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On Stabilization of Nonlinear Systems Affine in Control

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***Abstract*— In a recent paper, we developed a structural decomposition for MIMO nonlinear systems that are affine in control but otherwise general. In this paper we exploit the properties of such a decomposition for the purpose of solving the stabilization problem. In particular, this decomposition simplifies the conventional backstepping design, motivates a new backstepping design procedure that is able to stabilize some systems on which the conventional backstepping is not applicable, and allows the stabilization of non-square systems.**

I. Introduction and Problem Statement

Since the development of the normal form for affine-in-control nonlinear systems [1–14], there have been a surge of works that explore the nonlinear analogous of linear systems structural properties, in establishing the nonlinear equivalence of linear system structures, in identifying more intricate structural properties that linear systems do not display, and in applying the discovered structural properties to solve nonlinear control problems (see, e.g., [15–21]).

In this paper, we consider a nonlinear system of the form

$$\left\{\begin{array}{rcl} \dot{x} & = & f(x)+g(x)u, \\ y & = & h(x), \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output, respectively, and the mappings f, g and h are smooth with $f(0)=0$ and $h(0)=0$.

The work on structural decomposition of such a system started with the definition of relative degrees, the nonlinear equivalence of infinite zeros, and the normal form decomposition for the SISO case, *i.e.*, $m=p=1$ [4]. This definition of relative degrees was soon generalized to the case with $m=p>1$. In general, the system (1) with $m=p\geq 1$ has a vector relative degree [6, 8] $\{r_1, r_2, \cdots, r_m\}$ at $x=0$ if $L_{g_j}L_f^k h_i(x)=0$, $0\leq k<r_i-1$, $1\leq i,j\leq m$ in a neighborhood of $x=0$, and $\det A(0)\neq 0$, where $A(x)=\{L_{g_j}L_f^{r_i-1}h_i(x)\}_{m\times m}$. If the system (1) has a vector relative degree $\{r_1, r_2, \cdots, r_m\}$ at $x=0$, and with the assumption of the distribution spanned by the row vectors of $g(x)$ being involutive in a neighborhood of $x=0$, it can be described by

$$\left\{\begin{array}{rcll} \dot{\eta} & = & f_0(x), & \\ \dot{\xi}_{i,j} & = & \xi_{i,j+1}, & j=1,2,\cdots,r_i-1, \\ \dot{\xi}_{i,r_i} & = & v_i, & \\ y_i & = & \xi_{i,1}, & i=1,2,\cdots,m, \end{array}\right. \tag{2}$$

where $v_i=a_i(x)+b_i(x)u$, $i=1,2,\cdots,m$, with the matrix $\mathrm{col}\{b_1(x), b_2(x), \cdots, b_m(x)\}$ being smooth and nonsingular.

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Such a definition of relative degree and the resulting normal form are nonlinear equivalence of the notion of infinite zeros and the related canonical form for single input single output systems. For multiple input multiple output systems, the vector relative degree is a rather strong structural property that not even all square invertible linear systems, with the freedom of choosing coordinates for the state, output and input spaces, could possess [22].

A major generalization of the form (2) was made in [10, 13, 14], where square invertible systems are considered. By using the *Zero Dynamics Algorithm*, under the assumptions that the ranks of certain matrices are constant and that the distribution spanned by the row vectors of $g(x)$ is involutive, the system can be transformed into the following form

$$\left\{\begin{array}{rcl} \dot{\eta} & = & f_0(x), \\ \dot{\xi}_{i,j} & = & \xi_{i,j+1}+\displaystyle\sum_{l=1}^{i-1}\delta_{i,j,l}(x)v_l, \quad j=1,\cdots,n_i-1, \\ \dot{\xi}_{i,n_i} & = & v_i, \\ y_i & = & \xi_{i,1}, \qquad i=1,2,\cdots,m, \end{array}\right. \tag{3}$$

where $n_1\leq n_2\leq\cdots\leq n_m$, $v_i=a_i(x)+b_i(x)u$, $i=1,2,\cdots,m$, with the matrix $\mathrm{col}\{b_1(x), b_2(x), \cdots, b_m(x)\}$ being smooth and nonsingular.

As pointed out in [10], when all $\delta_{i,j,l}(x)=0$, the set of integers $\{n_1, n_2, \cdots, n_m\}$ in (3) corresponds to the vector relative degrees, which in this case, represent the infinite zero structure if the system is linear. These integers however are not related to the infinite zero structure of linear systems when $\delta_{i,j,l}(x)\neq 0$, and thus cannot be viewed as the nonlinear equivalence of and expected to play a similar role as infinite zeros (see [22] for an example showing this).

In a recent paper [22], we study the structural properties of affine-in-control nonlinear systems beyond the case of square invertible systems. We propose an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form representation that corresponds to these integers as well as to the system invertibility structure. This new normal form representation takes the following form

$$\left\{\begin{array}{rcl} \dot{\eta} & = & f_\triangle(\eta, z_{\mathrm{d}})+g_\triangle(\eta, z_{\mathrm{d}})u_\triangle, \\ \dot{\xi}_{i,j} & = & \xi_{i,j+1}+\displaystyle\sum_{l=1}^{i-1}\delta_{i,j,l}(x)v_{\mathrm{d},l}, \ j=1,2,\cdots,q_i-1, \\ \dot{\xi}_{i,q_i} & = & v_{\mathrm{d},i}, \\ y_\triangle & = & h_\triangle(\eta, z_{\mathrm{d}}), \\ y_{\mathrm{d},i} & = & \xi_{i,1}, \qquad i=1,2,\cdots,m_{\mathrm{d}}, \end{array}\right. \tag{4}$$

where $q_1\leq q_2\leq\cdots\leq q_{m_{\mathrm{d}}}$, $\xi_i=\{\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,q_i}\}$, $i=1,2,\cdots,m_{\mathrm{d}}$, $z_{\mathrm{d}}=\{\xi_1, \xi_2, \cdots, \xi_{m_{\mathrm{d}}}\}$, $v_{\mathrm{d},i}=a_i(x)+b_i(x)u$, with the matrix col $\{b_1(x), b_2(x), \cdots, b_{m_d}(x)\}$ being of full row rank and smooth, and

$$\delta_{i,j,l}(x) = 0, \quad \text{for } j < q_l, \ i = 1, 2, \cdots, m_d. \tag{5}$$

