2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Stochastic Stability for Discrete-Time Singular Systems with Markov Jump Parameters

A. L. P. Manfrim, M. H. Terra, E. F. Costa, and J. Y. Ishihara

Abstract— **In this paper, stochastic stability for discrete-time singular systems with Markov jump parameters is addressed. We present a set of coupled generalised Lyapunov equations (CGLEs) that serves as a necessary and sufficient condition for stochastic stability. A method for solving the obtained CGLEs is also presented, based on iterations of usual descriptor Lyapunov equations. An illustrative example is included.**

I. INTRODUCTION

Very recently, singular systems with Markovian jumping parameters have received attention in the literature [15], [19], [20]. The interest in these systems is motivated by the fact that they are sufficiently specialised to yield strong results and, simultaneously, provide useful models for applications as it comprises both the singular (also called descriptor) systems and the systems with Markovian jumping parameters. Singular systems applications include aircraft modeling [18], chemical processes [9], circuit systems [13], [14], economic systems [11], large-scale interconnected systems [10], [17], mechanical engineering systems [6], power systems [16], and robotics [12]. Markov jump systems are becoming more and more popular due to their capability to describe dynamic behavior of systems with abrupt changes in their structures. For applications modeled by Markov jump systems, see [1], [4] and the references therein.

In this paper, a Lyapunov-type analysis of discrete-time singular linear systems with Markov jump parameters (SLSMJP) is developed in order to check stochastic stability. The set of coupled generalised Lyapunov equations (CGLEs) developed in this paper, and its respective solution, extend the results presented in [7] and [8] for conventional singular systems, and the results presented in [2] for conventional linear systems with Markov jump parameters (LSMJP). For computation solution of the proposed CGLE, a recursive algorithm is proposed. The algorithm has the nice property that, if a sequence generated by the recursion converges, then it converges to a solution of the CGLE, whenever it exists, disregarding stability or any other condition. For the interested reader in tests methods for solving coupled algebraic Riccati equations that are available in the literature for Markov jump linear systems, we can refer [3] and [5].

The paper is organized as follows. In Section II we present the notation. The problem statement and previous concepts are presented in Section III. The coupled generalized Lyapunov equation is addressed in Section IV. An algorithm for solving the CGLE is presented and studied in Section V. In Section VI, a numerical example is presented.

This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) Grants 03/12574-0 and 05/59548-9, and Cnpq Grant 310852/2006-4, and in part by CNPq Grant 305657/2004-6

A. L. P. Manfrim and M. H. Terra are with the Department of Electrical Engineering - University of São Paulo at São Carlos - C.P.359, São Carlos, SP, 13566-590, Brazil {amanfrim, terra}@sel.eesc.usp.br

Eduardo F. Costa is with the Depto. de Matemática Aplicada e Estatística, Universidade de São Paulo (USP), C.P. 668, 13560-970, São Carlos, SP, Brazil, and currently is an academic visitor at the Electrical and Electronic Engineering Dept., Imperial College London SW72AZ, London, UK efcosta@icmc.usp.br

João Y. Ishihara is with the Department of Electrical Engineering - University of Brasília, CEP 70.919-970, Brasília/DF, Brazil ishihara@ene.unb.br

II. NOTATION

Let $\mathbb{R}^n$ be the Euclidean linear space formed by all n-vectors. Let $\mathcal{R}^{r,n}$ (respectively, $\mathcal{R}^r$) represented the normed linear space formed by all $r \times n$ real matrices (respectively, $r \times r$) and $\mathcal{R}^{r0}$ ($\mathcal{R}^{r+}$) the closed convex cone $\{U \in R^r : U = U' \geq 0\}$, (the open cone $\{U \in R^r : U = U' > 0\}$), where U' denotes the transpose of U; $U \geq V$ ($U > V$) signifies that $U - V \in \mathcal{R}^{r0}$ ($U - V \in \mathcal{R}^{r+}$). For $U \in \mathcal{R}^{n0}$, $\sigma(U)$ stands for the maximal eigenvalue of U. Let $\mathcal{M}^{r,n}$ denote the linear space formed by a number N of matrices such that $\mathcal{M}^{r,n} = \{U = (U_1, \ldots, U_N) : U_i \in \mathcal{R}^{r,n}, i = 1, \ldots, N\}$; also, $\mathcal{M}^r \equiv \mathcal{M}^{r,r}$. We denote by $\mathcal{M}^{r0}$ ($\mathcal{M}^{r+}$) the set $\mathcal{M}^r$ when it is made up of $U_i \in \mathcal{R}^{r0}$ ($U_i \in \mathcal{R}^{r+}$) for all $i = 1, \ldots, N$. For $U \in \mathcal{M}^{n0}$, we define $||U|| = \max_{0 \leq i \leq N} \sigma(U_i)$, and we denote as $U_i(j,l)$ the j,l-element of matrix U_i.

