Robust Stability for Iterative Learning Control

Richard Bradley, Mark French
School of Electronics and Computer Science
University of Southampton
Southampton, SO17 1BJ, UK
rsb05r@ecs.soton.ac.uk,
mcf@ecs.soton.ac.uk

Abstract—The phenomenon of long term instability of iterative learning control systems is examined. The concept of stability with respect to a specified trajectory is used to define a nonlinear biased gap metric. The resulting robust stability theorem, applied in a 2D setting, is used to prove the robust stability of a set of ILC algorithms engaged in trajectory tracking. The robust stability guarantee is dependent on the gain of the system from input and output disturbances to internal signals. This gain is calculated for a set of linear ILC algorithms to provide a guide to the trade-offs between tracking accuracy and robust stability.

I. INTRODUCTION

Humans learn tasks by repeating them and learning from mistakes; iterative learning control (ILC) attempts to employ this approach to improve system performance as it repeats a task. The system is provided with an input signal in an attempt to follow a given trajectory, and the resulting error is measured. After a finite period of time the system is reset and an ‘improved’ input signal is calculated using data from the previous trial. This update at each iteration aims to reduce the error at the output.

One of the features of ILC is the 2-dimensional attribute of ILC systems. Within classical feedback control, all the system information propagates in one direction: time. As time progresses the output develops depending on the input in a manner governed by the feedback loop. For ILC this occurs for only a finite time period, then the system is reset and a separate rule defines how the results from that trial will affect the system along the following trial. The two control laws are therefore entwined together, increasing the complexity of any analysis. This also renders traditional definitions of stability insufficient; for example, within any finite length of time, a plant’s continuous output will remain bounded regardless of any conventional interpretation of its stability.

Learning controllers are prone to long term stability problems [1]: the error decreases for many iterations before suddenly increasing, leading to instability. This type of instability is not very well understood and so is a serious drawback of using learning control. A theory is therefore needed to guarantee stability of these type of controllers. This paper considers the possibility of using the gap metric to try to examine this.

In order to design a controller for a plant a mathematical model is usually derived and then the controller built to

fit this model. No model is a perfect reproduction of the underlying plant, and so the modelled closed loop system must be sufficiently robust to accept the perturbations encountered when the controller is implemented on the real plant. If a system is stable, the stability margin around the plant depicts the amount that the plant can differ and still have stability guaranteed. The gap metric introduces a method of measuring the distance between two plants therefore providing a quantitative description of the subset of plants able to be stabilised by the controller [2].

This paper is primarily grounded on gap metric results from [2] concerning robustness analysis based on the closeness of input-output trajectories. The analysis in [2] was restricted to systems where zero input gave zero output. In [3] this was extended, relaxing this restriction by introducing a bias into norm definitions.

Here this biased induced norm theorem is adapted to the 2D ILC setting to provide a robust stability tool for use with tracking problems. If the bias is taken to be a reference trajectory, the stability (often analogous to convergence) of an ILC system can be examined with respect to this reference. This provides a robust stability guarantee in 2D for an ILC system engaged in tracking a reference trajectory.

The ILC robustness problem has been tackled before from a number of directions, often with some limitations. Frequency domain robustness results (such as [4]) are popular but results are approximations when applied to ILC due to the Fourier transform’s assumption of infinite (and ILC’s inhereantly finite) time intervals.

Causality is a requirement for some robustness results (e.g. [5]); although it is argued in [6] and [7] that all causal ILC controllers have an equivalent LTI controller, and in [8] that the only way to attain perfect tracking in some cases is to adopt non-causal ILC. Also, in [9] it is pointed out that for Arimoto-style algorithms a derivative in the controller is needed for a plant with single relative degree. A simple corollary of this is the requirement of information from one time-step ahead from the previous iteration for the stability of a discrete time relative degree one plant, which is a non-causal controller.

The main advantage of the method presented here over existing methods is the generality of the uncertainty model utilised [2]. In contrast, studies of ILC under various, more limited forms of uncertainty models have been published

including: multiplicative perturbations in [10], [11] and [12]; additive and multiplicative in [13], [14] and [15]; parametric, frequency-like and stochastic uncertainty in [16]; and interval model uncertainty in [17] and [18]. In particular, [10] provides a condition of positive real multiplicative uncertainties limited by $\pm 90^\circ$ over all frequencies, derived from a requirement that the modelling error is restricted to being expressed as a positive-real system. Although a strong result, this appears to be very conservative; it would not be expected that shifts in phase greater than 90° at high frequencies would cause a problem if the gain was sufficiently small.

