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Convergent discrete-time nonlinear systems: the case of PWA systems

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Abstract— **In this paper we extend the notion of convergence, as defined for continuous-time dynamical systems, to the realm of discrete-time systems. A system is said to be convergent if it exhibits a unique, globally asymptotically stable solution that is defined and bounded on the entire time axis. The convergence property is highly instrumental in solving output regulation, tracking, synchronization and observer design problems. First, we provide a general sufficient condition for the convergence of nonlinear discrete-time systems. Next, we propose constructive sufficient conditions for convergence of discrete-time piecewise affine (PWA) systems. These conditions are given in the form of matrix inequalities. The proposed results are illustrated by an example in which a tracking control problem for a discrete-time PWA system is tackled.**

I. Introduction

In this paper, we extend the notion of convergence as introduced by [1] for nonlinear continuous-time systems (see also [2], [3]) to the the case of discrete-time systems described by nonlinear maps. A system is called convergent if it has a unique solution that is bounded on the whole time axis and this solution is globally asymptotically stable. Obviously, if such a solution does exist, then all other solutions converge to this solution regardless of their initial conditions. This solution can be considered as a steady-state solution. As shown in e.g. [3], [4], the convergence property is highly instrumental in solving output regulation, tracking, synchronization and observer design problems. Moreover, it is beneficial for frequency domain analysis of nonlinear systems, see e.g. [5].

In this paper we, firstly, present a general sufficient condition for the convergence property of discrete-time systems described by nonlinear maps. This result serves as a discrete-time counterpart of the well-known Demidovich's sufficient condition [1] for the convergence of continuous-time nonlinear systems. Next, we study the convergence property for the class of discrete-time piecewise affine (PWA) systems. For this class of systems we present constructive sufficient conditions to check convergence. The conditions are given in terms of matrix inequalities.

Piecewise affine systems have been receiving wide attention due to the fact that the PWA framework [6] provides a means to describe dynamic systems exhibiting switching between a multitude of linear dynamic regimes, see e.g. [7], [8]. Moreover, PWA system models can be exploited to approximate more complex nonlinear dynamics. Here, we consider the class of discrete-time PWA systems. As it has been shown in [9], discrete-time PWA systems are equivalent to other hybrid systems modelling formalisms, such as linear complementarity systems, min-max plus scaling systems and mixed logical dynamical systems, and therefore represent a large class of hybrid systems. Applications can be found in many fields, such as the modelling of genetic regulatory networks [10] or electronic throttles [11].

The stability analysis (of fixed points) of discrete-time PWA systems has been studied extensively, see [12] for a recent survey discussing the use of piecewise affine, piecewise quadratic and piecewise polynomial Lyapunov functions. In [13], the stability analysis for discrete-time PWA systems without logic states is addressed, while in [14], [15] systems with logic states are addressed by considering the concept of Lagrange stability (the stability of continuous-valued part of the state only). Furthermore, in these papers ([13], [14], [15]) also H_∞ analysis results providing bounds between norms of time-varying inputs and outputs, which is typically useful in the context of disturbance attenuation problems, are presented.

In the context of control synthesis techniques, results on the optimal control of discrete-time PWA systems are presented in [16], [17], [18]. The model predictive control for discrete-time PWA systems is studied in [19], [20], where the results in [19] are confined to continuous PWA systems and [20] tackles the discontinuous case. However, in these works, exact regulation problems, such as the tracking problem or the synchronization problem, and the observer design problem have not been addressed. As mentioned above, the convergence property is highly instrumental in solving these control problems; see e.g. [21] for a convergence-based approach towards solving the master-slave synchronization problem for continuous-time PWA systems. Sufficient conditions for continuous-time PWA systems have been proposed in [22] for both the case of continuous and discontinuous vectorfields. In this paper, we propose sufficient conditions, in terms of matrix inequalities, for convergence of discrete-time PWA systems described by continuous PWA maps.

The outline of this paper is as follows. In Section II, the concept of convergence is extended to discrete-time nonlinear systems and a discrete-time equivalent of the well-known Demidovich's result [1], [2] is formulated. Sufficient conditions for convergence of discrete-time PWA systems are proposed in Section III. Section IV presents an illustrative example in which the convergence property is exploited to solve a tracking control problem for a discrete-time PWA

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system. Concluding remarks and an outlook on future work is given in Section V.