We define the operators $\mathcal{L} : \mathcal{M}^{n0} \to \mathcal{M}^{n0}$, $\bar{\mathcal{L}} : \mathcal{M}^{n0} \to \mathcal{M}^{n0}$ and $\tilde{\mathcal{L}} : \mathcal{M}^{n0} \to \mathcal{M}^{n0}$ as follows:

$$\begin{aligned} \mathcal{L}_i(U) &= \sum_{j=1}^{N} p_{ij} U_j; \\ \bar{\mathcal{L}}_i(U) &= \sum_{j \neq i}^{N} p_{ij} U_j; \end{aligned} \tag{1}$$

for $i = 1, \ldots, N$.

Remark 1: The operator $\mathcal{L}$ is monotonically increasing in the positive semidefinite sense, i.e., for matrices U, $V \in \mathcal{M}^{n0}$,

$$\mathcal{L}(V) \leq \mathcal{L}(U) \quad \text{whenever} \quad V \leq U. \tag{2}$$

III. PROBLEM STATEMENT AND PREVIOUS CONCEPTS

We consider the discrete-time SLSMJP, described by

$$\Phi : \begin{cases} E_{\theta(k+1)}x(k+1) = A_{\theta(k)}x(k), \ k = 0, 1, \dots \\ \qquad x(0) = x_0, \quad \theta(0) = \theta_0 \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state variable, θ is the state of an underlying discrete-time homogeneous Markov chain $\Theta = \{\theta(k); k \geq 0\}$ having $\mathcal{N} = \{1, \dots, N\}$ as state space and $\mathbb{P} = [p_{ij}], \ i, j = 1, \dots, N$ as the transition matrix. The system is defined in a fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Matrices A_i, $E_i \in \mathcal{R}^n$, $i = 1, \dots, N$, belong to the collections of N real constant matrix: $A = (A_1, \dots, A_N)$ and $E = (E_1, \dots, E_N)$ may be singular, with rank$(E_i) = r_{E_i} \leq n$.

Definition 1: A matrix V is stable if and only if all its eigenvalues have magnitude less than 1.

The following notion is standard for stochastic systems, and have been considered in the literature of singular systems as well, see e.g. [19].

Definition 2: The discrete SLSMJP in (3) is said to be stochastically stable (SS) if there exist a scalar $M(x_0, \theta_0) > 0$ such that

$$\lim_{N\to\infty} \mathcal{E}\left\{ \sum_{k=0}^{N} ||x(k, x_0, \theta_0)||^2 \Big| \, x_0, \theta_0 \right\} \leq M(x_0, \theta_0) \tag{4}$$

where $x(k, x_0, \theta_0)$ denotes the solution to the System Φ at time k under the initial conditions x_0 and θ_0.

IV. Coupled Generalized Lyapunov equation

Let $x = \{x(k)\}_{k=0}^{\infty}$ be a solution to the System Φ. Consider the following Lyapunov function candidate

$$V\left(x(k), \theta(k)\right) := x'(k)E'_{\theta(k)}X_{\theta(k)}E_{\theta(k)}x(k) \geq 0 \tag{5}$$

where $X_i \geq 0$ for $\theta(k) = i$, $i = 1, \dots, N$. Then, for a set of matrices $X_i = X_i' \geq 0$, $X_i \in \mathcal{R}^n$, $R_i \in \mathcal{R}^{(n-r), n}$, and $W \in \mathcal{M}^{n0}$ we define the following CGLEs as follows

$$\begin{aligned} A_i' \left(\sum_{j=1}^{N} p_{ij} X_j \right) A_i + A_i' \left(\sum_{j=1}^{N} p_{ij} \bar{E}_j \right) R_i \\ + R_i' \left(\sum_{j=1}^{N} p_{ij} \bar{E}_j' \right) A_i - E_i' X_i E_i + W_i = 0. \end{aligned} \tag{6}$$

for $i = 1, \dots, N$, where E_i, $A_i \in \mathcal{R}^n$, and $\bar{E}_i \in \mathcal{R}^{n, (n-r)}$ is a full column rank matrix such that $E_i'\bar{E}_i = 0$ and $r = \text{rank}(E_i)$. With these assumptions, we are in position to state the following theorem

Theorem 1: The System Φ is stochastically stable if and only if for $W_i \in \mathcal{R}^{n0}$ there exist solutions $E_i'X_iE_i \geq 0$ and R_i, $i = 1, \dots, N$, to the CGLEs in (6).

