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Optimal PID Controller Design using Standard Optimal Control Techniques

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***Abstract*—An optimal PID controller design procedure is formulated for 2nd order systems where the computation of the PID gains is equivalent to a state feedback design problem. As a result, any optimal state feedback control design method can be used. Furthermore, this method can be extended to higher order systems using model reduction techniques. The procedure is verified experimentally using a ball and beam apparatus.**

I. Introduction

THE Proportional-Integral-Derivative (PID) controller is the standard for industrial control with over 90% of industrial control systems using PID control [1]. The ubiquitous nature of PID control stems from its simple structure, the distinct effect of each of the three PID terms, its established use in industry, and an engineer's preference to improve existing methods before adopting a new method [1]. This paper presents a novel and relatively simple approach to optimal PID controller design that does not require advanced optimization techniques.

For a 2nd order system, a state feedback regulator (with an integrator augmented to the original dynamics) yields a PID controller structure. Therefore, any optimal control method can be applied to obtain an optimal PID controller. For higher order systems, model reduction techniques can be used either to reduce the plant to a 2nd order system or to reduce the higher-order optimal controller to a PID structure [2]. In this paper, this optimal PID controller design problem is formulated for a 2nd order system and verified using an experimental Ball and Beam apparatus from Quanser, Inc.[1]

There has been considerable research into the design of an optimal PID controller. This research can be divided into two categories: parametric optimization and tuning (see [3] for a review of tuning methods and [4] for tuning methods using Internal Model Control). While the method in this paper does not involve parametric optimization, it is comparable to the papers in the first category.

In the optimization category, linear quadratic Gaussian methods are used in [5] and a mixed H_2/H_∞ method is used in [6]. Genetic algorithms and fuzzy logic are used in [7, 8]. Model matching methods are employed in [9], based on least squares approximation of the desired impulse response, and in [10], based on approximation of the desired loop shape using a PID controller.

In this paper, the PID gains are computed directly (i.e., without parametric optimization) using a linear quadratic regulator (LQR) approach. As a result, practicing engineers should find the method more accessible because the optimal PID controller is relatively simple to compute. Furthermore, any optimal state feedback control method can be used.

The remainder of the paper is organized as follows. Section II contains a description of the proposed optimal PID control design method for 2nd order systems. Section III describes the extension of this procedure to higher order systems. Section IV describes the experimental apparatus and procedure used to verify the theoretical results. Section V contains conclusions and areas for future work.

II. Optimal PID Controller design for 2nd Order Systems

Consider a standard 2nd order system where y is the position and $\dot{y}$ is the velocity. The system model takes the form

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K_{DC}\omega_n^2 \end{bmatrix} u \tag{1}$$

with $\underline{x} = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$. To design a state feedback regulator with integral action, the plant in (1) is augmented with an integral to form

$$\underline{\dot{w}} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_n^2 & -2\zeta\omega_n & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{w} + \begin{bmatrix} 0 \\ K_{DC}\omega_n^2 \\ 0 \end{bmatrix} u \tag{2}$$

where $\underline{w} = \begin{bmatrix} y & \dot{y} & \int y \end{bmatrix}^T$. For the system in (2), a state feedback controller has the form

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[1] http://www.quanser.com

$$u = -K\underline{w} = -\begin{bmatrix} K_p & K_d & K_i \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \\ \int y \end{bmatrix} \tag{3}$$
$$= -\left(K_p y + K_d \dot{y} + K_i \int y\right)$$

and, clearly, has a PID structure. Therefore, an optimal state feedback control design method will yield an optimal PID controller. A similar approach is used in [6] where the PID design problem is formulated as an output feedback problem and can accommodate higher order systems. However, the resulting output feedback problem in [6] is more difficult, in general, to solve than the state feedback problem in the proposed design method.

