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Model Predictive Controller Tuning via Eigenvalue Placement

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Abstract— **This paper presents an analytical study of the effect of model predictive control (MPC) tunable parameters over a wide range, on the closed-loop performance quantified in terms of the location(s) of closed-loop eigenvalue(s) of a large set of widely common, single-input single-output, linear plants whose constraints are inactive. Symbolic manipulation capabilities of MATHEMATICA are used to obtain analytical expressions describing the dependence of closed-loop eigenvalues on the tunable parameters. This work is first to investigate how MPC tuning-parameters affect the location of the eigenvalues of the closed-loop system of a plant in the discrete-time setting. It is to provide theoretical basis/justification for many of the existing qualitative MPC tuning rules and propose new tuning guidelines for MPC. For example, as the prediction horizon is increased while other tunable parameters remain constant, a subset of the closed-loop eigenvalues (poles) move non-monotonically towards the open-loop eigenvalues (poles) of the plant. If a prediction horizon much longer than the reference-trajectory time-constant is used, the value of reference-trajectory time-constant has little effect on the closed-loop performance. As the weights on the magnitude or the rate of change of the manipulated input are increased, the closed-loop eigenvalues move towards the open-loop eigenvalues. As the control horizon is increased from one, the dominant eigenvalue of the closed-loop system initially moves towards the origin and then away from the origin to a location that does not change with a further increase in the control horizon.**

I. INTRODUCTION

Model predictive control (MPC) has been implemented widely in the process industries since its introduction in the mid 1970's. Among factors contributing to the success of MPC are (i) the ability of MPC to handle constraints on manipulated, state and controlled variables in a systematic way during the design and implementation of the controller, and (ii) its enormous flexibility that allows the use of models, objective functions and constraint functionalities in a variety of forms [1], [2]. The theory of MPC has received increasingly attention for the past decade, especially from researchers outside process systems engineering.

A typical model predictive controller has a great number of tunable parameters. These parameters include the prediction horizon(s), control horizon(s), model horizon, output penalties, input magnitude and rate of change penalties, and reference trajectory time constant(s). For the past three decades, a qualitative understanding of the effect of these controller tunable parameters on the stability and performance of the closed-loop system has been gained. This understanding together with closed-loop simulation studies has made possible the wide-spread implementation of MPC. It is known that the effect of these parameters on the closed-loop performance is sometimes non-monotonic.

There have been several studies on MPC tuning, including [3], [4], [5], [6], [7], [8]. Lee and Yu [3] derived expressions for sensitivity equations for state-space models. Using an optimal Kalman filter and a long horizon control law, they studied MPC tuning for robust stability and performance. Shridhar and Cooper [4] derived an analytical expression that calculates move suppression coefficients as a function of plant model parameters, other MPC design parameters, and partitioned block condition numbers of the system matrix. Their tuning method is applicable to unconstrained multivariable plants, including non-square systems. They also proposed an expression for optimal move suppression coefficients using a first-order-plus-dead-time approximation of process model, and tuning heuristics for prediction horizon, model horizon, and control horizon [5]. Al-Ghazzawi et al. [6] presented an approach to tuning MPC on-line based on sensitivity equations derived from a step response model with linear constraints. The sensitivity equations were based on finding optimal move suppression and controlled variable damping coefficients from process parameters. Wojsznis et al. [7] developed a heuristic approach to MPC tuning. They proposed setting penalties on control moves as a function of plant dead time as the primary factor, with some correction from plant gain. Trierweiler and Farinab [8] also presented a tuning strategy for MPC based on the attainable performance of a system and its degree of directionality.

The dynamics of a plant in closed-loop is described mainly by eigenvalues of the Jacobian of the closed-loop system, if the plant is linear, and by the eigenvalues and nonlinearity of the closed-loop system, if the plant is nonlinear. In MPC, there has been very little or no attempt to study how a controller places closed-loop eigenvalues, while in other control methods, closed-loop performance, stability, and robust stability have been studies in terms of the location of closed-loop eigenvalues.

This paper presents an analytical study of the effect of the MPC tunable parameters over a wide range, on the closed-loop performance quantified in terms of the location(s) of closed-loop eigenvalue(s) of a large set of widely common, single-input single-output, linear plants whose constraints are inactive. The tunable parameters include control horizon, prediction horizon, controlled variable weights, time constant of reference trajectory, and weights on the magnitude and rate of change of manipulated variable. Symbolic manipulation capabilities of MATHEMATICA are used to obtain analytical expressions describing the dependence of closed-

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loop eigenvalues on the tunable parameters. This work is first to investigate how MPC tuning-parameters affect the location of the eigenvalues of the closed-loop system of a plant in the discrete-time setting. It is to provide theoretical basis/justification for many of the existing qualitative MPC tuning rules and propose new tuning guidelines for MPC.