Proof: (Necessity). Consider the sequence $q_T(k, x, i)$, $k \geq 0$, defined by

$$\begin{aligned} q_T(k, x, i) :=& \mathcal{E}\Bigg\{ \sum_{t=k}^{k+T} x'(t)W_{\theta(t)}x(t) \Big| \, x(k) = x, \\ & \qquad \theta(k) = i \Bigg\}. \end{aligned} \tag{7}$$

We can show that there exist $X_i(T)$, $i = 1, \dots, N$, $T \geq 0$, satisfying

$$x(k)\, E_i' X_i(T) E_i \, x(k) = q_T(k, x, i), \tag{8}$$

for any $x \in \mathbb{R}^n$, $i = 1, \dots, N$, and $k \geq 0$. Since Φ is stochastically stable, we can evaluate

$$\begin{aligned} q_T(k, x, i) =& \mathcal{E}\Bigg\{ \sum_{t=k}^{k+T} x'(t)W_{\theta(t)}x(t) \Big| \, x(k) = x, \\ & \qquad \theta(k) = i \Bigg\} \\ \leq& ||W||\mathcal{E}\Bigg\{ \sum_{t=k}^{k+T} x'(t)W_{\theta(t)}x(t) \Big| \, x(k) = x, \\ & \qquad \theta(k) = i \Bigg\} < \infty, \end{aligned} \tag{9}$$

Equations (8) and (9) yield

$$\lim_{T\to\infty} x'E_i'X_i(T)E_ix \leq \infty, \tag{10}$$

for any x and $i = 1, \dots, N$, hence

$$X_i = \lim_{T\to\infty} X_i(T) \in \mathcal{R}^n.$$

¿From (7) and (9) we can write

$$\begin{aligned} q_T(0, x(0), \theta(0)) &= x'(0)E'_{\theta(0)}X_{\theta(0)}(T)E_{\theta(0)}x(0) \\ &= \mathcal{E}\left\{ \sum_{t=0}^{T} x'(t)W_{\theta(t)}x(t) \Big| \, x(0), \theta(0) \right\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} & q_{T-1}(1, x(1), \theta(1)) \\ &\quad = x'(1)E'_{\theta(1)}X_{\theta(1)}(T-1)E_{\theta(1)}x(1) \\ &\quad = x'(1)E'_{\theta(1)}X_{\theta(1)}(T-1)E_{\theta(1)}x(1) \\ &\qquad + x'(1)E'_{\theta(1)}\bar{E}_{\theta(1)}R_{\theta(0)}x(0) \\ &\qquad + x'(0)R'_{\theta(0)}\bar{E}'_{\theta(1)}E_{\theta(1)}x(1) \\ &\quad = \mathcal{E}\left\{ \sum_{t=1}^{T} x'(t)W_{\theta(t)}x(t) \Big| \, x(1), \theta(1) \right\}. \end{aligned} \tag{12}$$

By subtracting (12) from (11), we have that

$$\begin{aligned} & x'(1)E'_{\theta(1)}X_{\theta(1)}(T-1)E_{\theta(1)}x(1) \\ &\quad + x'(1)E'_{\theta(1)}\bar{E}_{\theta(1)}R_{\theta(0)}x(0) \\ &\quad + x'(0)R'_{\theta(0)}\bar{E}'_{\theta(1)}E_{\theta(1)}x(1) \\ &\quad - x'(0)E'_{\theta(0)}X_{\theta(0)}(T)E_{\theta(1)}x(1) \\ &\qquad = -\mathcal{E}\left\{ x'(0)W_{\theta(0)}x(0) \Big| \, x(0), \theta(0) \right\} \end{aligned} \tag{13}$$

and taking the expectation in (13) yelds

$$
\begin{aligned}
&\mathcal{E}\Big\{x'(0)\Big[A'_{\theta(0)}X_{\theta(1)}(T-1)A_{\theta(0)} \\
&\quad + A'_{\theta(0)}\bar{E}_{\theta(1)}R_{\theta(0)} + R'_{\theta(0)}\bar{E}'_{\theta(1)}A_{\theta(0)} \\
&\quad - E'_{\theta(0)}X_{\theta(0)}(T)E_{\theta(0)}\Big]x(0)\,\Big|\,x(0),\theta(0)=i\Big\} \\
&= -x'(0)W_i x(0),
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&x'(0)\Bigg[A'_i\sum_{j=1}^{N}\left(p_{ij}X_j(T-1)\right)A_i \\
&\quad + A'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}_j\right)R_i + R'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}'_j\right)A_i \\
&\quad - E'_iX_i(T)E_i\Bigg]x(0) = -x'(0)W_i x(0),
\end{aligned}
\tag{14}
$$