The implementation of the control law (3) requires full state feedback. If the velocity can not be measured, a reduced order filter can be used to maintain a standard controller structure. Using the measurement of y and the computation of $\int y$, the reduced order observer for the plant in (2) has the form (see [11, pp. 272-274])

$$\begin{aligned} \dot{z} &= -\left(2\zeta\omega_n + L_1\right)z - K_{DC}\omega_n^2\left(K_p y + K_d \hat{\dot{y}} + K_i \int y\right) \\ &-\left(\omega_n^2 + \left(2\zeta\omega_n + L_1\right)L_1 + L_2\right)y - \left(2\zeta\omega_n + L_1\right)L_2 \int y \\ \hat{\dot{y}} &= L_1 y + L_2 \int y + z \end{aligned} \tag{4}$$

where $\hat{\dot{y}}$ is the estimated velocity and $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$ is the observer gain. If the control law in (3) is implemented using the velocity estimator in (4), the resulting control law has a transfer function of the form

$$\frac{U(s)}{Y(s)} = -\left(\frac{b_2 s^2 + b_1 s + b_0}{s\left(s+a\right)}\right) \tag{5}$$

and has a PI-lead structure. Note the H_2 and H_∞ optimal reduced order filtering problems have been addressed in [12] and [13], respectively.

III. Optimal PID Controller design for Higher Order Systems

To consider a general (i.e., higher than 2nd order) system as in [6], model reduction techniques can be employed to reduce the plant to a 2nd order system and the results of Section II can be applied directly. Alternatively, a full order optimal controller can be designed using the full plant model and then model reduction techniques can be applied to reduce this controller to a PID structure [2]. Fig. 1 shows the two options for optimal PID controller design using the state feedback formulation from Section II. Note that any optimal model reduction technique can be employed.

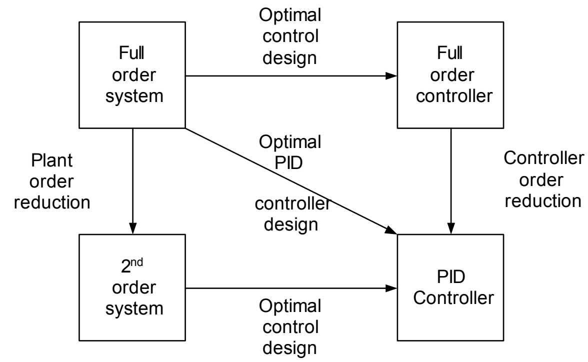

Fig. 1. Two options for optimal PID controller design for higher (>2) order systems

The use of approximation (i.e., model reduction) in optimal PID controller design is similar to the approach in [10] where the optimal PID controller is computed using the solution of a weighted H_∞ approximation. Specifically, the approximation problem is to minimize the norm

$$\left\| W\left(\hat{L} - L\right)\right\|_\infty \tag{6}$$

over the set of approximate loop gains $\hat{L}$ where L is the desired loop gain (determined from the performance criteria), $W = \left(1+L\right)^{-1}$ is the desired sensitivity. In [10], the approximate loop gain is defined as $\hat{L} = GC_{PID}$ where G is the plant, and C_{PID} is the PID controller. Note the PID controller is computed by performing a parametric optimization (over the PID gains) to determine the solution of the minimization problem.

The two options shown in Fig. 1 can be viewed as suboptimal solutions of this problem. If the plant is reduced to 2nd order, the approximate loop gain is defined as $\hat{L} = \hat{G}_2 C_{PID}$ where $\hat{G}_2$ is the 2nd order approximation of the G and C_{PID} is the optimal PID controller designed using the method in Section II. If a full order controller C is designed and reduced to a PID structure $\hat{C}_{PID}$, the approximate loop gain is defined as $\hat{L} = G\hat{C}_{PID}$.

The proposed optimal PID controller design method has two advantages over the methods in [6] and [10]. The design method is uses standard optimization methods and is more likely to be accepted by practicing engineers. In addition, this method avoids parametric optimization and, therefore, the optimal PID controller is relatively simple to compute.

IV. Experimental Verification of Optimal PID Controller

A. Experimental apparatus and procedure

The optimal PID controller design method in Sections II and III is verified using a ball and beam apparatus from Quanser, Inc. shown in Fig. 2. The objective of the experiment is to verify that the optimal PID controller design maintains the properties of the optimal design method. In this case, LQR methods are used and it is verified that the optimal PID controller produces a lower combined sum square error and sum square voltage (input).