The organization of this paper is as follows. The scope of the study and some mathematical preliminaries are given in Section II. A general moving-horizon optimization problem is formulated in Section III and is applied to linear plants in Section IV. Symbolic calculations are described in Section V. Several widely common plants are considered in Section VI. Concluding remarks are given in Section VII.

II. Scope and Preliminaries

Consider a single-input single-output (SISO) nonlinear plant in the form

$$\left.\begin{array}{rcll}\bar{x}(k+1) & = & \Phi[\bar{x}(k),u(k)], & \bar{x}(0)=0\\ \bar{y}(k) & = & h[\bar{x}(k)]+d & \end{array}\right\} \qquad (1)$$

where $\bar{x} \in X \subset \Re^n$ denotes the vector of the plant state variables, $u \in U \subset \Re$ is the plant manipulated variable (input), $\bar{y} \in h(X) \subset \Re$ is the plant controlled variable (output), and $d \in D \subset \Re$ is an unmeasurable constant disturbance. Here X and U are open connected sets, and D is a connected set. It is assumed that the discrete-time plant description is a result of using a sampling period in order of one-tenth of the dominant time constant of the plant.

We also make the following assumptions: (a) $(x_{ss}, u_{ss}) = (0, 0)$ is the nominal equilibrium pair, (b) the nominal equilibrium pair $(0, 0) \in X \times U$, (c) $\Phi(x, u)$ and $h(x)$ are smooth vector functions on $X \times U$ and X, respectively, and (d) for a plant of the form of (1), a discrete-time model of the following form is available:

$$\left.\begin{array}{rcll}x(k+1) & = & \Phi[x(k),u(k)], & x(0)=0\\ y(k) & = & h[x(k)] & \end{array}\right\} \qquad (2)$$

where $x \in X \subset \Re^n$ denotes the vector of *model* state variables and $y \in h(X) \subset \Re$ is the *model* controlled variable (output). Relative order (degree) of the controlled output y with respect to the manipulated input u is denoted by r, where the relative order r is the smallest integer for which $y(k+r)$ depends explicitly on $u(k)$.

There have been several studies on the connections between MPC and existing analytical control methods, leading to a better understanding of the effect of MPC tuning parameters on the closed-loop performance [9], [10]. A few special cases of unconstrained SISO MPC are listed in Table I. M and P denote control and prediction horizons, respectively. As the prediction horizon is increased and the control horizon is decreased, a more robust but less aggressive controller is obtained. It has been shown [9], [10] that in special cases shortest-prediction-horizon MPC leads to:

- Input-output linearizing control laws that inherently include optimal windup and directionality compensators,
- Model state feedback control (MSFC) and modified internal model control (IMC) laws that inherently include an optimal directionality compensator,
- Proportional–integral (PI) and proportional–integral–derivative (PID) controllers that inherently include optimal windup and directionality compensators.

TABLE I

UNCONSTRAINED SISO MODEL PREDICTIVE CONTROLLERS WITH NO PENALTY ON MANIPULATED INPUT MAGNITUDE OR RATE OF CHANGE [9].

P	M	Reference Trajectory	Model	Resulting Controller
r	1	No		Deadbeat
r	1	Yes		I-O Linearization
r	1	Yes	Linear	Modified IMC
r	1	Yes	Linear	MSFC
$r=1$	1	Yes	Linear	PI, PID
$>> r$	$>> 1$	Yes/No		Long Horizon MPC
∞	1	Yes/No		Steady State

III. Model Predictive Control Law

Consider a moving-horizon minimization problem of the form

$$\begin{array}{r}\min\limits_{U(k)} \Bigg\{ \sum\limits_{\ell=r}^{P} w_{y_\ell}\left[y_d(k+\ell)-\hat{y}(k+\ell)\right]^2 + \\ \sum\limits_{\ell=0}^{M-1} w_{u_{\ell+1}}\left[u(k+\ell)\right]^2 + \\ \sum\limits_{\ell=0}^{M-1} w_{\Delta u_{\ell+1}}\left[u(k+\ell)-u(k+\ell-1)\right]^2 \Bigg\}\end{array} \qquad (3)$$

subject to

$$\begin{array}{c} x(k+1)=\Phi[x(k),u(k)] \\ y(k)=h[x(k)] \\ u(k+\ell)=u(k+\ell-1), \qquad \ell=M,\cdots,P-r \end{array}$$

where

- $P \geq r$,
- $U(k) = [u(k) \cdots u(k+M-1)]^T$,
- $w_{y_r}, \cdots, w_{y_P}, w_{u_1}, \cdots, w_{u_M}, w_{\Delta u_1}, \cdots, w_{\Delta u_M} \geq 0$,
- y_d is a reference trajectory, given by

$$\begin{array}{l} y_d(k+j) = (1+\beta)^r y_{sp}(k+j) - \sum\limits_{\ell=0}^{r-1}\binom{r}{r-\ell}\beta^{r-\ell} \\ \qquad\qquad \times y_d(k+j-r+\ell), \quad j=r,\cdots,P \\ y_d(k+\ell) = \hat{y}(k+\ell), \quad \ell=0,\cdots,r-1 \end{array}$$