$x \in \mathbb{R}$, $i \in \mathcal{N}$, which

$$
\begin{aligned}
&A'_i\sum_{j=1}^{N}\left(p_{ij}X_j(T-1)\right)A_i + A'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}_j\right)R_i \\
&\quad + R'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}'_j\right)A_i - E'_iX_i(T)E_i + W_i = 0
\end{aligned}
\tag{15}
$$

Finally, we take the limit with $T \to \infty$ in (15), leads to

$$
\begin{aligned}
&A'_i\sum_{j=1}^{N}\left(p_{ij}X_j\right)A_i + A'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}_j\right)R_i \\
&\quad + R'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}'_j\right)A_i - E'_iX_iE_i + W_i = 0
\end{aligned}
\tag{16}
$$

when $W_i > 0$ for any $i = 1, \ldots, N$,

$$
\begin{aligned}
&A'_i\sum_{j=1}^{N}\left(p_{ij}X_j\right)A_i + A'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}_j\right)R_i \\
&\quad + R'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}'_j\right)A_i - E'_iX_iE_i < 0
\end{aligned}
\tag{17}
$$

(Sufficiency). For $x_0 = 0$, we can consider $x(k) = 0$, $k = 0, 1, \ldots$ and so, (4) holds trivially. Therefore, we consider $x_0 \neq 0$ in what follows. We can rewrite (6) equivalently as

$$
\begin{aligned}
&A'_i\sum_{j=1}^{N}\left(p_{ij}X_j\right)A_i + A'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}_j\right)R_i \\
&\quad + R'_i\sum_{j=1}^{N}\left(p_{ij}\bar{E}'_j\right)A_i - E'_iX_iE_i + W_i = 0
\end{aligned}
\tag{18}
$$

where we set stochastic Lyapunov function (5), we can evaluate $W_i = W'_i > 0$ conveniently. For

$$
\begin{aligned}
&\mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)\,\Big|\,x(k)=x,\,\theta(k)=i\right\} \\
&= \mathcal{E}\Big\{x'(k+1)E'_{\theta(k+1)}X_{\theta(k+1)}E_{\theta(k+1)}x(k+1) \\
&\quad + x'(k+1)E'_{\theta(k+1)}\bar{E}_{\theta(k+1)}R_i x \\
&\quad + x'R'_i\bar{E}'_{\theta(k+1)}E_{\theta(k+1)}x(k+1) \\
&\quad \Big|\,x(k)=x,\theta(k)=i\Big\} \\
&= \mathcal{E}\Big\{x'A'_iX_{\theta(k+1)}A_i x + x'A'_i\bar{E}_{\theta(k+1)}R_i x \\
&\quad + x'R'_i\bar{E}'_{\theta(k+1)}A_i x\,\Big|\,x(k)=x,\theta(k)=i\Big\}
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)\,|\,x(k)=x,\theta(k)=i\right\} \\
&= x'A'_i\mathcal{E}\left\{X_{\theta(k+1)}\,|\,x(k)=x,\theta(k)=i\right\}A_i x \\
&\quad + x'A'_i\mathcal{E}\left\{\bar{E}_{\theta(k+1)}\,|\,x(k)=x,\theta(k)=i\right\}R_i x \\
&\quad + x'R'_i\mathcal{E}\left\{\bar{E}'_{\theta(k+1)}\,|\,x(k)=x,\theta(k)=i\right\}A_i x \\
&= x'\left[A'_i\left(\sum_{j=1}^{N}p_{ij}X_j\right)A_i\right. \\
&\quad \left. + A'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}_j\right)R_i + R'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}'_j\right)A_i\right]x,
\end{aligned}
\tag{20}
$$

consider the function $\Delta V : \mathcal{R}^n \times \mathcal{N} \to \mathcal{R}^+$ defined by