In the paper we will use the following notations. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of natural, integer and real numbers, respectively. Given a matrix $P = P^T > 0$ and a vector x, $|x|_P$ denotes $|x|_P := \sqrt{x^T P x}$.

II. Convergent discrete-time systems

In this section we consider general discrete-time nonlinear systems described by equations of the form

$$x[k+1] = f(x[k], k), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$ and $k \in \mathbb{Z}$ reflects the discrete time variable. Below we give a definition of convergent discrete-time systems.

Definition 1: System (1) is called (uniformly, exponentially) convergent if

- there exists a unique solution $\bar{x}[k]$ that is defined and bounded on $\mathbb{Z}$,
- $\bar{x}[k]$ is globally (uniformly, exponentially) asymptotically stable.[1]

The solution $\bar{x}[k]$ is called a steady-state solution.

As follows from the definition of convergent systems, any solution of a convergent system "forgets" its initial condition and converges to some steady-state solution which is independent of the initial condition. The convergence property is an extension of stability properties of asymptotically stable linear time-invariant (LTI) systems. The next statement summarizes some properties of convergent systems (1) with periodic or time-invariant right-hand sides. These properties are natural for linear systems, whereas for nonlinear systems they, in general, do not hold.

Lemma 1 ([1]): Suppose system (1) convergent. If the right-hand side of (1) is independent of k, the corresponding steady-state solution $\bar{x}[k]$ is constant; if $f(x,k)$ is periodic with respect to k with period $T \in \mathbb{N}$ (i.e. $f(x,k) = f(x, k+T)$ for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$), then the corresponding steady-state solution $\bar{x}[k]$ is also periodic with the same period T.

The next theorem provides sufficient conditions under which system (1) is exponentially convergent. It is a discrete-time counterpart of the result on convergent continuous-time systems from [1] (see also [2]).

Theorem 1: Consider system (1). If there exist a matrix $P = P^T > 0$ and a number ρ such that $0 < \rho < 1$ and

$$|f(x_1, k) - f(x_2, k)|_P \le \rho |x_1 - x_2|_P, \tag{2}$$

[1]A solution $\bar{x}[k]$ of system (1) is called

- *globally asymptotically stable* if a) for any $\epsilon > 0$ and $k_0 \in \mathbb{Z}$ there exists $\delta = \delta(\epsilon, k_0) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < \delta$, then $|x[k] - \bar{x}[k]| < \epsilon$ for all $k \ge k_0$; b) any solution $x[k]$ satisfies $|x[k] - \bar{x}[k]| \to 0$, as $k \to +\infty$.
- *globally uniformly asymptotically stable* if a) for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < \delta$, then $|x[k] - \bar{x}[k]| < \epsilon$ for all $k \ge k_0$; b) for any $R > 0$ and any $\tilde{\epsilon} > 0$ there exists $T = T(R, \tilde{\epsilon}) > 0$ such that if $|x[k_0] - \bar{x}[k_0]| < R$, then $|x[k] - \bar{x}[k]| < \tilde{\epsilon}$ for all $k \ge k_0 + T$.
- *globally exponentially stable* if there exist $c > 0$ and $0 < \rho < 1$ such that $|x[k] - \bar{x}[k]| \le c\rho^{(k-k_0)} |x[k_0] - \bar{x}[k_0]|$ for all $k \ge k_0$.

for all $x_1, x_2 \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$, and

$$\sup_{k \in \mathbb{Z}} |f(0,k)|_P =: C < +\infty, \tag{3}$$

then system (1) is exponentially convergent. Moreover, the steady-state solution $\bar{x}[k]$ satisfies

$$\sup_{k \in \mathbb{Z}} |\bar{x}[k]|_P \le \frac{C}{1-\rho} \tag{4}$$

and any other solution $x[k]$ starting in $x[k_0]$ at time instant k_0 satisfies

$$|x[k] - \bar{x}[k]|_P \le \rho^{(k-k_0)} |x[k_0] - \bar{x}[k_0]|_P \tag{5}$$

for all $k \ge k_0$.

Proof: We will first prove the existence of $\bar{x}[k]$ defined and bounded on $\mathbb{Z}$. Then we will prove global exponential stability of $\bar{x}[k]$ and, finally, its uniqueness.