- β is a tunable scalar parameter that sets the speed of the reference trajectory and is chosen such that $-1 < \beta < 0$.
- $y_{sp} \in Y \subset \Re$ is the output set-point. Y is the set of all y_{sp} for which for every $d \in D$ there exist a $u_{ss} \in U$ and an $x_{ss} \in X$ satisfying $x_{ss} = \Phi(x_{ss}, u_{ss})$ and $y_{sp} = h(x_{ss}) + d$.
- $$\binom{a}{b} \triangleq \frac{a!}{b!(a-b)!}$$

- $\hat{y}$ is the predicted future value of the controlled output, given by

$$\begin{aligned}
\hat{y}(k+\ell) &\triangleq \bar{y}(k) + h^{\ell}[x(k)] - h[x(k)], \quad \ell = 0, \cdots, r-1 \\
\hat{y}(k+r) &\triangleq \bar{y}(k) + h^{r}[x(k), u(k)] - h[x(k)] \\
\hat{y}(k+\ell) &\triangleq \bar{y}(k) + h^{\ell}[x(k), u(k), \cdots, u(k+\ell-r)] - \\
&\qquad h[x(k)], \qquad \ell = r+1, \cdots, P
\end{aligned} \tag{4}$$

-

$$\begin{aligned}
h^{0}[x(k)] &\triangleq h[x(k)] \\
h^{\ell}[x(k)] &\triangleq h^{\ell-1}[\Phi(x(k), u(k))], \\
&\qquad \ell = 1, \cdots, r-1 \\
h^{r}[x(k), u(k)] &\triangleq h^{r-1}[\Phi(x(k), u(k))] \\
h^{r+1}[x(k), u(k), u(k+1)] &\triangleq h^{r}[\Phi(x(k), u(k)), u(k+1)] \\
&\vdots \\
h^{P}[x(k), u(k), \cdots, u(k+P-r)] &\triangleq \\
h^{P-1}[\Phi(x(k), u(k)), &u(k+1), \cdots, u(k+P-r)]
\end{aligned}$$

When measurements of the state variables are available, in the preceding prediction equations we set $x = \bar{x}$. Otherwise, the values of the state variables have to be estimated, for example, via on-line simulation of the plant model (use of an open-loop state estimator) or by using an extended Kalman filter. An estimate of the unmeasured output disturbance, $\hat{d}$, is then calculated from $\hat{d} = \bar{y} - h(x)$. The use of this disturbance estimate in the model predictive control method leads to a model predictive controller with integral action. For the sake of simplicity, the state estimator dynamics are not accounted for in the results presented herein.

Solving the moving-horizon minimization problem of (3) leads to a feedforward/state feedback. It is assumed that for every $x \in X$, every $y_{sp} \in Y$, and every $d \in D$, the optimization problem of (3) is feasible; that is, there is a $u \in U$ that minimizes the performance index in (3) globally.

IV. Application to Linear Plants

Consider the class of time-invariant, linear plants with a model of the form

$$\left.\begin{aligned}
x(k+1) &= Ax(k) + bu(k), \qquad x(0) = 0 \\
y(k) &= cx(k)
\end{aligned}\right\} \tag{5}$$

where A, b and c are $n \times n$, $n \times 1$ and $1 \times n$ constant matrices respectively. This class of plants is a special case of (1).

For this class of plants, the predicted future values of the controlled output are given by:

$$\begin{aligned}
\hat{y}(k+1) &= \bar{y}(k) + [cA - c]x(k) + cbu(k) \\
\hat{y}(k+2) &= \bar{y}(k) + [cA^{2} - c]x(k) + cAbu(k) + cbu(k+1) \\
&\vdots \\
\hat{y}(k+P) &= \bar{y}(k) + [cA^{P} - c]x(k) + cA^{P-1}bu(k) + \cdots \\
&\quad + cA^{r-1}bu(k+P-r) + \cdots + cbu(k+P-1)
\end{aligned} \tag{6}$$

and the current and future values of the reference trajectory by

$$\begin{aligned}
y_d(k) &= \bar{y}(k) \\
y_d(k+1) &= \bar{y}(k) + [cA - c]x(k) \\
&\vdots \\
y_d(k+r-1) &= \bar{y}(k) + [cA^{r-1} - c]x(k) \\
y_d(k+r) &= (1+\beta)^{r} y_{sp}(k+r) - \sum_{\ell=0}^{r-1} \binom{r}{r-\ell} \beta^{r-\ell} \\
&\qquad \times y_d(k+\ell) \\
&\vdots \\
y_d(k+P) &= (1+\beta)^{r} y_{sp}(k+P) - \sum_{\ell=0}^{r-1} \binom{r}{r-\ell} \beta^{r-\ell} \\
&\qquad \times y_d(k+P-r+\ell)
\end{aligned}$$