$$
\begin{aligned}
&\Delta V(x(k)=x,\theta(k)=i) \\
&= \mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)|\,x(k)=x,\theta(k)=i\right\} \\
&\quad - V\left(x(k)=x,\theta(k)=i\right)
\end{aligned}
$$

then,

$$
\begin{aligned}
&\Delta V\left(x(k)=x,\theta(k)=i\right) \\
&= \mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)|x(k)=x,\theta(k)=i\right\} \\
&\quad - V\left(x(k)=x,\theta(k)=i\right) \\
&= x'\left[A'_i\left(\sum_{j=1}^{N}p_{ij}X_j\right)A_i + A'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}_j\right)R_i\right. \\
&\quad \left. + R'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}'_j\right)A_i - E'_iX_iE_i\right]x \\
&= -x'W_i x.
\end{aligned}
\tag{21}
$$

We now split the proof in two parts. First, assume x and i are such that $x'E'_iX_iE_ix > 0$, and consequently $V(\cdot) \neq 0$. Defining the scalar

$$
\alpha := 1 - \min_{i=1,\ldots,N}\left\{\frac{\lambda_{\min}(W_i)}{\lambda_{\max}\left(E'_{\theta(k)}X_{\theta(k)}E_{\theta(k)}\right)}\right\}
\tag{22}
$$

we have that

$$\frac{\mathcal{E}\left\{V\left(\theta(k+1),x(k+1)\right)\big|\,x(k),\theta(k)\right\}-V\left(x(k),\theta(k)\right)}{V\left(\theta(k),x(k)\right)}$$
$$= -\frac{x'(k)W_i x(k)}{x'(k)E'_{\theta(k)}X_{\theta(k)}E_{\theta(k)}x(k)}$$
$$\leq \alpha-1 \tag{23}$$

Note that, when W_i and X_i are definite positive matrices, then $0<\alpha<1$, from (23) we have that

$$\frac{\mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)\big|\,x(k),\theta(k)\right\}}{V\left(x(k),\theta(k)\right)}\leq\alpha \tag{24}$$

and from (24) we conclude that

$$\begin{aligned}0<\mathcal{E}\left\{V\left(x(k+1),\theta(k+1)\right)\big|\,x(k),\theta(k)\right\}\\ \leq\alpha\,V\left(x(k),\theta(k)\right)\end{aligned} \tag{25}$$

where $0<\alpha<1$. Now, for $xE'_iX_iE_ix=0$, we have that $V(\cdot)=0$ and (25) holds. Employing basic properties of the operator $\mathcal{E}\{\cdot\}$, from (25) we obtain with $k=0$

$$\mathcal{E}\left\{V\left(x(1),\theta(1)\right)\big|\,x(0),\theta(0)\right\}\leq\alpha\,V\left(x(0),\theta(0)\right)$$

and with $k=1$

$$\begin{aligned}&\mathcal{E}\left\{V\left(x(2),\theta(2)\right)\big|\,x(0),\theta(0)\right\}\\ &\quad=\mathcal{E}\left\{V\left(x(2),\theta(2)\right)\big|\,x(0),x(1),\theta(0),\theta(1)\right\}\\ &\quad\leq\alpha\,\mathcal{E}\left\{V\left(x(1),\theta(1)\big|\,x(0),\theta(0)\right)\right\}\end{aligned}$$

yielding

$$\mathcal{E}\left\{V\left(x(2),\theta(2)\right)\big|\,x(0),\theta(0)\right\}\leq\alpha^2\,V\left(x(0),\theta(0)\right). \tag{26}$$

Similarly as above, for a general $k\geq 0$, we have

$$\mathcal{E}\left\{V\left(x(k),\theta(k)\right)\big|\,x(0),\theta(0)\right\}\leq\alpha^k\,V\left(x(0),\theta(0)\right) \tag{27}$$

hence,

$$\begin{aligned}&\mathcal{E}\left\{\sum_{k=0}^{T}V\left(x(k),\theta(k)\right)\big|\,x(0),\theta(0)\right\}\\ &\quad=\mathcal{E}\left\{V\left(x(0),\theta(0)\right)+\ldots\right.\\ &\qquad\left.+V\left(x(T),\theta(T)\right)\big|\,x(0),\theta(0)\right\}\\ &\quad=\mathcal{E}\left\{V\left(x(0),\theta(0)\right)\big|\,x(0),\theta(0)\right\}+\ldots\\ &\qquad+\mathcal{E}\left\{V\left(x(T),\theta(T)\right)\big|\,x(0),\theta(0)\right\}\\ &\quad\leq V\left(x(0),\theta(0)\right)+\alpha^1\,V\left(x(0),\theta(0)\right)+\ldots\\ &\qquad+\alpha^T\,V\left(x(0),\theta(0)\right)\\ &\quad=\left(1+\alpha^1+\alpha^2+\ldots+\alpha^T\right)\,V\left(x(0),\theta(0)\right)\end{aligned} \tag{28}$$