Existence: To prove the existence of $\bar{x}[k]$ let us first show that the set $\mathcal{D} := \{x : |x|_P \le \frac{C}{1-\rho}\}$ is a positively invariant set for system (1). Consider the inequality

$$\begin{aligned} |f(x,k)|_P &\le |f(x,k) - f(0,k)|_P + |f(0,k)|_P \\ &\le \rho |x|_P + C. \end{aligned} \tag{6}$$

In the latter inequality we have used (2) and (3). It follows from (6) that if $|x|_P \le \frac{C}{1-\rho}$, then $|f(x,k)|_P \le \frac{C}{1-\rho}$. Hence, from (1) we conclude that the set $\mathcal{D}$ is positively invariant. Moreover, it is compact. To prove the existence of a solution $\bar{x}[k]$ defined and bounded on $\mathbb{Z}$ and satisfying (4), we will use the following lemma (see Appendix for its proof.)

Lemma 2: Consider system (1). Let $f(x,k)$ be such that for each $k \in \mathbb{Z}$ it is continuous with respect to x. Suppose $\mathcal{D}$ is a compact positively invariant set of system (1). Then there is a solution $\bar{x}[k]$ defined on $\mathbb{Z}$ and satisfying $\bar{x}[k] \in \mathcal{D}$ for all $k \in \mathbb{Z}$.

Notice that due to condition (2), for every $k \in \mathbb{Z}$ the function $f(x,k)$ is continuous. Applying Lemma 2, we conclude that there exists a solution $\bar{x}[k]$ that is defined and bounded on $\mathbb{Z}$ and $\bar{x}[k] \in \mathcal{D}$ for all $k \in \mathbb{Z}$, i.e. condition (4) holds.

Global exponential stability: Let $x_1[k]$ and $x_2[k]$ be two solutions of (1) defined for $k \ge k_0$ for some $k_0 \in \mathbb{Z}$. As follows from (1) and (2),

$$|x_1[k+1] - x_2[k+1]|_P \le \rho |x_1[k] - x_2[k]|_P, \quad k \ge k_0.$$

This implies that

$$|x_1[k] - x_2[k]|_P \le \rho^{(k-k_0)} |x_1[k_0] - x_2[k_0]|_P. \tag{7}$$

Therefore, every solution of system (1) is globally exponentially stable. If $\bar{x}[k]$ is a solution defined and bounded on $\mathbb{Z}$, then by substituting $\bar{x}[k]$ for $x_2[k]$ in (7), we obtain (5).

Uniqueness: Suppose $\bar{x}[k]$ and $\tilde{x}[k]$ are two solutions of (1) that are defined and bounded on $\mathbb{Z}$. Then, according to (7),

$$|\bar{x}[k] - \tilde{x}[k]|_P \le \rho^{(k-k_0)} |\bar{x}[k_0] - \tilde{x}[k_0]|_P, \tag{8}$$

for all $k \ge k_0$. Since $\bar{x}[k]$ and $\tilde{x}[k]$ are bounded on $\mathbb{Z}$, there exists $L > 0$ such that $\sup_{k_0 \in \mathbb{Z}} |\bar{x}[k_0] - \tilde{x}[k_0]|_P \le L$. Hence,

$$|\bar{x}[k] - \tilde{x}[k]|_P \le \rho^{(k-k_0)} L. \tag{9}$$

Since $0 < \rho < 1$, in the limit for $k_0 \to -\infty$ we obtain $|\bar{x}[k] - \tilde{x}[k]|_P \leq 0$, which implies $\bar{x}[k] = \tilde{x}[k]$. Due to the arbitrary choice of k, we obtain that $\bar{x}[k] = \tilde{x}[k]$ for all $k \in \mathbb{Z}$. This completes the proof of the theorem. $\square$

In the scope of control problems, time dependency of the right-hand side of system (1) is usually due to some input. This input may represent, for example, a disturbance or a feedforward control signal. In this case the system takes the form

$$x[k+1] = f(x[k], u[k]) \tag{10}$$

with state $x[k] \in \mathbb{R}^n$ and input $u[k] \in \mathbb{R}^m$. Below we define the convergence property for systems with inputs from a certain class $\mathcal{I}$.

Definition 2: System (10) is said to be (uniformly, exponentially) convergent for a class of inputs $\mathcal{I}$ that are defined on $\mathbb{Z}$ if it is (uniformly, exponentially) convergent for every input $u \in \mathcal{I}$. In order to emphasize the dependency on the input $u[k]$, the steady-state solution is denoted by $\bar{x}_u[k]$.