We note here that m_d is the largest integer for which the system assumes the above form. The system is left invertible if $u_\triangle$ is non-existent, right invertible if $y_\triangle$ is non-existent, and invertible if both are non-existent. In the case that the system is square and invertible, *i.e.*, the system that was considered in [10, 13, 14], $m = p = m_d$ and the parts containing $y_\triangle$ and $u_\triangle$ drop off. Thus, the normal form (4) simplifies to

$$\begin{cases} \dot{\eta} &= f_\triangle(\eta, z_d), \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_l, \ j = 1, \cdots, q_i - 1, \\ \dot{\xi}_{i,q_i} &= v_i, \\ y_i &= \xi_{i,1}, \qquad i = 1, 2, \cdots, m, \end{cases} \tag{6}$$

where $q_1 \le q_2 \le \cdots \le q_m$, and

$$\delta_{i,j,l}(x) = 0, \quad \text{for } j < q_l, \ i = 1, 2, \cdots, m. \tag{7}$$

We note that the normal form (6) is the same as (3) except for the additional structural property (7). The $\dot{\xi}_{i,j}$ equation in (6) displays a triangular structure of the control inputs that enter the system. The property (7) imposes additional structure within each chain of integrators on how control inputs enter the system. With this additional structural property, the set of integers $\{q_1, q_2, \cdots, q_{m_d}\}$ indeed represent infinite zero structure when the system is linear.

Control design techniques and structural decompositions of nonlinear systems have been developed interweavingly. The discovery of structural properties and the corresponding normal form representation motivates new control designs. On the other hand, the desire for achieving more stringent closed-loop performances for a larger class of systems entails the exploitation of more intricate structural properties. For example, various stabilization results have been obtained in this process. In this paper, we would like to revisit the problem of stabilization. We will show how our new normal form (4) simplifies the conventional backstepping design, motivates a new backstepping design technique that is able to stabilize some systems that cannot be stabilized by the conventional backstepping technique, and allows the stabilization of non-square systems.

II. Preliminary Results

In the section, we recall some results on the backstepping design methodology [9, 13, 18]. The backstepping design method is readily applicable to systems that have vector relative degrees and are represented in the form (2), which contains m chains of integrators. Each of these chains independently controlled by a separate input. If the zero dynamics is only dependent on the states of the leading integrators of each chain, *i.e.*, $\dot{\eta} = f_0(\eta, \xi_{11}, \xi_{21}, \cdots, \xi_{m1})$, and there exist smooth functions, $v_1^\star(\eta), v_2^\star(\eta), \cdots, v_m^\star(\eta)$, with $v_1^\star(0) = v_2^\star(0) = \cdots = v_m^\star(0) = 0$, such that $\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star \eta), \cdots, v_m^\star(\eta))$ is globally asymptotically stable at $\eta = 0$, then it is straightforward to design a globally stabilizing feedback law $v_1(x), v_2(x), \cdots, v_m(x)$, recursively, by viewing the next integrators as a new intermediate input. Such a design procedure is thus referred to as "backstepping."

The technique of backstepping, however, cannot as easily been implemented if the system does not have a vector relative degree. Additional assumptions are required. In what follows, we recall from [13] such additional assumptions on the normal form (3) and the backstepping design procedure that is implemented under these assumptions.

Assumption 1: The dynamics η is driven only by $\xi_{i,1}$, $i = 1, 2, \cdots, m$, *i.e.*,

$$\dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}), \tag{8}$$

and there exist smooth functions $v_i^\star(\eta)$, with $v_i^\star(0) = 0$, $i = 1, 2, \cdots, m$, such that $\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta), \cdots, v_m^\star(\eta))$ is globally asymptotically stable at its equilibrium $\eta = 0$.

We will also need the following additional assumption, which requires the coefficient functions $\delta_{i,j,l}$ to display a certain "triangular" dependency on the state variables.

Assumption 2: The functions $\delta_{i,j,l}$ depend only on variable ξ_{ℓ_p,ℓ_b}, with 1) $1 \le \ell_p \le m$ and $\ell_b = 1$; or, 2) $\ell_p \le i-1$; or, 3) $\ell_p = i$ and $\ell_b \le j$.

Under Assumptions 1 and 2, a feedback law $v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_1, \xi_2, \cdots, \xi_i), i = 1, 2, \cdots, m$ that globally stabilizes the whole system can be constructed from $v_i^\star(\eta)$, $i = 1, 2, \cdots, m$, through a backstepping procedure. The procedure commences with the subsystem (8), and is followed by backstepping n_1 times through the variables in first chain of integrators to obtain $u_1^\star(\eta; \xi_{1,1}; v_2^\star(\eta), v_3^\star(\eta), \cdots, v_m^\star(\eta); \xi_1)$, and backstepping n_2 times through the variables in the second chain of integrators to obtain the feedback law $u_2^\star(\eta; \xi_{1,1}, \xi_{2,1}; v_3^\star(\eta), v_4^\star(\eta), \cdots, v_m^\star(\eta); \xi_1, \xi_2)$. This procedure is continued chain by chain for $i = 1$ through m, each backstepping n_i times through i-th chain of integrators to discover the feedback law $u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}; v_{i+1}^\star(\eta), v_{i+2}^\star(\eta), \cdots, v_m^\star(\eta); \xi_1, \xi_2, \cdots, \xi_i)$. As the backstepping is implemented on the integrators chain by chain, we will refer to the above backstepping procedure as the chain-by-chain backstepping.

