,, \tag{28}$$
$$z^0(t) := [|\eta(t_i)|, |x_\varrho(t_i)|] e^{-\gamma(t - t_i)} \,,$$

where $t_i \in [0, t_M)$ is a generic time instant and z^0 denotes the *transient state* [11], [14] due to state conditions (at $t = t_i$) of the stable systems corresponding to the η-dynamics and the filters used in the modulation function design.

The following theorem is independent of the strategy used to obtain χ and, thus, it holds for both proposed schemes.

Theorem 1: Consider the error system (7) and (18) with control law (11) and modulation function (27). Assume that (A1)–(A3) hold. Then, for sufficiently small $\mu > 0$, the complete error system, with state $z(t)$, is globally/semi-globally exponentially stable *w.r.t.* a small residual set of order $\mathcal{O}(\mu)$ independent of the initial conditions. Moreover, under these conditions, all signals in the closed loop system are uniformly bounded. In particular, if (A4) holds the above statements are still valid for the modulation function (23).

Proof: See Appendix. ■

Remark 1: (Smaller Residual Set) The residual set in Theorem 1 is of order $\mathcal{O}(\mu)$ while in GBC approach [3], [4] this set is of order $\mathcal{O}(\sqrt{\mu})$.

Remark 2: (Ideal Sliding Mode) If, additionally to the assumptions of Theorem 1, $\varrho \geq |K_m \xi_m - k_m r| + \delta$ with $\delta > 0$ then the sliding mode $\hat{\sigma}(t) \equiv 0$ is reached in finite time. This implies that finite frequency chattering is avoided.

Remark 3: (Stability Domain and Transient Behavior) In Theorem 1, the control system is semi-global in only one parameter (μ), while the GBC approach [3], [4] has one more parameter to be adjusted – the saturation level. We conjecture that this two-parameter dependence is the reason for smaller stability domains observed in the simulations with the GBC. Indeed, it is interesting to note that increasing the saturation level, while μ is kept constant, the GBC stability domain is *reduced*. In order to recover the domain using GBC it is necessary to reduce μ, i.e., to increase the HGO gain. However, while the stability domain is increased, the transient behavior is degraded due to large peaks transmitted to the plant allowed by the larger saturation level.

VIII. Simulation Results

Consider the nonlinear plant ($\rho = 3$):

$$\begin{aligned} \dot{x}_1 &= x_2 + \alpha y^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -3x_3 - 3x_2 - \beta x_1 + \varepsilon x_2^3 + k_p u \\ y &= x_1 \end{aligned}$$

where only y is measured and the uncertain parameters are $|\alpha| \leq 0.5$, $-1.5 \leq \beta \leq -0.5$, $|\varepsilon| < 2$ and $0.5 \leq k_p \leq 2$. In this plant, the η-dynamics is absent. Thus, computing the time derivatives $\dot{y}(t)$, $\ddot{y}(t)$ and $\dddot{y}(t)$, one can obtain the global diffeomorphism (T, T^{-1}), the ξ-dynamics and the input disturbance d_ϕ in (A1). Moreover, in (A2)–(A3): $\varphi_{T1}(a) = 2\bar{\alpha}^2 a^3 + 3\bar{\alpha} a^2 + 3a$; $\varphi_{T2}(a) = 3\bar{\alpha} a^2 + 3a$; $k_{T1} = k_{T2} = 0$; $\varphi_d(a) = 6\bar{\alpha}^3 a^4 + (8\bar{\alpha}^2 + \bar{\varepsilon}) a^3 + 4\bar{\alpha} a^2 + (6 + \bar{\beta}) a$, where $\bar{\alpha} = 0.5$, $\bar{\beta} = 1.5$ and $\bar{\varepsilon} = 2$ are the upper bounds for $\alpha, \beta, \varepsilon$, respectively.

The control objective is to track the output y_m of the reference model $M(s) = \frac{1}{(s+1)^3}$ with state space realization (5), $r(t) = 0.5 \sin(0.2\pi t)$ and $K_m = [-1\ -3\ -3]^T$. The HGO is implemented with $\mu = 0.01$, $N(s) = (s+5)^3$, $L_o = [15\ 75\ 125]^T$ and $k_p^{nom} = 1$.

(a) Linear growth condition: If $\varepsilon = 0$, (A4) is trivially satisfied. In this case, if the peaking is injected to the plant input, a strong degradation of the system transient will occur (curves not shown). To avoid peaking, one could apply the strategy based on *state filters* described in Section V-B. Due to the lack of space we will restrict our attention only to the more general case when $\varepsilon \neq 0$.