The property of convergence can be exploited in several ways. A convergent system excited by a periodic input has a unique globally asymptotically stable periodic solution with the same period time as the period time of the input (see Lemma 1). In bifurcation analysis such a property allows one to significantly reduce computational efforts for constructing the bifurcation diagram. Namely, if the system is convergent, only period-1 steady-state solutions can exist, while other responses (and thus bifurcations giving rise to such responses), such as period-k, k = 2, 3, . . ., solutions or quasi-periodic behavior, can not occur. In control problems, the convergence property is very useful in tackling such problems as output regulation, tracking, synchronization and observer design. One may design a controller with a feedback and feedforward components, where the feedback component guarantees convergence (the existence and global asymptotic stability of a bounded steady-state solution), while the feedforward component shapes this steady-state solution to achieve certain desired properties. For example, in the tracking problem, the steady-state solution must be equal to the desired state trajectory. This will be illustrated with an example in Section IV. In the observer design problem the convergence property, which is achieved by a proper output error injection, guarantees that solutions of the observer converge to a steady-state solution, which, in this case, equals the state of the observed system.

III. CONVERGENCE FOR PWA SYSTEMS

The result of Theorem 1 provides general conditions for convergence, but these conditions may be difficult to check. In this section we present constructive sufficient conditions for convergence for a class of nonlinear discrete-time systems, namely, for discrete-time piecewise affine (PWA) systems characterized by continuous PWA maps.

Consider the state space $\mathbb{R}^n$ that is divided into polyhedral cells Λ_i, $i = 1, \ldots, l$, by hyperplanes given by equations of the form $H_{ij}^T x + h_{ij} = 0$, such that $\Lambda_i \subset \{x \in \mathbb{R}^n : H_{ij}^T x + h_{ij} \geq 0\}$ and $\Lambda_j \subset \{x \in \mathbb{R}^n : H_{ij}^T x + h_{ij} < 0\}$, with $H_{ij} \in \mathbb{R}^n$ and $h_{ij} \in \mathbb{R}$ for $\{i, j\} = 1, \ldots, l$ and $i \neq j$. We will consider piecewise-affine systems of the form

$$x[k+1] = A_i x[k] + b_i + B u[k], \text{ for } x[k] \in \Lambda_i, \ i = 1, \ldots, l. \tag{11}$$

Here $A_i \in \mathbb{R}^{n \times n}$, B and $b_i \in \mathbb{R}^n$, $i = 1, \ldots, l$, are constant matrices and vectors, respectively. The vectors $x[k] \in \mathbb{R}^n$ and $u[k] \in \mathbb{R}^m$ are the state and the input vectors at time k, respectively. The hyperplanes $H_{ij}^T x + h_{ij} = 0$ are the switching surfaces. We assume that the inputs $u[k]$ are defined on the whole time axis $\mathbb{Z}$.

In the sequel we will deal with piecewise affine systems which have continuous right-hand sides. This continuity requirement on the right-hand side of system (11) can be characterized by the following simple algebraic lemma. Its proof can be found, for example, in [3].

Lemma 3: Consider system (11). The right-hand side of system (11) is continuous iff the following condition is satisfied: for any two cells Λ_i and Λ_j having a common boundary $H_{ij}^T x + h_{ij} = 0$ the corresponding matrices A_i and A_j and the vectors b_i and b_j satisfy the equalities

$$\begin{aligned} G_{ij} H_{ij}^T &= A_i - A_j \\ G_{ij} h_{ij} &= b_i - b_j, \end{aligned} \tag{12}$$

for some vector $G_{ij} \in \mathbb{R}^n$.

The following theorem establishes sufficient conditions for exponential convergence of system (11) for all bounded inputs $u[k]$.

Theorem 2: Consider system (11). Suppose the right-hand side of (11) is continuous and there exist a matrix P and a number α such that $0 < \alpha < 1$ and

$$\begin{aligned} &P = P^T > 0 \\ &A_i^T P A_i \leq \alpha P, \quad i = 1, \ldots, l. \end{aligned} \tag{13}$$

Then system (11) is exponentially convergent for the class of inputs $u[k]$ that are defined and bounded on $\mathbb{Z}$.