and we evaluate

$$\begin{aligned}&\lim_{T\to\infty}\left(\mathcal{E}\left\{\sum_{k=0}^{T}V\left(x(k),\theta(k)\right)\big|\,x(0),\theta(0)\right\}\right)\\ &\quad\leq\lim_{T\to\infty}\frac{\left(1-\alpha^T\right)}{1-\alpha}\,V\left(x(0),\theta(0)\right)\\ &\quad=\left(\frac{1}{1-\alpha}\right)\,V\left(x(0),\theta(0)\right)\\ &\quad=\left(\frac{1}{1-\alpha}\right)\,x'(0)E'_{\theta(0)}X_{\theta(0)}E_{\theta(0)}x(0).\end{aligned} \tag{29}$$

Remember (7), (8), and that by hypotheses $W_i=W'_i>0$, $i=1,\ldots,N$. Then, we have that

$$\begin{aligned}&x(k)E'_iX_i(T)E_ix(k)\\ &\qquad=\mathcal{E}\left\{\sum_{t=k}^{k+T}x'(t)W_{\theta(t)}x(t)\big|\,x,i\right\}\end{aligned}$$

then, we can evaluate

$$\begin{aligned}&\lim_{T\to\infty}\mathcal{E}\left\{\sum_{k=0}^{T}x'(k)x(k)\big|\,x(0),\theta(0)\right\}\\ &\leq\lim_{T\to\infty}\mathcal{E}\left\{\sum_{k=0}^{T}x'(k)W_{\theta(k)}x(k)\big|\,x(0),\theta(0)\right\}\\ &\leq\lim_{T\to\infty}\mathcal{E}\left\{\sum_{k=0}^{T}x'(k)E'_{\theta(k)}X_{\theta(k)}E_{\theta(k)}x(k)\big|\,x(0),\theta(0)\right\}\\ &=\lim_{T\to\infty}\mathcal{E}\left\{\sum_{k=0}^{T}V\left(x(k),\theta(k)\right)\big|\,x(0),\theta(0)\right\}\\ &\leq\left(\frac{1}{1-\alpha}\,x'(0)E'_{\theta(0)}X_{\theta(0)}E_{\theta(0)}x(0)\right)\\ &:=\ M(x(0),\theta(0))\end{aligned} \tag{30}$$

■

Remark 2: It is a straightforward task to check that, when $N=1$, the CGLE given in (6) reduces to the GLE of standard singular linear systems, see for instance [8]. Note that, in this situation, $\mathbb{P}=1$, and we can rewrite (6) as

$$\begin{aligned}A'_1X_1A_1+A'_1\bar{E}_1R_1+R'_1\bar{E}'_1A_1\\ -E'_1X_1E_1+W_1=0.\end{aligned}$$

Remark 3: The following set of coupled generalized Lyapunov inequality (CGLI) has been considered in [19],

$$\begin{aligned}E'_iX_iE_i\geq 0,\\ A'_i\left(\sum_{j=1}^{N}p_{ij}X_j\right)A_i-E'_iX_iE_i<0.\end{aligned}$$

From the proposed CGLE in this paper, we can further associate for stochastic stability characterization, an alternative CGLI described as following

$$\begin{aligned}A'_i\left(\sum_{j=1}^{N}p_{ij}X_j\right)A_i+A'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}_j\right)R_i\\ +R'_i\left(\sum_{j=1}^{N}p_{ij}\bar{E}'_j\right)A_i-E'_iX_iE_i<\ 0.\end{aligned} \tag{31}$$

V. An algorithm for solving the CGLE

In this section we consider a method for solving the following CGLE. The CGLE consists of a set of CGLs with unknowns $X_i\in\mathcal{R}^n$, $i=1,\ldots,N$ and associated matrices $A_1,\ldots,A_N$ and similarly E, $\bar{E}$, R and W, as in (6).