The ball and beam system is an excellent platform for testing the proposed design method because its dynamics are 4th order but can be approximated by a 2nd order system. The block diagram of the closed-loop system is shown in Fig. 3. The dominant dynamics are two integrators due to the nearly frictionless rolling that occurs when the beam moves off the horizontal. The servo motor loop used to actuate the beam adds two additional states to the system dynamics. Typically, the servo loop is designed so that these dynamics are much faster than the desired closed-loop dynamics. In this work, a PI controller, $C(s)=\left(5.36s+32.6\right)/s$, is used to achieve a gain-crossover frequency of 6 rad/sec and a phase margin of 50°. The resulting servo (inner loop) response has a settling time of about 1.1 sec and the 2nd modeling assumption is valid because the ball position (outer loop) response has a settling time of over 6 sec. In this case, the model reduction from the 4th order system to the 2nd order system is performed by inspection and more sophisticated techniques are not required.

Fig. 2. Ball and Ball Apparatus from Quanser, Inc.

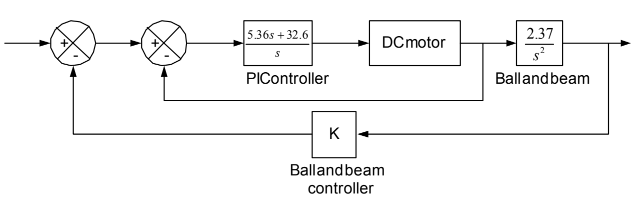

Fig. 3. Closed-loop Ball and Ball System

The performance of the optimal PID controller is compared with a PID controller design using classical design methods and with a PID design using standard tuning methods [14]. In the classical design methods, the desired time response characteristics are used to define the desired dominant closed-loop poles and the PID controller is designed to place the dominant closed-loop at the desired locations. In the tuning methods, the PID gains are computed from measured time response parameters.

B. Experimental results

1) Tuning

The standard tuning method is the Ziegler-Nichols' tuning method (see [14, pp. 134-151]). The first step in this method is to find the open loop step response, and compute a linear approximation of it as the system approaches a constant velocity (see Fig. 3).

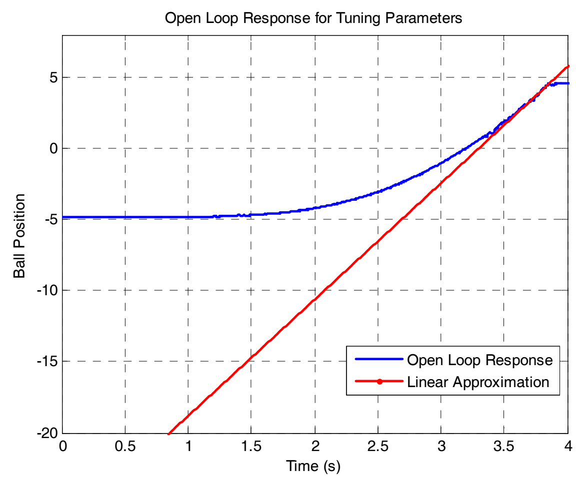

Fig. 3. Open-loop ball and beam response

The ball and beam system has a constant acceleration in open loop that makes the linear fit difficult. However, since the beam length limits the ball's motion, the ball does approach a constant velocity as it nears the end of the beam. This maximum velocity is best fit by the linear approximation $x=8.2t-27$ as shown in Fig. 3.

Since the movement starts at $t=1.25\,\text{sec}$, the measurement of a and L reference to that time (see [14, p.135]). The line crosses $x=-4.8\text{cm}$ at $t=2.71\text{sec}$ and it follows that $L=2.71-1.25=1.46$. The line crosses time $t=1.25\text{sec}$ at $x=-16.78\text{cm}$ and it follows that $a=-4.79-\left(-16.78\right)=12$. According to Ziegler-Nichols method, the PID gains are

$$K_p = \frac{1.2}{a} = 0.1$$
$$K_i = \frac{1.2}{2aL} = 0.0342 \quad (7)$$
$$K_d = \frac{1.2L}{2a} = 0.073$$

The Chen, Hrones, and Reswick setpoint method [14] improves upon the Ziegler-Nichols method using the time constant, T, of the open-loop system. For a double integrator system, however, the time constant is infinite and, as a result, the integral gain is zero (i.e., a PD controller is obtained) since the integral gain is inversely proportional to T. To obtain a PID controller, the Chen, Hrones, and Reswick method is applied for increasing values for $T = 1, 3, 10, 30\,\text{sec}$.