(b) No growth condition: For $\varepsilon \neq 0$, the system has a strong nonlinearity in the unmeasured state x_2 and the simple state filters approach cannot be applied. Thus, we use the dwell-time strategy to generate a peaking free control law. The modulation function (27) is implemented with $\delta = 1$, $\hat{d}$ given in (22) with $\hat{\varphi}(a) = c_1 \varphi_{T1}(2a) + \varphi_d(2a)$, $c_r \geq |r|/|k^*|$ and χ given in (26) with $\hat{\varphi}_{T2}(a) = \varphi_{T2}(2a)$ and $\Delta = 1$. Note that, the first order filter in (26) can be neglected since the η-dynamics was dropped, i.e., the norm bound χ is just a function of $|\hat{\xi}(t)|$ and $c_0 = 1$ in (26). In this example, if the peaking from HGO states is completely transmitted to the plant, finite time escape can be provoked. Indeed, considering the initial state $x(0) = [1\ 0\ 0]^T$, finite time escape occurs at $t = 0.262$ (curves not shown). Fig. 2 shows the remarkable performance obtained with the dwell-time strategy ($\tau_D = 10\mu$) and the same initial conditions.

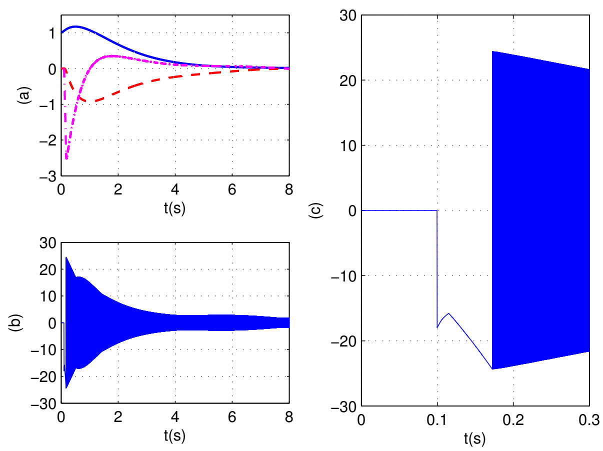

Fig. 2. HGO plus dwell-time: (a) state tracking error trajectories x_e, (b) control signal u and (c) zoom in u plot showing the dwell-time $\tau_D = 0.1$.

One alternative way to avoid peaking consists in applying a GBC [4] using a proper saturation level u_{sat}, which would accommodate for the necessary control strength. However, as expected, saturation may reduce the domain of stability. When we fix the parameter μ, the domain of stability is reduced by increasing u_{sat} due to the high levels of peaking transmitted to the plant. Indeed, for $x(0) = [x_1(0)\ 0\ 0]^T$, $\mu = 0.01$ and $u_{sat} = 500$, stability is achieved with $|x_1(0)| \leq 2$. On the other hand, in our scheme the domain of stability is considerable bigger ($|x_1(0)| \leq 6$) with the same μ. In addition, when $x_1(0) = 2$, our scheme presents the maximum control amplitude equal to 500. Thus, to guarantee a fair comparison between both controllers, we consider $x_1(0) = 2$ and $\mu = 0.01$, while $u_{sat} = 500$ in the GBC.

In Fig. 3 (a), it can be observed that for $x_1(0) = 2$, the performance is similar to both controllers. However, for smaller initial conditions ($x_1(0) = 1; 0.01$), a significant transient degradation in the tracking error results with the GBC approach compared to the transient in the HGO plus dwell-time scheme ($\tau_D = 0.1$) as illustrated in Fig. 3 (b)-(c). This comes from the fact that in the latter case the magnitude of control is automatically adjusted and becomes smaller for smaller initial conditions leading to a more *uniform transient* behavior in the whole stability domain. In contrast, in the GBC approach, a larger stability domain requires larger saturation level, as well as higher observer gain. This does not prevent the peaking deleterious effect for smaller initial conditions precluding such an uniform transient behavior.