III. Backstepping Design for Invertible Systems

In this section we focus on systems that are square invertible and discuss about their stabilization by the backstepping technique. We will first show that the conventional chain-by-chain backstepping design technique as described in [13] and recalled in Section II is applicable to our new normal form (6), and its implementation on this new normal form is simpler than on the earlier normal form (3). We then propose a new backstepping procedure which we refer to as the level-by-level backstepping. In the level-by-level backstepping design procedure, the backstepping is first implemented on the first integrators of all chains and then on the second integrators of all chains, and so on. We will show that the level-by-level backstepping will allow the backstepping to be implemented on some systems for which the chain-by-chain backstepping procedure is not applicable. We will also show that the chain-by-chain backstepping and the level-by-level backstepping can be mixed and implemented on a same system to allow stabilization of a larger class of systems.

A. Stabilization by Chain-by-Chain Backstepping

Since the normal form (6) is a special case of the normal form (3), backstepping is applicable to it. As explained in [13], the chain-by-chain backstepping requires the system (6) to satisfy Assumptions 1 and 2. Under these two assumptions, the normal form (6) is much simpler than the normal form (3). This simpler form make the implementation of the chain-by-chain backstepping simpler.

Example 3.1: A three input three output system in the form (3) with $\{n_1, n_2, n_3\} = \{3, 3, 4\}$ and satisfying Assumptions 1 and 2 will take the following form,

$$\begin{cases} \dot{\eta} &=& f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,j} &=& \xi_{1,j+1}, \quad j = 1, 2, \\ \dot{\xi}_{1,3} &=& v_1, \\ \dot{\xi}_{2,1} &=& \xi_{2,2} + \delta_{2,1,1}(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}) v_1, \\ \dot{\xi}_{2,2} &=& \xi_{2,3} + \delta_{2,2,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}) v_1, \\ \dot{\xi}_{2,3} &=& \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{2,3}, \xi_{3,1}) v_1, \\ \dot{\xi}_{2,4} &=& v_2, \\ \dot{\xi}_{3,1} &=& \xi_{3,2} + \delta_{3,1,1}(\eta, \xi_1, \xi_2, \xi_{3,1}) v_1 \\ && \quad + \delta_{3,1,2}(\eta, \xi_1, \xi_2, \xi_{3,1}) v_2, \\ \dot{\xi}_{3,2} &=& \xi_{3,3} + \delta_{3,2,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}) v_1 \\ && \quad + \delta_{3,2,2}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}) v_2, \\ \dot{\xi}_{3,3} &=& \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}) v_1 \\ && \quad + \delta_{3,3,2}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}) v_2, \\ \dot{\xi}_{3,4} &=& v_3. \end{cases} \tag{9}$$

On the other hand, under the same assumptions, the normal form (6)-(7), would take the following simpler form

$$\begin{cases} \dot{\eta} &=& f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,j} &=& \xi_{1,j+1}, \\ \dot{\xi}_{1,3} &=& v_1, \\ \dot{\xi}_{2,j} &=& \xi_{2,j+1}, \\ \dot{\xi}_{2,3} &=& \xi_{2,4} + \delta_{2,3,1}(\eta, \xi_1, \xi_{2,1}, \xi_{2,2}, \xi_{2,3}, \xi_{3,1}) v_1, \\ \dot{\xi}_{2,4} &=& v_2, \\ \dot{\xi}_{3,j} &=& \xi_{3,j+1}, \quad j = 1, 2, \\ \dot{\xi}_{3,3} &=& \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}) v_1, \\ \dot{\xi}_{3,4} &=& v_3. \end{cases} \tag{10}$$

B. Stabilization by Level-by-Level Backstepping

In the backstepping procedure described in Section II, we first carry out backstepping n_1 times through the variables in the first chain of integrators to arrive at the desired control action for the first chain of integrators, and then carry out backstepping n_2 times through the variables in the second chain of integrators to arrive at the desired control action for the second chain of integrators, and repeat this procedure on the remaining chains of integrators.

Let us call all $\xi_{i,1}$, *i.e.*, the "leading" variables in each chain of integrators which connect an input to an output, the first level integrators, and call all $\xi_{i,2}$ the second level integrators, and so on. As an alternative to the chain-by-chain backstepping, we here propose to carry out the backstepping on all first level integrators, and then repeat the procedure on all second level integrators until we reach to last level of integrators. We will refer to such a backstepping procedure as the level-by-level backstepping, in contrast with the chain-by-chain backstepping procedure.

To make the level-by-level backstepping possible, the coefficients $\delta_{i,j,l}$ in (6) should satisfy the following assumption:

Assumption 3: The functions $\delta_{i,j,l}$ depend only on variable $\xi_{\ell_{\rm p},\ell_{\rm b}}$, with 1) $\ell_{\rm b} \le j-1$; or 2) $\ell_{\rm b} = j$ and $\ell_{\rm p} \le i$.

We will say that the coefficients $\delta_{i,j,l}$ in the form (3) or (6) have the chain-by-chain triangular dependency on state variables if they satisfy Assumption 2, and have the level-by-level triangular dependency on state variables if they satisfy Assumption 3.

Under Assumptions 1 and 3, the level-by-level backstepping procedure for (6) can be described as follows. We will start with $\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta), \cdots, v_m^\star(\eta))$. After the first-level backstepping, we obtain the feedback laws $v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), i = 1, 2, \cdots, \alpha_1$, where α_1 is the number of chains that contain exactly one integrator, *i.e.*, $q_1 = q_2 = \cdots = q_{\alpha_1} = 1$. For chains that contain more than one integrator, we have $\xi_{i,2} = \phi_{i,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{i,1}), i = \alpha_1 + 1, \alpha_1 + 2, \cdots, m$. Here, $\xi_{i,2}$ are viewed as inputs. We next proceed with backstepping on the second level integrators. After the second level backstepping, we obtain the feedback laws $v_i = u_i^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2}, \cdots, \xi_{i,2})$, $i = \alpha_1 + 1, \alpha_1 + 2, \cdots, \alpha_2$, where $\alpha_2 - \alpha_1$ is the number of chains that contain exactly two integrators, *i.e.*, $q_{\alpha_1+1} = q_{\alpha_1+2} = \cdots = q_{\alpha_2} = 2$. For chains with lengths greater than 2, we obtain $\xi_{i,3} = \phi_{i,3}^\star(\eta; \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2} \cdots, \xi_{i,2})$, $i = \alpha_2 + 1, \alpha_2 + 2, \cdots, m$. Here, $\xi_{i,3}$ are viewed as inputs. Continuing in this way, we finally obtain $v_i = u_i^\star(\eta;\ \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m,1}; \xi_{1,2}, \xi_{2,2}, \cdots, \xi_{m,2}; \cdots; \xi_{1,q_m-1}, \xi_{2,q_m-1}, \cdots, \xi_{m,q_m-1}; \xi_{i,q_m})$, for chains that contain q_m integrators.