In Table I, the Ziegler-Nichols method is the first iteration and the four cases for T are iterations 2-5. The sum squared error ($\sum e^2$) is a measure of tracking performance and sum squared voltage ($\sum v^2$) is a measure of control input energy consumed. The experimental responses for these iterations (except iteration 2) are shown in Fig. 4. Note that as the time constant T in iterations 2-5, the steady-state error (SSE) increases as the integral gain is reduced.

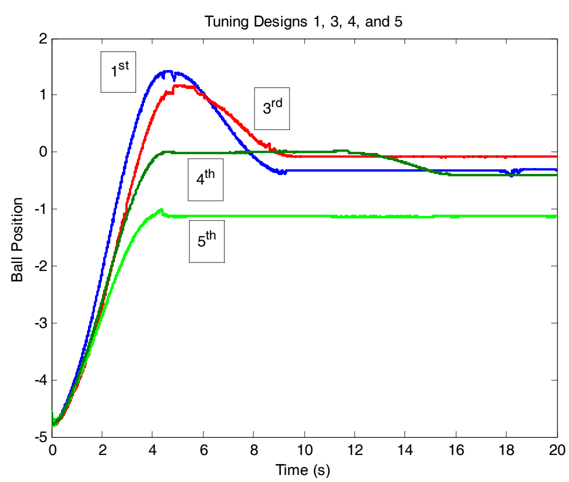

Fig. 4. Closed-loop ball and beam responses using PID tuning methods

TABLE I
TUNING ITERATIONS

Run	$\sum e^2$	$\sum v^2$	PO	t_s	SSE
1	37.0	3.6	33.33	8.39	0.30
2	207.	6.0	96.19	>>20	N/A
3	44.3	3.1	24.6	8.87	0.08
4	41	3.1	0.30	14.88	0.40
5	125.3	2.4	0.00	3.77	1.13

2) Classical and optimal designs

For both the classical and optimal design methods, a model of the ball and beam system is required. The model has the form of (2) where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2.37 \\ 0 \end{bmatrix} \quad (8)$$

The performance goal is to minimize the percent overshoot while keeping the settling time (as defined in the sequel) under 7 seconds.

For classical control, the position is measured (i.e., $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$) and the transfer function is $G_{bb}(s) = \frac{2.37}{s^2}$. The dominant desired poles are specified as

$s_d = -\zeta\omega_n \pm j\omega_n\sqrt{\left(1-\zeta^2\right)}$ from the desired natural frequency, ω_n, and damping coefficient, ζ. The third pole is placed at $s = -\omega_n$ in an adhoc attempt to limit control input usage. Finally, the PID gains are computed using Ackermann's formula. Table II lists the results for the classical design method.

Fig. 5 depicts the progression of classical designs outputs. The tabulated values are the average value from three tests of each iteration. Due to the static friciton in the system, the 2% settling times are exaggerated and, as a result, an alternative settling time is defined as the elapsed time until the response is within 10% of the full (initial) displacement (or $\pm 0.48\text{cm}$). Using this definition, this settling time is independent of the value of the system at the end of the experiment (i.e., $t = 20\,\text{sec}$) because the integral term will eventually force the ball to $x = 0\text{cm}$. The 7th classical design yields the best performance.