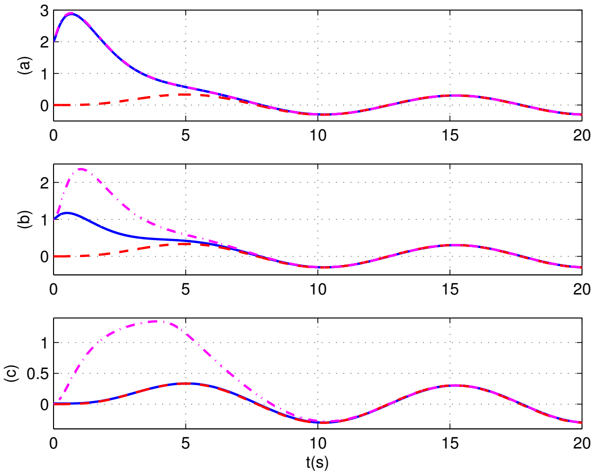

Fig. 3. Non uniform transient for different initial conditions: (a) $x_1(0) = 2$, (b) $x_1(0) = 1$ and (c) $x_1(0) = 0.01$. Plant output x_1 [HGO plus dwell-time: solid-line; GBC: dot-dash] and y_m [dash] trajectories.

IX. Conclusions

The variable structure tracking controllers for uncertain nonlinear SISO systems developed in this paper use high gain observers and norm observers. The estimated HGO states are used in the computation of the switching law while the modulation of the control law is generated using signals from the state filters or HGO states associated with a dwell-time strategy. The proposed schemes were shown to be peaking free, independently of the observer gain. They were also shown to lead to global or semi-global exponential stability, *w.r.t.* a small residual set, without the need for globally bounding the control signal. The only parameter required to increase the domain of stability is the observer gain. In order to avoid peaking after the initial time, a *monitoring function*, similar to the one of [15], to detect the onset of peaking is under development.

References

[1] F. Mazenc, L. Praly, and W. P. Dayawansa. Global stabilization by output feedback: examples and counterexamples. *Systems & Contr. Letters*, 23:119–125, 1994.

[2] H. J. Sussmann and P. V. Kokotović. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. Aut. Contr.*, 36(4):424–440, 1991.

[3] S. Oh and H. K. Khalil. Output feedback stabilization using variable structure control. *Int. J. Contr.*, 62(4):831–848, 1995.

[4] S. Oh and H. K. Khalil. Nonlinear output-feedback tracking using high-gain observer and variable structure control. *Automatica*, 33(10):1845–1856, 1997.

[5] E. D. Sontag and Wang Y. Output-to-state stability and detectability of nonlinear systems. *Systems & Contr. Letters*, 29:279–290, 1997.

[6] J. P. Hespanha, D. Liberzon, and A. S. Morse. Overcoming the limitations of adaptive control by means of logic-based switching. *Systems & Contr. Letters*, 49:49–65, 2003.

[7] L. B. Freidovich and H. K. Khalil. Lyapunov-based switching control of nonlinear systems using high-gain observers. *Automatica*, (43):150–157, 2007.

[8] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3^{rd} ed., 2002.

[9] Z. P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals and Systems*, (7):95–120, 1994.

[10] A. F. Filippov. Differential equations with discontinuous right-hand side. *American Math. Soc. Translations*, 42(2):199–231, 1964.

[11] L. Hsu, J. P. V. S. Cunha, R. R. Costa, and F. Lizarralde. Multivariable output-feedback sliding mode control. In X. Yu and J.-X. Xu, editors, *Variable Structure Systems: Towards the 21st Century*, pages 283–313. Springer-Verlag, 2002.

[12] P. A. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice-Hall, 1996.

[13] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, 3^{rd} ed., 1995.

[14] A. J. Peixoto, L. Hsu, R. R. Costa, and F. Lizarralde. Global tracking sliding mode control for uncertain nonlinear systems based on variable high gain observer. In *Proc. IEEE Conf. on Decision and Control*, pages 2041–2046, New Orleans, LA, December 2007.

[15] T. R. Oliveira, A. J. Peixoto, E. V. L. Nunes, and L. Hsu. Control of uncertain nonlinear systems with arbitrary relative degree and unknown control direction using sliding modes. *Int. J. Adaptive Contr. Signal Process.*, 21:692–707, 2007.

[16] L. Hsu, R. R. Costa, and J. P. V. S. Cunha. Model-reference output-feedback sliding mode controller for a class of multivariable nonlinear systems. *Asian Journal of Control*, 5(4):543–556, 2003.

X. Appendix

A. Proof of Proposition 1

The proof of (25) follows from [3, Lemma 1]. Applying [16, Lemma 2] to the η-dynamics in (A1), one has $|\eta(t)| \leq \frac{k_0}{s+\lambda_0}\varphi_0(|\xi(t)|, t)$, *modulo* exponentially decaying term, where $k_0, \lambda_0 > 0$ are appropriate constants. Thus, from (A1)–(A2) and (25), we can obtain (26), with Δ accounting for the effect of the $\mathcal{O}(\mu)$ estimation error (25) and the constant k_{T2} in (A2).