V. Symbolic Calculations

After entering values for A, b, and c matrices (unless the plant is first order), the following *symbolic* calculations are performed:

(a) After setting values of P and M, calculate the performance index $J(x(k), y_{sp}(k), u(k), \cdots, u(k+M-1))$.
(b) Calculate $[u(k), \cdots, u(k+M-1)]^{T} = Q(x(k), y_{sp}(k))$ that globally minimizes $J(x(k), y_{sp}(k), u(k), \cdots, u(k+M-1)$. This involves taking partial derivatives of J with respect to $u(k), \cdots, u(k+M-1)$, setting the resulting partial derivatives to zero, and then solving for $u(k), \cdots, u(k+M-1)$.
(c) Calculate the Jacobian of the closed-loop system; i.e., $J_{cl} = A + bk_1$, where k_1 is the first row of the $M \times n$ matrix $\partial Q(x, y_{sp})/\partial x$.
(d) Calculate the eigenvalues of J_{cl}.

This approach can be extended to calculate eigenvalues of the closed-loop system of an input-constrained plant. For such a plant, steps (b) and (c) are different; the optimization in step (b) is constrained, and when an input constraint is active, $k_1 = 0$; $J_{cl} = A$, which is the Jacobian of the plant in open-loop (J_{ol}). A sufficient condition for asymptotic stability of an input-constrained plant is that all eigenvalues of (a) J_{cl} when the same plant is unconstrained and (b) $J_{ol} = A$ lie inside the unit circle. In the case that the order of the plant is higher than one and the values of the matrices A, b, and c are not set, MATHEMATICA calculates symbolic expressions for the closed-loop Jacobian that may be hundreds of pages long.

Sample analytical expressions for the closed-loop Jacobian (eigenvalue), J_{cl}, of a general first-order plant of the form (5) for several low values of P and M are given in Table II. They show how M, P, $w_{y_1}, \cdots, w_{y_n}$, $w_{u_1}, \cdots, w_{u_M}$, $w_{\Delta u_1}, \cdots, w_{\Delta u_M}$, and β affect the location of the closed-loop eigenvalue. Closed-loop Jacobian expressions for higher values of P and M are not presented here because of their larger sizes and the 6-page limit on the length of ACC

TABLE II

ANALYTICAL EXPRESSIONS FOR THE CLOSED-LOOP EIGENVALUE OF A GENERAL FIRST-ORDER LINEAR PLANT.