TABLE II
CLASSICAL ITERATIONS

Run	ω_n	ζ	$\sum e^2$	$\sum v^2$	PO	t_s
1	1	.707	24.0	340	36.2	5.57
2	.5	.707	30.0	60.0	9.6	3.53
3	1	.850	24.0	360	35.2	5.5
4	.75	.950	24.5	325	23.7	7.0
5	.6	.950	26.0	260	15.1	9.2
6	.25	.707	100.	22.0	0	>20
7	.35	.707	45.6	32.6	0.9	6.25
8	.32	.707	56.0	25.0	0	10.3

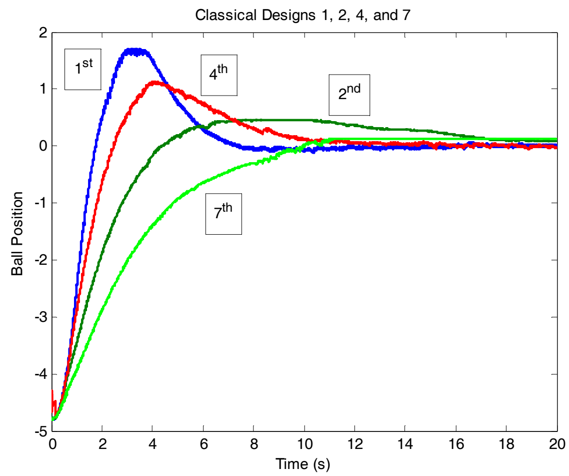

Fig. 5. Closed-loop ball and beam responses using classical PID design methods

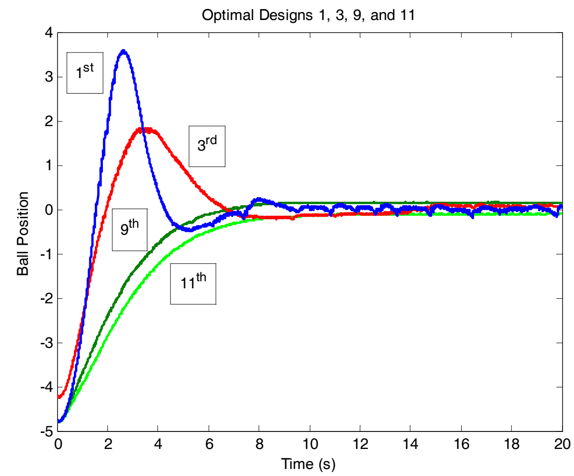

Fig. 6. Closed-loop ball and beam responses using optimal PID design methods

For the optimal PID design, LQR methods [2] are used with the procedure discussed in Secitons II and III. The design inputs are the unit cost, r, and a velocity weight, v. The unit cost governs the use of control energy and alows the engineer to compromise between performance and energy conservation. A large cost implies the control input is expensive and limits its use. A small cost admits a faster response through the use of larger inputs. All three PID gains tend to decrease as the unit cost, r, increases. The velocity weight provides a means to adjust the damping the response. For the plant in (8), the output matrix is $C = \begin{bmatrix} 0 & v & 1 \end{bmatrix}$ so that the velocity is weighted to control damping and overshoot and the integral of the position (or regulation error) is weighted to control steady-state error. Note that the integral weight is independent of v and the proportional and derivative gains tend to increases as v increases. Table III lists the results for the optimal design iterations. The optimal design is iterated to yield comparible performance with the classical design in terms of sum squred error and settling time.

TABLE III
OPTIMAL ITERATIONS

Run	r	v	$\sum e^2$	$\sum v^2$	PO	t_s
1	1	0	35	390	74.6	3.97
2	3	0.5	33	355	63.1	6.68
3	20	1	30	250	37.5	5.6
4	20	0	26	175	35.2	6.26
5	100	0	27	105	18.3	9.04
6	20	-1	24	200	22.5	7.47
7	100	-1	25	90	17.0	8.81
8	1000	-1	34	30	3.4	4.72
9	1000	-2	35	34	3.3	4.64
10	1000	-10	33.6	66	4.75	4.76
11	5000	-2	42.3	14.5	0.00	6.11

As expected, a comparison of Tables I, II, and III shows that the optimal PID controller produces a lower combined $\sum e^2$ and $\sum v^2$. The best results of each of the three methods (tuning, classical and optimal) are shown in Table IV. While the tuning methods provide a stable closed-loop response, the classical and optimal methods provide give the engineer much more control over the closed loop performance assuming a model of the system is available.