Example 3.2: Consider a system in the form of (6) with three chains of integrators of lengths $\{3, 4, 4\}$,

$$\begin{cases} \dot{\eta} &=& f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,j} &=& \xi_{1,j+1}, \\ \dot{\xi}_{1,3} &=& v_1, \\ \dot{\xi}_{2,j} &=& \xi_{2,j+1}, \\ \dot{\xi}_{2,3} &=& \xi_{2,4} + \delta_{2,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}) v_1, \\ \dot{\xi}_{2,4} &=& v_2, \\ \dot{\xi}_{3,j} &=& \xi_{3,j+1}, \quad j = 1, 2, \\ \dot{\xi}_{3,3} &=& \xi_{3,4} + \delta_{3,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}, \xi_{3,3}) v_1, \\ \dot{\xi}_{3,4} &=& v_3. \end{cases} \tag{11}$$

Clearly, this system satisfies Assumption 3. In what follows, we will illustrate how to implement the level-by-level backstepping on this system.

Let Assumption 1 be satisfied, *i.e.*, there exist smooth functions $v_i^\star(\eta)$, with $v_i^\star(0) = 0$, $i = 1, 2, 3$, such that the equilibrium $\eta = 0$ of the subsystem

$$\dot{\eta} = f_0(\eta, v_1^\star(\eta), v_2^\star(\eta), v_3^\star(\eta)) \tag{12}$$

is globally asymptotically stable. The backstepping procedure starts with the subsystem (12). To carry out the backstepping on the first level variables, we first consider

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, v_2^\star(\eta), v_3^\star(\eta)), \\ \dot{\xi}_{1,1} &= \xi_{1,2}, \end{cases}$$

with $\xi_{1,2}$ as the input. This subsystem can be globally asymptotically stabilized by a control of the form

$$\xi_{1,2} = \phi_{1,2}^\star(\eta; \xi_{1,1}). \tag{13}$$

Hence, for the subsystem

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, v_3^\star(\eta)), \\ \dot{\xi}_{1,1} &= \phi_{1,2}^\star(\eta; \xi_{1,1}), \end{cases}$$

with $\xi_{2,1}$ as the input, $v_2^\star(\eta)$ globally asymptotically stabilizes its equilibrium col$\{\eta, \xi_{1,1}\} = 0$. We next look at

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, v_3^\star(\eta)), \\ \dot{\xi}_{1,1} &= \phi_{1,2}^\star(\eta; \xi_{1,1}), \\ \dot{\xi}_{2,1} &= \xi_{2,2}, \end{cases}$$

with $\xi_{2,2}$ as the input. This subsystem can be globally asymptotically stabilized by a control of the form

$$\xi_{2,2} = \phi_{2,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}). \tag{14}$$

That is, the equilibrium col$\{\eta, \xi_{1,1}, \xi_{2,1}\} = 0$ of

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, v_3^\star(\eta)), \\ \dot{\xi}_{1,1} &= \phi_{1,2}^\star(\eta; \xi_{1,1}), \\ \dot{\xi}_{2,1} &= \phi_{2,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}) \end{cases}$$

is globally asymptotically stable. Similarly, the subsystem

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,1} &= \phi_{1,2}^\star(\eta; \xi_{1,1}), \\ \dot{\xi}_{2,1} &= \phi_{2,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}), \\ \dot{\xi}_{3,1} &= \xi_{3,2}, \end{cases}$$

with $\xi_{3,2}$ as the input, can be globally asymptotically stabilized by a control of the form

$$\xi_{3,2} = \phi_{3,2}^\star(\eta; \xi_{1,1}, \xi_{2,1}, \xi_{3,1}). \tag{15}$$

Thus, after the first level backstepping, the subsystem

$$\begin{cases} \dot{\eta} &= f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{j,1} &= \xi_{j,2}, \quad j = 1, 2, 3 \end{cases} \tag{16}$$

can be written as

$$\dot{\eta}_{\mathrm{I}} = f_{\mathrm{I}}(\eta_{\mathrm{I}}, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \tag{17}$$

where $\eta_{\mathrm{I}} = $ col$\{\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}\}$. The equilibrium $\eta_{\mathrm{I}} = 0$ of this system (17) is globally asymptotically stabilized by the virtual inputs $\xi_{1,2}$, $\xi_{2,2}$ and $\xi_{3,2}$ as given by (13) -(15).

To start the second level backstepping, consider

$$\begin{cases} \dot{\eta}_{\mathrm{I}} &= f_{\mathrm{I}}(\eta_{\mathrm{I}}, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}), \\ \dot{\xi}_{j,2} &= \xi_{j,3}, \quad j = 1, 2, 3, \end{cases} \tag{18}$$

and view $\xi_{1,3}$, $\xi_{2,3}$ and $\xi_{3,3}$ as its inputs. Following the same procedure as in the first level backstepping, we find the controls of the form

$$\begin{cases} \xi_{1,3} &= \phi_{1,3}^\star(\eta_{\mathrm{I}}; \xi_{1,2}), \\ \xi_{2,3} &= \phi_{2,3}^\star(\eta_{\mathrm{I}}; \xi_{1,2}, \xi_{2,2}), \\ \xi_{3,3} &= \phi_{3,3}^\star(\eta_{\mathrm{I}}; \xi_{1,2}, \xi_{2,2}, \xi_{3,2}) \end{cases} \tag{19}$$

that globally asymptotically stabilize the equilibrium $\eta_{\mathrm{II}} = $ col$\{\eta_{\mathrm{I}}, \xi_{1,2}, \xi_{2,2}, \xi_{3,2}\} = 0$ of the subsystem (18). The subsystem (18) can be written as $\dot{\eta}_{\mathrm{II}} = f_{\mathrm{II}}(\eta_{\mathrm{II}}, \xi_{1,3}, \xi_{2,3}, \xi_{3,3})$, whose equilibrium $\eta_{\mathrm{II}} = 0$ is globally asymptotically stabilized by the virtual inputs $\xi_{1,3}$, $\xi_{2,3}$ and $\xi_{3,3}$ given by (19).