P=1, M=1	$Jcl := -\frac{-A\,Wu1 - A\,Wdeltau1 + c^2 b^2\,Wy1\,\beta}{c^2 b^2\,Wy1 + Wu1 + Wdeltau1}$
P=2, M=1	$Jcl := \frac{A\,Wu1 + A\,Wdeltau1 - c^2 b^2\,Wy1\,\beta + c^2 b^2 A\,Wy2\,\beta^2}{c^2 A^2 b^2\,Wy2 + Wu1 + c^2 b^2\,Wy1 + Wdeltau1}$
P=3, M=1	$Jcl := \big(Wy2\,c^2 A^2 b^2 + Wy2\,c^2 A\,b^2 + Wy3\,c^2 A^4 b^2 + 2\,Wy3\,c^2 A^3 b^2 + 2\,Wy3\,c^2 A^2 b^2 + Wy3\,c^2 A\,b^2 + A\,Wu1 + A\,Wdeltau1 - c^2 b^2\,Wy1\,\beta + c^2 b^2\,Wy2\,\beta^2 A + c^2 b^2\,Wy2\,\beta^2 - c^2 b^2\,Wy3\,\beta^3 A^2 - c^2 b^2\,Wy3\,\beta^3 A - c^2 b^2\,Wy3\,\beta^3\big) \big/ \big(c^2 b^2\,Wy1 + Wy2\,c^2 A^2 b^2 + 2\,Wy2\,c^2 A\,b^2 + Wy2\,c^2 b^2 + Wy3\,c^2 A^4 b^2 + 2\,Wy3\,c^2 A^3 b^2 + 3\,Wy3\,c^2 A^2 b^2 + 2\,Wy3\,c^2 A\,b^2 + Wy3\,c^2 b^2 + Wu1 + Wdeltau1\big)$
P=4, M=1	$Jcl := \big(Wy2\,c^2 A^2 b^2 + Wy2\,c^2 A\,b^2 + Wy3\,c^2 A^4 b^2 + 2\,Wy3\,c^2 A^3 b^2 + 2\,Wy3\,c^2 A^2 b^2 + Wy3\,c^2 A\,b^2 + A\,Wu1 + A\,Wdeltau1 - c^2 b^2\,Wy1\,\beta + c^2 b^2\,Wy2\,\beta^2 A + c^2 b^2\,Wy2\,\beta^2 - c^2 b^2\,Wy3\,\beta^3 A^2 - c^2 b^2\,Wy3\,\beta^3 A - c^2 b^2\,Wy3\,\beta^3\big) \big/ \big(c^2 b^2\,Wy1 + Wy2\,c^2 A^2 b^2 + 2\,Wy2\,c^2 A\,b^2 + Wy2\,c^2 b^2 + Wy3\,c^2 A^4 b^2 + 2\,Wy3\,c^2 A^3 b^2 + 3\,Wy3\,c^2 A^2 b^2 + 2\,Wy3\,c^2 A\,b^2 + Wy3\,c^2 b^2 + Wu1 + Wdeltau1\big)$
P=5, M=1	$Jcl := -\big(-Wy2\,c^2 A^2 b^2 - Wy2\,c^2 A\,b^2 - Wy3\,c^2 A^4 b^2 - 2\,Wy3\,c^2 A^3 b^2 - 2\,Wy3\,c^2 A^2 b^2 - Wy3\,c^2 A\,b^2 - Wy4\,c^2 A^6 b^2 - 2\,Wy4\,c^2 A^5 b - 3\,Wy4\,c^2 A^4 b^2 - 3\,Wy4\,c^2 A^3 b^2 - 2\,Wy4\,c^2 A^2 b^2 - Wy4\,c^2 A\,b^2 - c^2 b^2\,Wy2\,\beta^2 A + c^2 b^2\,Wy1\,\beta - c^2 b^2\,Wy2\,\beta^2 + c^2 b^2\,Wy3\,\beta^3 - c^2 b^2\,Wy4\,\beta^4 - A\,Wu1 - A\,Wdeltau1 + c^2 b^2\,Wy3\,\beta^3 A^2 + c^2 b^2\,Wy3\,\beta^3 A - c^2 b^2\,Wy4\,\beta^4 A^3 - c^2 b^2\,Wy4\,\beta^4 A^2 - c^2 b^2\,Wy4\,\beta^4 A\big) \big/ \big(c^2 b^2\,Wy1 + Wy2\,c^2 A^2 b^2 + 2\,Wy2\,c^2 A\,b^2 + Wy2\,c^2 b^2 + Wy3\,c^2 A^4 b^2 + 2\,Wy3\,c^2 A^3 b^2 + 3\,Wy3\,c^2 A^2 b^2 + 2\,Wy3\,c^2 A\,b^2 + Wy3\,c^2 b^2 + Wy4\,c^2 A^6 b^2 + 2\,Wy4\,c^2 A^5 b^2 + 3\,Wy4\,c^2 A^4 b^2 + 4\,Wy4\,c^2 A^3 b^2 + 3\,Wy4\,c^2 A^2 b^2 + 2\,Wy4\,c^2 A\,b^2 + Wy4\,c^2 b^2 + Wu1 + Wdeltau1\big)$

proceedings papers. For example, the closed-loop Jacobian (eigenvalue) for $P = 5$ and $M = 3$ is more than four pages long. As can be seen in the sample expressions, for every combination of P and M values, sum of the 2^M terms $A(\omega_{1_j} \times \cdots \times \omega_{M_j})$, $j = 1, 2$, where $\omega_{i_1} = w_{u_i}$ and $\omega_{i_2} = w_{\Delta u_i}$, $i = 1, \cdots M$, appear in the numerator of J_{cl}, and sum of the 2^M terms $(\omega_{1_j} \times \cdots \times \omega_{M_j})$, $j = 1, 2$, in the denominator. This explains clearly the previously-known qualitative and intuitive understanding that:

- As $w_{u_1}, \cdots, w_{u_M} \longrightarrow \infty$, $J_{cl} \longrightarrow A = J_{ol}$.
- As $w_{\Delta u_1}, \cdots, w_{w_{\Delta u_M}} \longrightarrow \infty$, $J_{cl} \longrightarrow A = J_{ol}$.

VI. ILLUSTRATIVE EXAMPLES

The application of the approach described above to several specific simple plants is presented in this section.