Note that modifying the CGLEs, adding and subtracting $\kappa_i p_{ii} A_i' X_i A_i$, we obtain

$$\begin{aligned}&\kappa_i p_{ii} A_i' X_i A_i - E_i' X_i E_i + W_i \\ &\quad + A_i' \left[(1-\kappa_i) X_i \sum_{j=1, j\neq i}^{N} p_{ij} X_j \right] A_i \\ &\quad + A_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right] R_i + R_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right]' A_i = 0.\end{aligned}$$

The method for solving the CGLE generates a sequence $X(k) \in \mathcal{M}^{n0}$, $k \geq 0$, as follows:

Step 1. Set $X(0) \in \mathcal{M}^{n0}$ and $W \in \mathcal{M}^{n0}$ as $X_i(0) = 0$ and $W_i = I$, $i \geq 1$. For each $i \in \mathcal{N}$, set κ_i as the largest scalar such that the matrix $(\kappa_i p_{ii})^{1/2} A_i$ is stable, see Definition 1; if $\kappa_i > 1$ then set $\kappa_i = 1$. Let $k = 1$.

Step 2. For each $i \in \mathcal{N}$, solve[1] the following equality in the variable $R_i(k)$,

$$\begin{aligned}0 = R_i' &\left[\sum_{j=1}^{N} p_{ij} \bar{E}_j' \right] A_i + A_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right] R_i \\ &+ \kappa_i p_{ii} A_i' X_i(k-1) A_i - E_i' X_i(k-1) E_i + \\ &+ A_i' \tilde{\mathcal{L}}_i(X(k-1)) A_i + W_i\end{aligned} \tag{32}$$

We define the operator $\tilde{\mathcal{L}} : \mathcal{M}^{n0} \to \mathcal{M}^{n0}$ as follows: where

$$\tilde{\mathcal{L}}_i(X) = (1-\kappa_i) p_{ii} X_i + \bar{\mathcal{L}}_i(X),$$

for $i = 1, \ldots, N$.

Step 3. For each $i \in \mathcal{N}$, solve the following generalized algebraic Lyapunov equation in the variable $X(k)$,

$$\kappa_i p_{ii} A_i' X_i(k) A_i - E_i' X_i(k) E_i + \tilde{Q}_i = 0 \tag{33}$$

where

$$\begin{aligned}\tilde{Q}_i &= A_i' \tilde{\mathcal{L}}_i(X(k-1)) A_i + W_i \\ &\quad + A_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right] R_i + R_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right]' A_i.\end{aligned}$$

Step 4. If $X(k)$ satisfies the stopping criterion (e.g. $\|X(k) - X(k-1)\| \leq \epsilon$ for some pre-specified ϵ), then stop, else set $k = k+1$ and go to Step 2.

Theorem 2: If $X(k)$ converges to some $X_\infty \in \mathcal{M}^{n0}$ as $k \to \infty$ then X_∞ is a solution to the CGLE.

Proof: Assuming that $X(k)$ converges to some X_∞ as $k \to \infty$, employing (33) we have that

$$\kappa_i p_{ii} A_i' X_\infty(k) A_i - E_i' X_\infty(k) E_i + \tilde{Q}_i = 0 \tag{34}$$

where

$$\begin{aligned}\tilde{Q}_i &= A_i' \tilde{\mathcal{L}}_i(X_\infty) A_i + W_i \\ &\quad + A_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right] R_i + R_i' \left[\sum_{j=1}^{N} p_{ij} \bar{E}_j \right]' A_i\end{aligned}$$

that satisfies the CGLE in (6). ■

VI. Numerical Example

Example 1: Consider the System Φ for the following numerical example, with $N = 2$,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix},$$
$$W_1 = W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \bar{E}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \bar{E}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\mathbb{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}$, in $k = 10$ steps R converges to $R_1 = \begin{bmatrix} 0 & 0.6235 & -5.6111 \end{bmatrix}$ and $R_2 = \begin{bmatrix} -90 & 0 & -10 \end{bmatrix}$, and X converges to semidefinite matrices

$$X_1 = \begin{bmatrix} 1.1942 & 0 & 0 \\ 0 & 1.0225 & 0 \\ 0 & 0 & 1.1111 \end{bmatrix}$$
$$\text{and} \qquad X_2 = \begin{bmatrix} 183.4164 & 0 & 0 \\ 0 & 0.7978 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then for this transition matrix of Markov chain, the system is stochastically stable.