Proof: We will use Theorem 1 to show that system (11) is convergent. Notice that condition (3) holds for every bounded input $u[k]$. Therefore, we only need to verify condition (2). The right-hand side of (11) equals $f(x, k) = \tilde{f}(x) + Bu[k]$, where

$$\tilde{f}(x) := A_i x + b_i, \quad \text{for } x \in \Lambda_i, \ i = 1, \ldots l. \tag{14}$$

From this expression we see that in order to prove (2) we only need to show that

$$|\tilde{f}(x_1) - \tilde{f}(x_2)|_P \leq \rho |x_1 - x_2|_P \tag{15}$$

holds for some ρ, $0 < \rho < 1$, and all $x_1, x_2 \in \mathbb{R}^n$.

First, consider the case when both x_1 and x_2 belong to the same cell Λ_i (including its borders) with the corresponding mapping $\tilde{f}(x) = A_i x + b_i$. Then $\tilde{f}(x_1) - \tilde{f}(x_2) = A_i(x_1 - x_2)$ and, therefore,

$$|\tilde{f}(x_1) - \tilde{f}(x_2)|_P^2 = (x_1 - x_2)^T A_i^T P A_i (x_1 - x_2). \tag{16}$$

Taking into account inequality (13), we obtain that

$$|\tilde{f}(x_1) - \tilde{f}(x_2)|_P^2 \leq \alpha |x_1 - x_2|_P^2. \tag{17}$$

Hence, (15) is satisfied for the given P, $\rho = \sqrt{\alpha}$ and for all x_1, x_2 from the closure of Λ_i. Due to the arbitrary choice of the cell Λ_i, inequality (15) holds for any x_1 and x_2 both lying in the closure of any cell Λ_i, $i = 1, \ldots, l$.

Next, we consider the case of arbitrary x_1 and x_2. Consider the line segment $[x_1, x_2]$ connecting these two points. Denote $y_1 := x_1$, $y_p := x_2$ and y_i, $i = 2, \ldots, p-1$, – the points of intersection of the line segment $[x_1, x_2]$ with the switching surfaces such that any pair of points y_i, $y_i + 1$ belongs to the same cell Λ_j (including its borders), $y_i \neq y_{i+1}$, $i = 1, \ldots, p-1$, and the sequence $y_1, y_2, \ldots, y_p$ is ordered, see Fig. 1 Then

$$\begin{aligned} |\tilde{f}(x_1) - \tilde{f}(x_2)|_P &= \left| \sum_{i=1}^{p-1} \tilde{f}(y_i) - \tilde{f}(y_{i+1}) \right|_P \\ &\leq \sum_{i=1}^{p-1} |\tilde{f}(y_i) - \tilde{f}(y_{i+1})|_P. \end{aligned} \tag{18}$$

Since each pair of points y_i and y_{i+1}, $i = 1, \ldots, p-1$, simultaneously belongs to a particular cell, from the first step of the proof we obtain

$$|\tilde{f}(y_i) - \tilde{f}(y_{i+1})|_P \leq \rho |y_i - y_{i+1}|_P \tag{19}$$

for all $i = 1, \ldots, p-1$. Applying the last inequality to (18), we obtain

$$|\tilde{f}(x_1) - \tilde{f}(x_2)|_P \leq \rho \sum_{i=1}^{p-1} |y_i - y_{i+1}|_P. \tag{20}$$

Since all points y_i, $i = 1, \ldots, p$, lie on the same line segment $[x_1, x_2]$ and they are ordered,

$$\sum_{i=1}^{p-1} |y_i - y_{i+1}|_P = |y_1 - y_p|_P = |x_1 - x_2|_P. \tag{21}$$

This fact together with (20) implies (15). Hence all conditions of Theorem 1 are satisfied and, by this theorem, system (11) is convergent. □

It may seem that the existence of a common quadratic Lyapunov function for all the linear modes (condition (13)) is such a strong requirement that it only by itself, i.e. without the continuity assumption on the right-hand side, guarantees the convergence of system (11) for arbitrary bounded inputs $u[k]$. In general, this is not the case, as illustrated by the following example.

Consider the following scalar system

$$x[k+1] = \begin{cases} -\dfrac{1}{2} x[k] + u[k], & \text{for } x(k) \leq 2, \\ \dfrac{1}{2} x[k] + u[k], & \text{for } x(k) > 2. \end{cases} \tag{22}$$

This system is of the form (11) with $A_1 = -\frac{1}{2}$, $A_2 = \frac{1}{2}$, $b_1 = b_2 = 0$. Obviously, condition (13) is satisfied with $P = 1$ and $\rho = \frac{1}{4}$. But, at the same time, the system is not convergent for all bounded inputs $u[k]$, because for $u[k] \equiv 2$ it has two fixed points $x = \frac{4}{3}$ and $x = 4$. This contradicts the convergence property. This example illustrates the importance of the continuity condition on the right-hand side of (11) in Theorem 2.