For the third level backstepping, we define

$$\begin{cases} \dot{\eta}_{\mathrm{II}} &= f_{\mathrm{II}}(\eta_{\mathrm{II}}, \xi_{1,3}, \xi_{2,3}, \xi_{3,3}), \\ \dot{\xi}_{1,3} &= v_1, \\ \dot{\xi}_{2,3} &= \xi_{2,4} + \delta_{2,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}) v_1, \\ \dot{\xi}_{3,3} &= \xi_{3,4} + \delta_{3,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}, \xi_{3,3}) v_1, \end{cases} \tag{20}$$

with v_1, $\xi_{2,4}$ and $\xi_{3,4}$ as its inputs. This system can be globally asymptotically stabilized by the controls of the form

$$v_1 = u_1^\star(\eta; \xi_1, \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}). \tag{21}$$

The subsystem (20) under the control (21) can be written as $\dot{\eta}_{\mathrm{III}} = f_{\mathrm{III}}(\eta_{\mathrm{III}}; \xi_{2,4}, \xi_{3,4})$, and its equilibrium $\eta_{\mathrm{III}} = $ col$\{\eta_{\mathrm{II}}, \xi_{1,3}\} = 0$ is globally asymptotically stabilized by the virtual inputs $\xi_{2,4}$ and $\xi_{3,4}$ as given by

$$\begin{cases} \xi_{2,4} &= \phi_{2,4}^\star(\eta_{\mathrm{II}}; \xi_{1,3}, \xi_{2,3}), \\ \xi_{3,4} &= \phi_{3,4}^\star(\eta_{\mathrm{II}}; \xi_{1,3}, \xi_{2,3}, \xi_{3,3}). \end{cases}$$

Finally, define

$$\begin{cases} \dot{\eta}_{\mathrm{III}} &= f_{\mathrm{III}}(\eta_{\mathrm{III}}; \xi_{2,4}, \xi_{3,4}), \\ \dot{\xi}_{2,4} &= v_2, \\ \dot{\xi}_{3,4} &= v_3, \end{cases}$$

on which we carry out the last level of backstepping to obtain

$$\begin{aligned} v_2 &= u_2^\star(\eta; \xi_1; \xi_2; \xi_{3,1}, \xi_{3,2}, \xi_{3,3}), \\ v_3 &= u_3^\star(\eta; \xi_1; \xi_2; \xi_3). \end{aligned}$$

The inputs v_1, v_2 and v_3 globally asymptotically stabilize the equilibrium col$\{\eta, \xi_1, \xi_2, \xi_3\} = 0$ of the system (11).

Remark 3.1: The structural property (7) of the normal form (6) makes the level-by-level backstepping possible. It is not possible to implement the level-by-level backstepping technique on the normal form (3). For example, in the system (9), which is in the form (3), backstepping the virtual input from $\xi_{2,1} = v_2^\star(\eta)$ to $\xi_{2,2}$ by the dynamical equation $\dot{\xi}_{2,1} = \xi_{2,2} + \delta_{2,1,1}(\eta, \xi_1, \xi_{2,1}, \xi_{3,1}) v_1$ is infeasible. At this stage, v_1 is not yet available.

Remark 3.2: The chain-by-chain and level-by-level backstepping procedures require different dependency of the coefficients $\delta_{i,j,l}$ on the state variables. For example, both (10) and (11) are in the form (6). In (11), $\delta_{2,3,1}$ allows dependency on $\xi_{3,2}$. As Assumption 2 does not allow such dependency, the chain-by-chain backstepping is not applicable to the system (10). On the other hand, $\delta_{3,3,1}$ in system (10) allows dependency on $\xi_{2,4}$. However, Assumption 3 does not allow such dependency. Hence, the level-by-level backstepping is not applicable to the system (10).

C. Stabilization by Mixed Chain-by-Chain and Level-by-Level Backstepping

A system with a vector relative degree is a special case of both systems (3) and (6) with all $\delta_{i,j,l} = 0$. Thus, both chain-by-chain backstepping and level-by-level backstepping can be implemented on it. Furthermore, backstepping can be switched across chains and levels as long as a variable of lower level in a chain is backstepped earlier than variables of higher levels in the same chain.

Example 3.3: Consider the system (2) with a vector relative degree $\{2, 3, 3\}$,

$$\begin{cases} \dot{\eta} &=& f_0(\xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,1} &=& \xi_{1,2}, \\ \dot{\xi}_{1,2} &=& v_1, \\ \dot{\xi}_{2,j} &=& \xi_{2,j+1}, \\ \dot{\xi}_{2,3} &=& v_2, \\ \dot{\xi}_{3,j} &=& \xi_{3,j+1}, \quad j = 1, 2, \\ \dot{\xi}_{3,3} &=& v_3. \end{cases} \tag{22}$$

Let Assumption 1 hold. We can carry out backstepping in the order of $\Xi = \{\xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{1,1}, \xi_{2,3}, \xi_{3,2}, \xi_{1,2}, \xi_{3,3}\}$. After the backstepping of $\xi_{2,3}$, we obtain v_2, thus, v_2 depends on all the variables from $\xi_{2,1}$ to $\xi_{2,3}$ in Ξ, *i.e.*, $v_2 = u_2^\star(\eta, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{1,1}, \xi_{2,3})$. Similarly, we obtain

$$\begin{aligned} v_1 &= u_1^\star(\eta, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{1,1}, \xi_{2,3}, \xi_{3,2}, \xi_{1,2}), \\ v_3 &= u_3^\star(\eta, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{1,1}, \xi_{2,3}, \xi_{3,2}, \xi_{1,2}, \xi_{3,3}). \end{aligned}$$