For $\mathbb{P} = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}$, in $k = 13$ steps and R converges to

$$R_1 = \begin{bmatrix} 0 & 454.5 & -50.5 \end{bmatrix}$$
$$\text{and} \quad R_2 = \begin{bmatrix} -0.1235 & 0 & -1.1111 \end{bmatrix},$$

and X converges to

$$X_1 = \begin{bmatrix} 1.1942 & 0 & 0 \\ 0 & 82.8200 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$
$$\text{and} \qquad X_2 = \begin{bmatrix} 21.4411 & 0 & 0 \\ 0 & -8.0911 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then, for this second transition matrix of Markov chain, the system is not stochastically stable.

[1]Note that (32) forms a set of equations that are linear in the variables $R_i[j, \ell]$, which can be easily solved by Gaussian elimination.

VII. CONCLUSION

In this paper, a new characterization of stochastic stability for discrete-time SLSMJP was proposed. The set of coupled generalized Lyapunov equations (CGLEs), developed in this paper, reduces to the usual generalized Lyapunov equation for descriptor systems when the Markov chain is degenerated to the case $N = 1$ [8], and to a standard coupled Lyapunov equation when there is no singularity, i.e., E is a positive definite matrix [4]. We further derived a recursive algorithm for solving the CGLEs, and so, the issue of practical implementation of the proposed stability test was also dealt with.

REFERENCES

[1] E. K. Boukas. *Stochastic Switching Systems Analysis and Design*. Birkhauser, Boston, 2005.

[2] E. F. Costa and J. B. R. do Val. Weak detectability and the linear-quadratic control problem of discrete-time Markov jump linear systems. *International Journal of Control*, 75:1282–1292, 2002.

[3] E. F. Costa and J. B. R. do Val. An algorithm for solving a perturbed algebraic Riccati equation. *European Journal of Control, Paris*, 10(6):576–580, 2004.

[4] O. L. V. Costa, M. Fragoso, and R. P. Marques. *Discrete-time Markov jump linear systems: Probability and its applications*. Spriger, 2005.

[5] J.B.R. do Val and E. F. Costa. Numerical solution for the linear-quadratic control problem of Markov jump linear systems and a weak detectability concept. *Journal of Optimization Theory and Applications*, 114(1):69–96, 2002.

[6] H. Hemami and B. F. Wyman. Modeling and control of constrained dynamic systems with application to biped locomotion in the frontal plane. *IEEE Trans. Automat. Control*, 24:526–535, 1979.

[7] J. Y. Ishihara and M. H. Terra. On the lyapunov theorem for singular systems. *IEEE Transactions on Automatic Control, Estados Unidos*, 47:1926–1930, 2002.

[8] J. Y. Ishihara and M. H. Terra. A new lyapunov equation for discrete-time descriptor systems. *American Control Conference - ACC2003, 2003*, 6:5078–5082, 2003.

[9] A. Kumar and P. Daoutidis. Feedback control of nonlinear differential-algebraic equation systems. *AIChE Journal*, 41:619–636, 1995.

[10] D. G. Luenberger. Time-invariante descriptor systems. *Automatica*, 14, 1978.

[11] D. G. Luenberger and Arbel. Singular dynamic leontief systems. *Econometrica*, 45, 1977.

[12] J. K. Mills and A. A. Goldenberg. Force and position control of manipulators during constrained motion tasks. *IEEE Trans. Robot. Automat.*, 5:30–46, 1989.

[13] R. W. Newcomb. The semistrate description of nonlinear time-variable circuits. *IEEE Trans. Circuits Syst.*, 28:62–71, 1981.

[14] R. W. Newcomb and B. Dziurla. Some circuits and systems applications of semistate theory. *Circuits, Syst, Sig. Proces.*, 8:235–260, 1989.

[15] J. Raouf and E. K. Boukas. Stabilization of discontinuous singular systems with Markovian switching and saturating inputs. *Proceedings of the 2007 American Control Conference*, pages 2442–2447, 2007.

[16] B. Scott. *Power system dynamic response calculations*. Proc. IEEE, 1979.

[17] S. Singh and R. W. Liu. Existence of state equation representation of linear large-scale dynamical systems. *IEEE Trans. Circuits Syst.*, 20:239–246, 1973.

[18] B. L. Stevens and F. L. Lewis. *Aircraft Modeling, Dynamics and Control*. New York: Wiley, 1991.

[19] S. Xu and J. Lam. *Robust Control and Filtering of Singular Systems*. Springer, 2006.

[20] F. Yan-Ming, D. Guang-Ren, and Q. Lei. Robust guaranteed cost observer design for uncertain descriptor systems with state delays and markovian jumping parameters. *IMA Journal of Mathematical Control and Information*, pages 1–10, 2005.