TABLE IV
TYPE COMPARISON

Type	Total runs	$\sum e^2$	$\sum v^2$	Best PO	t_s
Tuning	5	125.3	2.4	0.00	3.77
Classical	8	45.6	32.6	0.9	6.25
Optimal	11	42.3	14.5	0.00	6.11

V. CONCLUSION

This paper presented the formulation and experimental verification of a novel and relatively simple approach to optimal PID controller design that does not require advanced optimization techniques. For a 2nd order system, a state feedback regulator (with an integrator augmented to the original dynamics) yields a PID controller structure. Therefore, any optimal control method can be applied to obtain an optimal PID controller. For higher order systems, model reduction techniques can be used either to reduce the plant to a 2nd order system or to reduce the higher-order optimal controller to a PID structure. The proposed optimal design method was verified using an experimental ball and beam apparatus and compared with PID controllers designed using classical and standard tuning methods.

In future work, the authors intend to explore the generalization of this design method to higher order systems. Specifically, the both plant and controller reductions methods will be explore to determine their impact on performance and robustness of the resulting PID controller.

REFERENCES

[1] C. Knospe, "PID Control," *IEEE Control Systems Magazine*, vol. 26, pp. 30–31, Feb. 2006.

[2] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control.* Upper Saddle River, NJ: Prentice-Hall, 1996.

[3] P. Cominos and N. Munro, "PID Contrtollers: recent tuning methods and design to specification ," *IEE Proc.-Control Theory Applications*, vol. 1, pp. 46–53, Jan. 2002.

[4] Z. Ming-Guang, W. Peng, and, W. Zhao-Gang, "Study of Fuzzy Self-Tuning PID Internal Model Control Algorithm and Its Application," in *Proc. Intl Conf. Computational Intelligence and Security Workshops*, Harbin, Heilongjiang, China, 2007, pp. 31-34.

[5] M. J. Grimble and M.A. Johnson, "Algorithm for PID Controller Tuning Using LQG Cost Minimization," in *Proc. American Control Conf.*, San Diego, CA, 1999, pp. 4368–4372.

[6] R. H. C. Takahashi, P.L.D. Peres, and P.A.V. Ferreira, "Multiobjective H_2 / H_∞ Guaranteed Cost PID Design," *IEEE Control Systems Magazine*, vol. 17, pp. 37–47, Oct. 1997.

[7] B. Chen, Y. Cheng, and C. Lee, "A Genetic Algorithm Approach to Mixed H_2 / H_∞ Optimal PID Control," *IEEE Control Systems Magazine*, vol. 15, pp. 51–60, Oct. 1995.

[8] H. Li, L. Zhang, K. Cai, and G. Chen, "An Improved Robust Fuzzy-PID Controller with Optimal Fuzzy Reasoning," *IEEE Trans. System, Man, Cybernetics – Part B: Cybernetic*, vol. 35, pp. 1283–1294, Dec. 2005.

[9] G. Herjolfsson and A.S.Hauksdottir, "Direct computation of optimal PID controllers," in *Proc. Conf. Decision Control*, Maut, HI, 2003, pp. 1120–1125.

[10] E. Grassi, K.S. Tsakalis, S. Dash, S.V. Gaikwad, W. MacArthur, and G. Stein, "Integrated System Identification and PID Controller Tuning by Frequency Loop-Shaping," *IEEE Trans. Control Systems Technology*, vol. 9, pp. 285–294, Mar. 2001.

[11] W. J. Rugh, *Linear System Theory (2nd ed.)* Upper Saddle River, NJ: Prentice Hall, 1996.

[12] K. M. Nagpal, R. E. Helmick, and C. S. Sims, "Reduced order estimation, part I: Filtering,". Intl J Control, vol. 45, pp. 1867–1888, Jun. 1987.

[13] R. T O'Brien, Jr. and K. Kiriakidis, "Reduced-Order H_∞ Filtering for Discrete-Time, Linear, Time-Varying Systems," in *Proc. American Control Conf.*, Boston, MA, 2004, pp. 4120-4125.

[14] K. Astrom and T. Hagglund, *PID Controllers: Theory, Design, and Tuning (2nd Ed.).*Research Triangle Park, NC: Instrument Society of America, 1995.