Alternatively, we can also implement the backstepping in the order of $\{\xi_{3,1}, \xi_{3,2}, \xi_{2,1}, \xi_{3,3}, \xi_{1,1}, \xi_{2,2}, \xi_{1,2}, \xi_{3,2}\}$ to arrive at the following stabilizing feedback laws

$$\begin{aligned} v_1 &= u_1^\star(\eta, \xi_{3,1}, \xi_{3,2}, \xi_{2,1}, \xi_{3,3}, \xi_{1,1}, \xi_{2,2}, \xi_{1,2}), \\ v_2 &= u_2^\star(\eta, \xi_{3,1}, \xi_{3,2}, \xi_{2,1}, \xi_{3,3}, \xi_{1,1}, \xi_{2,2}, \xi_{1,2}, \xi_{3,2}), \\ v_3 &= u_3^\star(\eta, \xi_{3,1}, \xi_{3,2}, \xi_{2,1}, \xi_{3,3}). \end{aligned}$$

Backstepping in different orders leads to different dependency of controls on state variables, which can be exploited to meet certain constraints or performance requirement.

In the absence of a vector relative degree, both normal forms (10) or (11) contains coefficient functions $\delta_{i,j,l}$. The implementation of both chain-by-chain and level-by-level backstepping require structural dependency on state variables of $\delta_{i,j,l}$s. Such structural dependency constraint can be weakened by utilizing mixed chain-by-chain and level-by-level backstepping.

Example 3.4: Consider a system in the form of (6),

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi_{1,1}, \xi_{2,1}, \xi_{3,1}), \\ \dot{\xi}_{1,j} = \xi_{1,j+1}, \\ \dot{\xi}_{1,3} = v_1, \\ \dot{\xi}_{2,j} = \xi_{2,j+1}, \\ \dot{\xi}_{2,3} = \xi_{2,4} + \delta_{2,3,1}(\eta; \xi_1; \xi_{2,1}, \xi_{3,1}; \xi_{2,2}, \xi_{3,2}; \xi_{2,3}) v_1, \\ \dot{\xi}_{2,4} = v_2, \\ \dot{\xi}_{3,j} = \xi_{3,j+1}, \quad j = 1, 2, \\ \dot{\xi}_{3,3} = \xi_{3,4} + \delta_{3,3,1}(\eta, \xi_1, \xi_2, \xi_{3,1}, \xi_{3,2}, \xi_{3,3}) v_1, \\ \dot{\xi}_{3,4} = v_3. \end{cases} \tag{23}$$

Let Assumption 1 be satisfied. It is obvious that $\delta_{2,3,1}$ here is the same as that in (11), and $\delta_{3,3,1}$ is the same as that in (10). Neither Assumption 2 nor Assumption 3 is satisfied. As a result, neither the chain-by-chain nor the level-by-level backstepping can be implemented on this system. However, a mixed chain-by-chain and level-by-level backstepping will successfully stabilize this system. In particular, we can carry out backstepping in the order of $\xi_{1,1}$, $\xi_{1,2}, \xi_{1,3}, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}, \xi_{2,3}, \xi_{2,4}, \xi_{3,3}, \xi_{3,4}$ to obtain

$$\begin{aligned} v_1 &= u_1^\star(\eta, \xi_{1,1}, \xi_{1,2}, \xi_{1,3}), \\ v_2 &= u_2^\star(\eta, \xi_{1,1}, \xi_{1,2}, \xi_{1,3}, \xi_{2,1}, \xi_{2,2}, \xi_{3,1}, \xi_{3,2}, \xi_{2,3}, \xi_{2,4}), \\ v_3 &= u_3^\star(\eta, \xi_1, \xi_2, \xi_3). \end{aligned}$$

IV. Global Asymptotical Stabilization of Non-Square Nonlinear Systems

To achieve stabilization of the systems in the normal form (4), some further decomposition is required. In this section, we will first carry out such further decomposition on the normal form (4) and then discuss about the stabilization of the further decomposed system.

A. Further Decomposition of the Normal Form (4)

Consider the following subsystem of (4),

$$\begin{cases} \dot{\eta} &=& f_\triangle(\eta, z_\mathrm{d}) + g_\triangle(\eta, z_\mathrm{d}) u_\triangle, \\ y_\triangle &=& h_\triangle(\eta, z_\mathrm{d}). \end{cases} \tag{24}$$

Assume that there exists a transformation under which the dependency of $f_\triangle, g_\triangle$ and $h_\triangle$ on z_d is restricted to $y_\mathrm{d} = \mathrm{col}\{\xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m_\mathrm{d},1}\}$, which is part of the vector z_d. Such an assumption is standard in the backstepping literature.

Assumption 4: The subsystem (24) of (4) takes the form,

$$\begin{cases} \dot{\eta} &=& f_\triangle(\eta, y_\mathrm{d}) + g_\triangle(\eta, y_\mathrm{d}) u_\triangle, \\ y_\triangle &=& h_\triangle(\eta, y_\mathrm{d}). \end{cases} \tag{25}$$

To derive the zero dynamics of the system under Assumption 4, we let $y_\mathrm{d} = 0$ in (4). It then follows from the dynamical equations that $z_\mathrm{d} = 0$ and $v_\mathrm{d} = 0$. Consequently, the system reduces to

$$\begin{cases} \dot{\eta} &=& f_\triangle(\eta, 0) + g_\triangle(\eta, 0) u_\triangle, \\ y_\triangle &=& h_\triangle(\eta, 0), \end{cases} \tag{26}$$

from which the zero dynamics of the system can be derived.

Indeed, the derivation of the zero dynamics from the above dynamics has been done in [10, 14] for the case of $m = p = m_\mathrm{d}$. In this case, $y_\triangle$ and $u_\triangle$ are absent from (4), leading to (6), and the zero dynamics can be obtained as $\dot{\eta} = f_\triangle(\eta, 0)$.