IV. Illustrative example

Let us consider a bi-modal, two-dimensional PWA discrete-time system described by (11) with the following system matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.5 & 0.2 \\ 0 & 1 \end{bmatrix}, \; b_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{23}$$

and $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. The two-dimensional state-space is divided in two half-spaces by the line defined by $H_{12}^T x + h_{12} = 0$, with $H_{12}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $h_{12} = -1$ (i.e. $x_1 = 1$). Note that the PWA map is continuous across this line. This example is taken from [19]; in that work, a model predictive controller stabilizing a fixed point is designed in the face of state and input constraints. Here we will study, firstly, the convergence property of this system, secondly, how we can induce such a property by means of state feedback control and, finally, we will exploit the convergence property to solve a tracking control problem.

Let us first study the system for zero input ($u[k] \equiv 0$). In this case, the system exhibits the following set $\mathcal{E}$ of fixed points: $\mathcal{E} = \{(x_1, x_2) \mid (x_1 \leq 1 \wedge x_2 = 0) \vee (x_1 \geq 1 \wedge x_2 = \frac{5}{2} x_1 - \frac{5}{2}\}$. The fact that such a set of fixed points exists for zero input, as opposed to a single isolated fixed point, implies that the system is not convergent, see Definition 1. Let us now design a linear state-feedback controller of the form $u[k] = Kx[k] + v[k]$, with $K \in \mathbb{R}^{1 \times 2}$, such that the resulting closed system

$$x[k+1] = (A_i + BK)x[k] + b_i + Bv[k], \quad \text{for } x \in \Lambda_i, \; i = 1, 2, \tag{24}$$

with the new input $v[k]$, is exponentially convergent. The closed-loop system is again of the form (11) with a continuous right-hand side. Moreover, for the control gain $K = \begin{bmatrix} -0.1355 & -1.0246 \end{bmatrix}$ it satisfies condition (13) (for the system matrices $\tilde{A}_i = A_i + BK$, $i = 1, 2$) with the matrix $P = \begin{bmatrix} 0.0014 & 0.0002 \\ 0.0002 & 0.0010 \end{bmatrix}$ and $\alpha = 0.98$. According

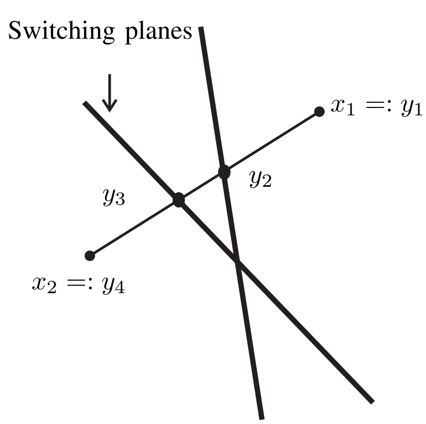

Fig. 1. The line segment (x_1, x_2) intersects the switching planes in the points $y_1, \ldots, y_4$.

to Theorem 2 this implies that the closed-loop system is exponentially convergent. Indeed, the closed-loop system has a unique globally exponentially stable fixed point at the origin for the input $v[k] \equiv 0$. Note, however, that Theorem 2 guarantees a much stronger property that for any bounded input $v[k]$ the closed-loop system exhibits a unique bounded on $\mathbb{Z}$ globally asymptotically stable solution.

We will now exploit this property to solve a tracking control problem for the example under study. Hereto, we consider the following periodic desired trajectory $x^d[k]$:

$$x^d[k] = \begin{bmatrix} \frac{1}{2} + \cos((k-1)\pi) \\ \frac{5}{8} + 9\frac{3}{8}\cos(k\pi) \end{bmatrix}. \tag{25}$$

We design a tracking controller, for system (11), (23), consisting of a feedforward $u^{ff}[k]$ and a linear tracking error feedback term:

$$u[k] = u^{ff}[k] + K(x[k] - x^d[k]), \tag{26}$$

where $u^{ff}[k] = 18\frac{3}{4}\cos((k-1)\pi)$. The idea behind this tracking controller is as follows:

1) firstly, the feedforward $u^{ff}[k]$ induces the desired solution in system (11), (23), i.e.

$$x^d[k+1] = A_i x^d[k] + b_i + Bu^{ff}[k], \text{ for } x^d[k] \in \Lambda_i, \tag{27}$$

2) secondly, the feedback renders the closed-loop system exponentially convergent and vanishes on the desired solution.