B. Proof of Theorem 1

Only a sketch of the proof is presented due to space limitations. In what follows Π denotes an exponential decaying term of the form $\Pi(t) := \Psi_\pi(|z(t_i)|)e^{-\gamma(t-t_i)}$, where Ψ_π is a generic class $\mathcal{K}_\infty$ function and $\gamma > 0$ a generic constant. Let $t^* \in [0, t_M)$ be the first time such that $|x|$ exits a given compact ball $\mathcal{B} := \{x : |x| \leq R\}$ of radius $R > |x(0)|$. Furthermore, $k_i > 0$ denotes constants not depending on the initial conditions, $k_i^R > 0$ denotes constants possibly depending on R and $\Psi_i(\cdot) \in \mathcal{K}_\infty$.

The analysis detailed in [14] can be directly applied here and the following ISS/ISpS properties can be demonstrated $\forall t \in [t_i, t^*)$:

$$|x_e(t)| \leq k_1 \|(\tilde{x}_e)_{t,t_i}\| + \Pi\,, \tag{29}$$

$$|\tilde{x}_e(t)| \leq \mu k_1^R \|(x_e)_{t,t_i}\| + \Pi + \mathcal{O}(\mu)\,. \tag{30}$$

In particular to obtain (30), the inequality

$$|\nu| \leq k_2^R \|(x_e)_{t,t_i}\| + \Pi + k_2\,, \tag{31}$$

must be satisfied, as shown in [14]. In such case, from ν (19), u (11), ϱ (27), $\hat{d}$ (22), χ (26), $\hat{\xi}$ (25) and noting that $\hat{d}$ is a valid upper bound for d_ϕ, one has $|\nu| \leq \Psi_1(\|(\xi)_{t,t_i}\|) + \Pi + k_3$. Thus, one can conclude that $|\nu| \leq \Psi_2(\|(x_e)_{t,t_i}\|) + \Pi + k_4$, reminding that $x_e := \xi - \xi_m$ and ξ_m is uniformly bounded. In addition, since $\forall t \in [0, t^*)$, $|x_e(t)| < k_3^R$, thus $\Psi_2(\|(x_e)_{t,t_i}\|) \leq k_4^R \|(x_e)_{t,t_i}\|$ whereby (31) results.

Due to the definition of $\varrho(\chi, t)$ in (27), we would obtain (29)-(30) valid only $\forall t \in [\tau_D, t^*)$, i.e., $t_i = \tau_D$. However, since peaking is not transmitted to the plant state x and x_e in (7) is ISS with respect to $[u + d(x,t)]$, one can conclude that x_e cannot escape far from the initial conditions set $\forall t \in [0, \tau_D(\mu))$ and μ sufficiently small. Thus, one has

$$|x_e(t)| \leq k_5^R |x_e(0)| + \mathcal{O}(\mu)\,, \quad \forall t \in [0, \tau_D) \subset [0, t^*)\,. \tag{32}$$

From the application of the small-gain theorem [9] to the inequalities (29)-(30) one can conclude that the $x_e(t)$ and the complete error state $z(t)$ are bounded $\forall t \in [t_D, t^*)$ provided $\mu < /(k_1^R k_1)$. Using (32), we can extend the results $\forall t \geq 0$. Moreover, $z(t) \to \mathcal{O}(\mu)$ exponentially and cannot escape in finite time, i.e., t^*, $t_M \to +\infty$. Since R ($> |x(0)|$) and thus $|x(0)|, |z(0)|$ can be chosen arbitrary large as $\mu \to 0$, semi-global exponential stability is concluded.

If the *linear growth condition* (A4) holds for ϕ in (1), $\varrho(\chi, t)$ in (11) can be implemented with (23), (22) and χ in (20) $\forall t \in [0, t^*)$, i.e., $t_i = 0$. Since χ is a function of ω and y, one can still prove (31) following the same steps from the proof of [15, Theorem 1]. Thus, (29)-(30) are valid $\forall t \in [0, t^*)$ and semi-global exponential stability can be directly obtained from the small-gain theorem [9] $\forall t \geq 0$ not using (32). If the nonlinearity ϕ satisfies a global Lipschitz condition, the constant k_1^R in (30) will not depend on R and the stability properties become *global*.