For the general case, by the structural decomposition algorithm of [22], we know that the system (26) does not contain any dynamics that is both controllable (by $u_\triangle$) and observable (through $y_\triangle$). Otherwise, we would have additional chains of integrators that connect inputs and outputs.

For the system (1), let $\mathcal{C}_0$ be the smallest distribution that is invariant for (1) and contains the distribution spanned by the column vectors of $g(0)$, and $d\mathcal{O}$ be the smallest codistribution that is invariant for (1) and contains the codistribution spanned by the row vectors of $dh(0)$ [7, 10].

Lemma 4.1: Consider the system (26). Assume that the distributions $\mathcal{C}_0$, $\ker d\mathcal{O}$ and $\mathcal{C}_0 + \ker d\mathcal{O}$ of (26) all have a constant dimension. Then there exist coordinates $\hat{z} = \text{col}\{z_{\rm a}, z_{\rm b}, z_{\rm c}\}$ such that (26) takes the following form

$$\left\{\begin{array}{rcl} \dot{z}_{\rm a} &=& f_{\rm a}(z_{\rm a}, z_{\rm b}), \\ \dot{z}_{\rm b} &=& f_{\rm b}(z_{\rm b}), \\ \dot{z}_{\rm c} &=& f_{\rm c}(z_{\rm a}, z_{\rm b}, z_{\rm c}) + g_{{\rm c}\Delta}(z_{\rm a}, z_{\rm b}, z_{\rm c})u_{\Delta}, \\ y_{\Delta} &=& h_{\Delta{\rm b}}(z_{\rm b}), \end{array}\right. \tag{27}$$

with $\mathcal{C}_0 = \text{span}\{\partial/\partial z_{\rm c}\}$ and $\ker d\mathcal{O} = \text{span}\{\partial/\partial z_{\rm a}, \partial/\partial z_{\rm c}\}$.

Now, by Lemma 4.1, (4) can be further decomposed into

$$\left\{\begin{array}{rcl} \dot{z}_{\rm a} &=& f_{\rm a}(z_{\rm a}, z_{\rm b}, y_{\rm d}), \\ \dot{z}_{\rm b} &=& f_{\rm b}(z_{\rm b}, y_{\rm d}), \\ \dot{z}_{\rm c} &=& f_{\rm c}(z_{\rm a}, z_{\rm b}, z_{\rm c}, y_{\rm d}) + g_{{\rm c}\Delta}(x)u_{\Delta}, \\ \dot{z}_{\rm d} &=& A_{\rm dd}z_{\rm d} + [B_{\rm dd} + B_{\delta}(x)]v_{\rm d}, \\ y_{\Delta} &=& h_{\Delta{\rm b}}(z_{\rm b}, 0), \\ y_{\rm d} &=& C_{\rm dd}z_{\rm d}, \end{array}\right. \tag{28}$$

where $v_{\rm d} = \text{col}\{v_{{\rm d},1}, v_{{\rm d},2}, \cdots, v_{m_{\rm d}}\}$, $A_{\rm dd} = \text{blkdiag}\{\aleph_{q_1}, \aleph_{q_2}, \cdots, \aleph_{q_{m_{\rm d}}}\}$, $B_{\rm dd} = \text{blkdiag}\{\vartheta_{q_1}, \vartheta_{q_2}, \cdots, \vartheta_{q_{m_{\rm d}}}\}$, $C_{\rm dd} = \text{blkdiag}\{\varrho_{q_1}, \varrho_{q_2}, \cdots, \varrho_{q_{m_{\rm d}}}\}$, with $\varrho_k = [1 \quad 0] \in \mathbb{R}^{1\times k}$, $\vartheta_k = [0 \quad 1]^T \in \mathbb{R}^k$, $\aleph_k = \begin{bmatrix} 0 & I_{k-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}$, and

$$B_{\delta}(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \delta_{2,*,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{m_{\rm d}-1,*,1} & \delta_{m_{\rm d}-1,*,2} & \cdots & 0 & 0 \\ \delta_{m_{\rm d},*,1} & \delta_{m_{\rm d},*,2} & \cdots & \delta_{m_{\rm d},*,m_{\rm d}-1} & 0 \end{bmatrix},$$

with $\delta_{i,*,l}(x) = \text{col}\{\delta_{i,1,l}(x), \delta_{i,2,l}(x), \cdots, \delta_{i,q_i-1,l}(x), 0\}$ and (5) hold.

By setting $y_{\Delta} = 0$ and noting that $z_{\rm b}$ is observable through y_{Δ}, it can be shown that $z_{\rm b} = 0$. Thus, the zero dynamics can be deduced from (28) as $\dot{z}_{\rm a} = f_{\rm a}(z_{\rm a}, 0, 0)$.

B. Global Asymptotical Stabilization of Systems in Form (28)

We first consider the case where $z_{\rm c}$ is non-existent in (28),

$$\left\{\begin{array}{rcl} \dot{z}_{\rm a} &=& f_{\rm a}(z_{\rm a}, z_{\rm b}, y_{\rm d}), \\ \dot{z}_{\rm b} &=& f_{\rm b}(z_{\rm b}, y_{\rm d}), \\ \dot{z}_{\rm d} &=& A_{\rm dd}z_{\rm d} + [B_{\rm dd} + B_{\delta}(z_{\rm a}, z_{\rm b}, z_{\rm d})]v_{\rm d}, \\ y_{\Delta} &=& h_{\Delta{\rm b}}(z_{\rm b}, 0), \\ y_{\rm d} &=& C_{\rm dd}z_{\rm d}. \end{array}\right. \tag{29}$$