A. Examples

Example 1. A minimum-phase, asymptotically stable, first-order, linear plant:

$$A = [0.905]\,,\ b = [1.903]\,,\ c = [1]$$

which has the transfer function

$$G(z) = \frac{1.903}{z - 0.905}$$

Example 2. A non-minimum phase, asymptotically stable, second-order, linear plant:

$$A = \begin{bmatrix} 0 & 1 \\ -0.125 & 0.75 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 2 & -1 \end{bmatrix};$$

$$G(z) = \frac{2 - z}{(z - 0.5)(z - 0.25)}$$

Example 3. A non-minimum phase, asymptotically stable, second-order, linear plant:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} 2 & 1 \end{bmatrix};$$

$$G(z) = \frac{2 + z}{z(z - 0.5)}$$

Example 4. A minimum phase, unstable, second-order, linear plant:

$$A = \begin{bmatrix} 0 & 1 \\ -1.6 & 2.8 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} -0.5 & 1 \end{bmatrix};$$

$$G(z) = \frac{z - 0.5}{(z - 2)(z - 0.8)}$$

Example 5. A non-minimum phase, unstable, second-order, linear plant:

$$A = \begin{bmatrix} 0 & 1 \\ -1.6 & 2.8 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ c = \begin{bmatrix} -1.5 & 1 \end{bmatrix};$$

$$G(z) = \frac{z - 1.5}{(z - 2)(z - 0.8)}$$

Example 6. A minimum phase, asymptotically stable, fourth-order, linear plant:

$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ b = \begin{bmatrix} 1.9 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ c = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix};$$

$$G(z) = \frac{1.9}{z^3(z - 0.9)}$$

B. Results

Results of the application of the approach to the plant examples are presented in Figure 1 and Tables III-VIII.

C. Discussions

Figure shows how the closed-loop eigenvalue(s) of Examples 1-5 and the dominant closed-loop eigenvalue of Example 6 change(s) with P. When $w_{u_1} = w_{\Delta u_1} = 0$ and $M = 1$, as P increases, the closed-loop eigenvalue(s) move towards the open-loop eigenvalue(s), irrespective of the value of β that sets the speed of the reference trajectory. Furthermore, if a prediction horizon much longer than the reference-trajectory time-constant is used, the value of the reference-trajectory time constant has little effect on the closed-loop performance (locations of the closed-loop eigenvalues). When $P = 1$, r closed-loop eigenvalues are placed at $z = -\beta$, and the remaining $(n - r)$ eigenvalues at the plant zeros. For this

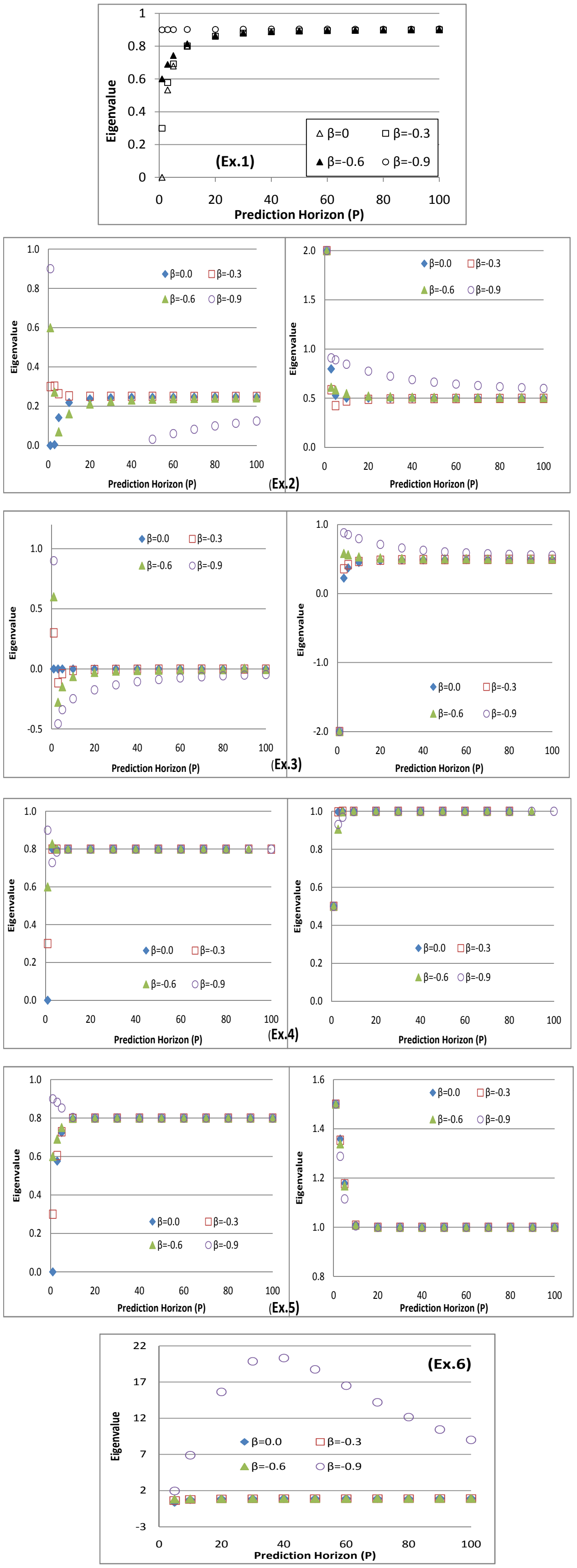

Fig. 1. Closed-loop eigenvalue(s) of Examples 1-5 and the dominant closed-loop eigenvalue of Example 6 for several values of P and β [$M=1$, $w_{u_1}=0$, $w_{\Delta u_1}=0$, $w_{y_r}=\cdots=w_{y_P}=1$].