Notice that system (11), (23) in closed loop with (26) is of the form (24) with $v[k] = u^{ff}[k] - Kx^d[k]$. It is exponentially convergent for the matrix K specified above. Since the closed-loop system is exponentially convergent, it can only exhibit *one* solution that is bounded on the entire time axis (the steady-state solution), and this solution is globally exponentially stable. Since the desired state trajectory $x^d[k]$ is a solution of the closed-loop system, see (27), and it is bounded on the entire time axis, the desired state trajectory is a globally exponentially stable solution of the closed-loop system and the tracking problem is solved.

In Figures 2 and 3, the results of a simulation of the closed-loop system are depicted. More specifically, in Figures 2 and 3, the evolution of, respectively, the first state component x_1 and the second state component x_2 of both the desired trajectory and a trajectory with initial condition $x[1] = \begin{bmatrix}10 & 1\end{bmatrix}^T$ are shown. Clearly, asymptotic tracking of the desired state trajectory is achieved.

Additional simulations show that if one only applies the feedforward (i.e. if one applies controller (26) with $K = 0$), then, depending on the initial condition, the solution will either escape to infinity or converge to other periodic solutions than the desired solution. Of course, for initial conditions starting on the desired solution, the resulting solutions will remain on the desired solution. The latter facts indicate that without the feedback term in the controller multiple steady-state solutions exist, which once more confirms that the open-loop system (with or without feedforward) is not convergent.

V. Conclusions

In this paper we extend the notion of convergence, as defined for continuous-time dynamical systems, to the case of discrete-time systems. The convergence property is highly instrumental in solving output regulation, tracking, synchronization and observer design problems. First, we obtain a general sufficient condition for convergence of discrete-time nonlinear systems. Then, for an important class of piecewise affine systems with continuous maps, we propose sufficient conditions for convergence given in terms of certain matrix inequalities. It is illustrated by a counter-example that the existence of a common quadratic Lyapunov function is, in general, not sufficient for convergence if the continuity requirement is dropped. The proposed results are applied to an example involving a discrete-time PWA system. Moreover, we illustrate that the convergence property can be readily exploited to tackle the tracking control problem for such

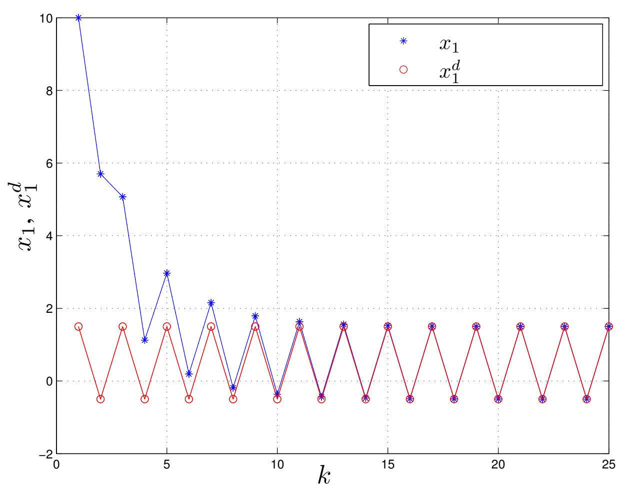

Fig. 2. First state component x_1 for both the desired trajectory and a trajectory with initial condition $x[1] = [10,\ 1]^T$.

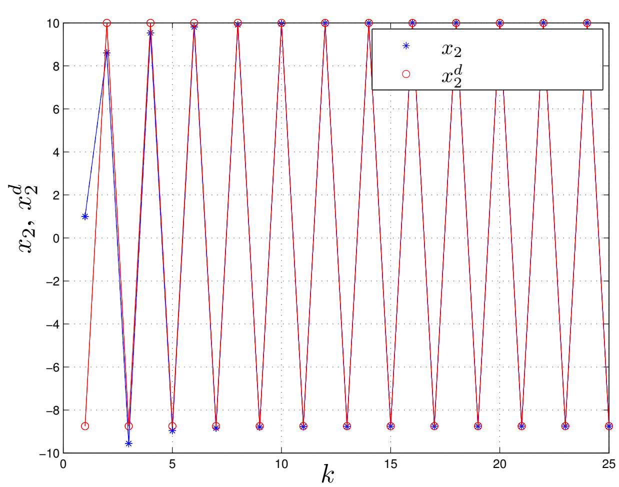

Fig. 3. Second state component x_2 for both the desired trajectory and a trajectory with initial condition $x[1] = [10,\ 1]^T$.

systems. Future research will focus on the extension of these results to discontinuous discrete-time PWA systems.