In this case, $p > m$. The dynamics of $z_{\rm a}$ and $z_{\rm b}$ are controlled only by $y_{\rm d}$. We view the following subsystem,

$$\left\{\begin{array}{rcl} \dot{z}_{\rm a} &=& f_{\rm a}(z_{\rm a}, z_{\rm b}, y_{\rm d}), \\ \dot{z}_{\rm b} &=& f_{\rm b}(z_{\rm b}, y_{\rm d}), \end{array}\right.$$

as $\dot{\tilde{\eta}} = f_o(\tilde{\eta}, \xi_{1,1}, \xi_{2,1}, \cdots, \xi_{m_{\rm d},1})$ with $\tilde{\eta} = \text{col}\{z_{\rm a}, z_{\rm b}\}$. Also, we ignore the equation y_{Δ} in (29). Thus, the stabilization for (29) is indeed the same as in the square invertible case. To implement backstepping, we first find smooth functions $v_i^{\star}(z_{\rm a}, z_{\rm b})$, with $v_i^{\star}(0,0) = 0$, $i = 1, 2, \cdots, m_{\rm d}$, such that the equilibrium $\text{col}\{z_{\rm a}, z_{\rm b}\} = 0$ of

$$\left\{\begin{array}{rcl} \dot{z}_{\rm a} &=& f_{\rm a}(z_{\rm a}, z_{\rm b}; v_1^{\star}(z_{\rm a}, z_{\rm b}), v_2^{\star}(z_{\rm a}, z_{\rm b}), \cdots, v_{m_{\rm d}}^{\star}(z_{\rm a}, z_{\rm b})), \\ \dot{z}_{\rm b} &=& f_{\rm b}(z_{\rm b}; v_1^{\star}(z_{\rm a}, z_{\rm b}), v_2^{\star}(z_{\rm a}, z_{\rm b}), \cdots, v_{m_{\rm d}}^{\star}(z_{\rm a}, z_{\rm b})) \end{array}\right.$$

is globally asymptotically stable. The backstepping techniques as described in Section III can now be implemented to arrive at feedback law $v_{\rm d} = u_{\rm d}^{\star}$, which globally asymptotically stabilizes the full system (29).

For the stabilization of the general system (28), we need an additional assumption.

Assumption 5: The function $B_{\delta}(x)$ is independent off $z_{\rm c}$, *i.e.*, $B_{\delta}(x) = B_{\delta}(z_{\rm a}, z_{\rm b}, z_{\rm d})$.

Under Assumption 5, we can first ignore the dynamics of $z_{\rm c}$ and find the controls $v_{\rm d}$ to stabilize (29). The full system can then be stabilized by finding the controls u_{Δ} that stabilizes the dynamics of $z_{\rm c}$.

V. Conclusions

We exploited the properties of a recently developed structural decomposition for the stabilization of general MIMO systems, and showed that this decomposition simplifies the conventional backstepping design, motivates a new backstepping design procedure that is able to stabilize some systems for which the conventional backstepping is not applicable, and allows the stabilization of non-square systems.

References

[1] R. M. Hirschorn, "Invertibility of multivariable nonlinear control systems," *IEEE Trans. Aut. Contr.*, vol. 24, pp. 855–865, 1979.

[2] A. J. Krener and A. Isidori, "Nonlinear zero distributions," in *Proc. 19th IEEE Conf. Decision Contr.*, 1980, pp. 665–668.

[3] S. N. Singh, "A modified algorithm for invertibility in nonlinear systems," *IEEE Trans. Aut. Contr.*, vol. 26, pp. 595–598, 1981.

[4] A. Isidori, A. J. Krener, C. Gori-Giorgi, and S. Monaco, "Nonlinear decoupling via feedback: a differential geometric approach," *IEEE Trans. Aut. Contr.*, vol. 26, pp. 331–345, 1981.

[5] C. I. Byrnes and A. Isidori, "A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control," in *Proc. 23rd IEEE Conf. Dec. Contr.*, 1984, pp. 1569–1573.

[6] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Aut. Contr.*, vol. 34, no. 11, pp. 1123–1131, 1989.

[7] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. Springer, 1990.

[8] C. I. Byrnes and A. Isidori, "Asymptotic stabilization of minimum phase nonlinear systems," *IEEE Trans. Aut. Contr.*, vol. 36, pp. 1122–1137, 1991.

[9] M. Krstić, P. V. Kokotović, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*. John Wiley & Sons, New York, 1995.

[10] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Springer, 1995.

[11] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive and Robust*. London: Prentice-Hall, 1996.

[12] G. Conte, C. H. Moog, and A. M. Perdon, *Nonlinear Control Systems: an Algebraic Setting*. Springer, 1999.

[13] A. Isidori, *Nonlinear Control Systems II*. Springer, 1999.

[14] B. Schwartz, A. Isidori, and T. J. Tarn, "Global normal forms for mimo nonlinear systems, with application stabilization and disturbance attenuation," *Math. Contr., Signals Syst.*, vol. 12, pp. 121–142, 1999.

[15] A. Teel, "Semi-global stabilization of minimum phase nonlinear systems in special normal forms," *Syst. Contr. Lett.*, vol. 19, no. 3, pp. 187–192, 1992.

[16] F. Esfandiari and H. K. Khalil, "Output feedback stabilization of fully linearizable systems," *Int. J. Contr.*, vol. 56, pp. 1007–1037, 1992.

[17] Z. Lin and A. Saberi, "Robust semiglobal stabilization of minimum-phase input-output linearizable systems via partial state and output feedback," *IEEE Trans. Aut. Contr.*, vol. 40, pp. 1029–1041, 1995.

[18] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Prentice Hall, 2002.

[19] C. I. Byrnes and A. Isidori, "Nonlinear internal models for output regulation," *IEEE Trans. Aut. Contr.*, vol. 49, pp. 2244–2247, 2004.

[20] D. Karagiannis, Z. P. Jiang, R. Ortega, and A. Astolfi, "Output-feedback stabilization of a class of uncertain non-minimum-phase nonlinear systems," *Automatica*, vol. 41, no. 9, pp. 1609–1615, 2005.

[21] G. Kaliora, A. Astolfi, and L. Praly, "Norm estimators and global output feedback stabilization of nonlinear systems with ISS inverse dynamics," *IEEE Trans. Aut. Contr.*, vol. 51, no. 3, pp. 493–498, 2006.

[22] X. Liu and Z. Lin, "On normal forms of nonlinear systems affine in control," in *Proc. 2007 Amer. Contr. Conf.*, 2007, pp. 2352–2357.