TABLE III

RESULTS FOR EXAMPLE 1.

(A) Closed-loop eigenvalue and the performance index J (with $y_{sp}=0$ and $x=5$) for several values of M, $\beta=-0.3$, and $P=7$, $w_{u_1}=0$, $w_{\Delta u_1}=0$, $w_{y_r}=\cdots=w_{y_P}=1$.

M	1	2	3	4	5
Eigenvalues	0.75	0.20	0.30	0.30	0.30
J	15.7928	0.59747	0.04267	0.00274	0.00015

M	6	7
Eigenvalues	0.30	0.30
J	0.00005	0.00000

(B) Closed-loop eigenvalue for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta=0.0$, $P=7$, $M=3$, $w_{y_r}=\cdots=w_{y_P}=1$.

w_{u_1}	Eigenvalue	$w_{\Delta u_1}$	Eigenvalue
0	0.00000	0	0.00000
1	0.17015	1	0.20051
2	0.26461	2	0.26825
3	0.32912	3	0.310569
4	0.37778	4	0.342283

TABLE IV

RESULTS FOR EXAMPLE 2.

(A) Closed-loop eigenvalues for several values of M, $\beta=-0.3$, and $P=7$, $w_{u_1}=0$, $w_{\Delta u_1}=0$, $w_{y_r}=\cdots=w_{y_P}=1$.

M	1	3	7
Eigenvalues	0.2553, 0.4509	0.2359, 0.2998	0.2373, 0.3000

(B) Closed-loop eigenvalues for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta=0.0$, $P=7$, $M=3$, $w_{y_r}=\cdots=w_{y_P}=1$.

w_{u_1} ($w_{\Delta u_1}$=0)	Eigenvalues	$w_{\Delta u_1}$ (w_{u_1}=0)	Eigenvalues
0	0.0000, 0.5013	0	0.0000, 0.5013
1	0.0495, 0.5010	1	0.0800, 0.5008
2	0.0821, 0.5009	2	0.1151, 0.5011
3	0.1054, 0.5007	3	0.1362, 0.5016
4	0.1230, 0.5006	4	0.1506, 0.5019

reason, prediction horizons greater than one should be used for non-minimum-phase plants (Examples 2, 3 and 5; Figure 1) to ensure closed-loop stability. When a plant (such as Example V) is unstable and non-minimum-phase, prediction and control horizons much longer that those that are adequate to stabilize unstable or non-minimum-phase plants, should be used to ensure that the closed-loop system is asymptotically stable (e.g., $P=11$ and $M=10$ for Example V). As the control horizon is increased from one, as shown in Table III(A), the dominant eigenvalue of the closed-loop system initially moves towards the origin and then away from the origin to a location that does not change with a further increase in the control horizon. Note that the value of the performance index, J, evaluated at the minimizing u, as expected, decreases monotonically as M increases. In other words, J monotonically decreases with M, while the eigenvalue first decreases, then increases, and finally does not change. Tables III(B)-VIII(B) show how closed-loop eigenvalue(s) vary as w_{u_1} and $w_{\Delta u_1}$ are increased while $\beta=0.0$, $P=7$, $M=3$, and $w_{y_r}=\cdots=w_{y_P}=1$.

VII. CONCLUDING REMARKS

This paper presented an analytical study of the effect of the MPC tunable parameters over a wide range, on the closed-loop performance quantified in terms of the location(s) of closed-loop eigenvalue(s) of a large set of widely common

TABLE V

RESULTS FOR EXAMPLE 3.

(A) Closed-loop eigenvalues for several values of M, $\beta = -0.3$, and $P = 7$, $w_{u_1} = 0$, $w_{\Delta u_1} = 0$, $w_{y_r} = \cdots = w_{y_P} = 1$.

M	1	3	7
Eigenvalues	-0.0217, 0.4477	0.3012, -0.6495	0.3000, -0.6953

(B) Closed-loop eigenvalues for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta = 0.0$, $P = 7$, $M = 3$, $w_{y_r} = \cdots = w_{y_P} = 1$.

w_{u_1} ($w_{\Delta u_1}$=0)	Eigenvalues	$w_{\Delta u}$ (w_{u_1}=0)	Eigenvalues
0	0.0000, 0.5011	0	0.0000, 0.5011
1	0.0000, 0.5008	1	0.0000, -0.0888
2	0.0000, 0.5006	2	0.0000, 0.0276
3	0.0000, 0.5005	3	0.0000, 0.0919
4	0.0000, 0.5004	4	0.0000, 0.1353

TABLE VI

RESULTS FOR EXAMPLE 4.