Appendix: Proof of Lemma 2

The proof of this lemma follows the ideas from [23]. Let us construct a sequence of sets F_j, $j = 0, 1, \ldots$, defined as

$$F_0 = \mathcal{D}, \quad F_j := \{x[0] : x[-j] \in \mathcal{D}\}.$$

Since for each $k \in \mathbb{Z}$ the function $f(x, k)$ is continuous with respect to x and since the set $\mathcal{D}$ is compact, then all the sets F_j are also compact. Due to the fact that $\mathcal{D}$ is positively invariant, if $x[-j] \in \mathcal{D}$, then $x[-j+1] \in \mathcal{D}$. Therefore, $F_j \subset F_{j-1}$ for all $j = 1, 2, \ldots$. This, together with the fact that all F_j are compact implies that there is a point $a_0 \in \cap_{j=0}^{\infty} F_j$. Define $\bar{x}[k]$ for $k \geq 0$ as the solution of (1) with the initial condition $\bar{x}[0] = a_0$. Since $a_0 \in \mathcal{D}$, the solution $\bar{x}[k]$ lies in $\mathcal{D}$ for all $k \geq 0$. It remains to construct $\bar{x}[k]$ for $k < 0$.

The fact that $a_0 \in \cap_{j=0}^{\infty} F_j$ implies that there exists a sequence of solutions $x_j[k]$, $j = 1, 2, \ldots$, defined on $[-j, 0]$ and satisfying $x_j[-j] \in \mathcal{D}$ and $x_j[0] = a_0$. Notice that since $\mathcal{D}$ is invariant, then $x_j[k] \in \mathcal{D}$ for all $k \geq -j$. Let $k = -1$. Consider the sequence of points $x_j[-1]$. Since $x_j[-1] \in \mathcal{D}$ for all $j \geq 1$, we can select a converging subsequence $x_{j_m}[-1]$. Let a_{-1} be the corresponding limit of this subsequence. Since $\mathcal{D}$ is compact, $a_{-1} \in \mathcal{D}$. Since $x_{j_m}[k]$ are solutions of (1), and since $x_{j_m}[0] = a_0$, then $a_0 = f(x_{j_m}[-1], -1)$. By continuity of $f(x, -1)$, we conclude that $a_0 = f(a_{-1}, -1)$, i.e. the sequence $\{a_{-1}, a_0\}$ is a solution of (1) defined on $\{-1, 0\}$ and lying in $\mathcal{D}$.

At the next step, truncate, if necessary, the subsequence of solutions $x_{j_m}[k]$ such that they are defined for $k \geq -2$. From the subsequence j_m we select a subsequence, which we again call j_m, such that $x_{j_m}[-2] \to a_{-2} \in \mathcal{D}$, as $m \to +\infty$. Since $x_{j_m}[k]$ are solutions of (1) defined for $k \geq -2$, then $x_{j_m}[-1] = f(x_{j_m}[-2], -2)$. By construction of the subsequence j_m, $x_{j_m}[-2] \to a_{-2}$ and $x_{j_m}[-1] \to a_{-1}$, as $m \to +\infty$. Hence, due to continuity of $f(x, -2)$, we conclude that $a_{-1} = f(a_{-2}, -2)$, i.e. the sequence $\{a_{-2}, a_{-1}\}$ is a solution of (1) defined on $\{-2, -1\}$ and lying in $\mathcal{D}$.

Continuing this process, we define $a_{-3}, a_{-4}, \ldots$, such that $a_{k+1} = f(a_k, k)$ and $a_k \in \mathcal{D}$ for all $k < 0$. Hence, $\bar{x}[k]$ defined for $k \leq 0$ as $\bar{x}[k] = a_k$ is a solution of (1) lying $\mathcal{D}$ for all $k \leq 0$. This completes the proof. □

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