(A) Closed-loop eigenvalues for several values of M, $\beta = -0.3$, and $P = 7$, $w_{u_1} = 0$, $w_{\Delta u_1} = 0$, $w_{y_r} = \cdots = w_{y_P} = 1$.

M	1	3	6
Eigenvalues	0.975 ± 0.075i	0.384 ± 0.050	0.300, 0.500

(B) Closed-loop eigenvalues for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta = 0.0$, $P = 7$, $M = 3$, $w_{y_r} = \cdots = w_{y_P} = 1$.

w_{u_1} ($w_{\Delta u_1}$=0)	Eigenvalues	$w_{\Delta u_1}$ (w_{u_1}=0)	Eigenvalues
0	0.0000, 0.4809	0	0.0000, 0.4908
1	0.4976, 0.6130	1	0.667 ± 0.214i
2	0.5660, 0.6468	2	0.737 ± 0.211i
3	0.6151, 0.6584	3	0.782 ± 0.199i
4	0.659 ± 0.011i	4	0.813 ± 0.184i

TABLE VII

RESULTS FOR EXAMPLE 5.

(A) Closed-loop eigenvalues for several values of M, $\beta = -0.3$, and $P = 7$, $w_{u_1} = 0$, $w_{\Delta u_1} = 0$, $w_{y_r} = \cdots = w_{y_P} = 1$.

M	1	3	6
Eigenvalues	0.8850, 1.2045	0.3177, 1.2452	0.3000, 1.4844

(B) Closed-loop eigenvalues for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta = 0.0$, $P = 7$, $M = 3$, $w_{y_r} = \cdots = w_{y_P} = 1$.

w_{u_1} ($w_{\Delta u_1}$=0)	Eigenvalues	$w_{\Delta u_1}$ (w_{u_1}=0)	Eigenvalues
0	-0.0002, 0.9239	0	-0.0002, 0.9238
1	0.3307, 0.7094	1	0.7025 ± 0.1088i
2	0.3991, 0.7262	2	0.7604 ± 0.1314i
3	0.4288, 0.7387	3	0.7936 ± 0.1277i
4	0.4452, 0.7478	4	0.8167 ± 0.1190i

TABLE VIII

RESULTS FOR EXAMPLE 6.

(A) Closed-loop eigenvalues for several values of M, $\beta = -0.3$, and $P = 7$, $w_{u_1} = 0$, $w_{\Delta u_1} = 0$, $w_{y_r} = \cdots = w_{y_P} = 1$.

M	1	3	7
Eigenvalues	-0.017 ± 0.121i	0.302 ± 0.002i	0.302 ± 0.002i
	0.077, 0.772	0.298 ± 0.002i	0.298 ± 0.002i

(B) Closed-loop eigenvalues for several values of w_{u_1} and $w_{\Delta u_1}$, and $\beta = 0.0$, $P = 7$, $M = 3$, $w_{y_r} = \cdots = w_{y_P} = 1$.

w_{u_1} ($w_{\Delta u_1}$=0)	Eigenvalues	$w_{\Delta u_1}$ (w_{u_1}=0)	Eigenvalues
0	0.0000, 0.0000	0	0.0000, 0.0000
	0.0000, 0.0000		0.0000, 0.0000
1	0.0000, 0.0000	1	0.0000, 0.0000
	0.0000, 0.1709		0.0000, 0.2980
2	-0.000 ± 0.000i	2	0.0000, 0.0000
	0.0000, 0.2669		0.0000, 0.4021
3	-0.000 ± 0.000i	3	0.0000, 0.0000
	0.0000, 0.3324		0.0000, 0.4614
4	0.0000, 0.0000	4	-0.000 ± 0.000i
	0.0000, 0.3816		0.0000, 0.5009

plants whose constraints are inactive. Symbolic manipulation capabilities of MATHEMATICA were used to obtain analytical expressions describing the dependence of closed-loop eigenvalues on the tunable parameters. This work is first to investigate how MPC tuning-parameters affect the location of the eigenvalues of the closed-loop system of a plant in the discrete-time setting. It provides theoretical basis/justification for many of the existing qualitative MPC tuning rules and propose new tuning guidelines for MPC. For example, as the prediction horizon is increased while other tunable parameters remain constant, a subset of the closed-loop eigenvalues (poles) move non-monotonically towards the open-loop eigenvalues (poles) of the plant. If a prediction horizon much longer than the reference-trajectory time-constant is used, the value of reference-trajectory time-constant has little effect on the closed-loop performance. As the weights on the magnitude or the rate of change of the manipulated input are increased, the closed-loop eigenvalues move towards the open-loop eigenvalues.

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