The gap metric and ILC were combined in [19] to prove the robust stability of an adaptive ILC algorithm. The analysis was restricted to stabilisation problems, and to a single ILC algorithm. Here that analysis is adapted to examine the more general case of tracking control, although the work in the latter part of this paper does restrict the calculation of stability margin to a linear time invariant case.

II. NOTATION

The plants considered here will be modelled as single input single output, within feedback interconnections of the form shown in fig. 1. The closed loop system will be denoted by

Fig. 1. Feedback configuration

$[P, C]$, and defined as the set of equations:

$$y_0 = y_1 + y_2 \quad (1a)$$

$$u_0 = u_1 + u_2 \quad (1b)$$

$$y_1 = Pu_1 \quad (1c)$$

$$u_2 = Cy_2. \quad (1d)$$

Define $\mathcal{U}$ and $\mathcal{Y}$ as signal spaces, where a signal space is a normed vector space of signals $\mathcal{T} \rightarrow \mathbb{R}^n$ where $\mathcal{T}$ is a totally ordered set, e.g. $\mathcal{T} = \mathbb{N}, \mathbb{R}_+$. (Later in this paper we will use a 2D signal space applicable to ILC where $\mathcal{T}$ is defined to take account of time and iteration.)

Let $u_0 \in \mathcal{U}$, $y_0 \in \mathcal{Y}$ and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. The system governed by the set of equations $[P, C]$ is said to be globally well posed if for all $(u_0, y_0) \in \mathcal{W}$ there exists $((u_1, y_1), (u_2, y_2)) \in \mathcal{W}_e \times \mathcal{W}_e$. Here the extended spaces $\mathcal{U}_e, \mathcal{Y}_e$ are defined via the truncation operator T_τ :

$$T_\tau f(t) := \begin{cases} f(t) & t \leq \tau \\ 0 & t > \tau \end{cases}, \quad (2)$$

as follows:

$$\mathcal{X}_e := \{f : T_\tau f \in \mathcal{X} \quad \forall \tau > 0\}, \quad (3)$$

where $\mathcal{X} = \mathcal{U}, \mathcal{Y}$ or $\mathcal{W}$. Define $\mathcal{M}$ and $\mathcal{N}$ as the graphs of the nominal plant and controller respectively

$$\begin{aligned} \mathcal{M} &:= \mathcal{G}_P := \left\{ \begin{pmatrix} u_1 \\ Pu_1 \end{pmatrix} : u_1 \in \mathcal{U}, Pu_1 \in \mathcal{Y} \right\} \\ \mathcal{N} &:= \mathcal{G}_C := \left\{ \begin{pmatrix} Cy_2 \\ y_2 \end{pmatrix} : Cy_2 \in \mathcal{U}, y_2 \in \mathcal{Y} \right\}, \end{aligned} \quad (4)$$

and the maps $\Pi_{\mathcal{M}/\mathcal{N}}$ and $\Pi_{\mathcal{N}/\mathcal{M}}$ as the parallel projections of (u_0, y_0) onto (u_1, y_1) and (u_2, y_2) .

$$\begin{aligned} \Pi_{\mathcal{M}/\mathcal{N}} : \mathcal{W} &\rightarrow \mathcal{W}_e : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \\ \Pi_{\mathcal{N}/\mathcal{M}} : \mathcal{W} &\rightarrow \mathcal{W}_e : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \end{aligned} \quad (5)$$

In [2] a theorem is described relating the maximum distance between plant graphs allowed in order to guarantee robust stability, based on the induced norm of $\Pi_{\mathcal{M}/\mathcal{N}}$.

First, several definitions are required.

Let $\mathcal{U}[0, T]$ and $\mathcal{Y}[0, T]$ be signal spaces $\mathcal{U}$ and $\mathcal{Y}$ restricted to signals on $[0, T]$.

A map $A : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is gain stable iff

$$\|A|_{\mathcal{X}_1}\| = \sup_{x \in \mathcal{X}_1, \|x\| \neq 0} \frac{\|Ax\|}{\|x\|} < \infty. \quad (6)$$

A mapping $Q : \mathcal{U}_e \rightarrow \mathcal{Y}_e$ is said to be causal iff

$$\begin{aligned} \forall x, y \in \mathcal{U}_e \quad \forall \tau \in \text{dom}(x) \cap \text{dom}(Qx) \\ : \quad [T_\tau x = T_\tau y \Rightarrow T_\tau(Qx) = T_\tau(Qy)]. \end{aligned} \quad (7)$$

We assume throughout that all system operators are causal.

A causal plant P is stabilisable if for all $T > 0$ and for all $(u, y) \in \mathcal{U}[0, T] \times \mathcal{Y}[0, T]$ satisfying $T_T y = T_T P T_T u$ there exists $(\tilde{u}, \tilde{y}) \in \mathcal{U} \times \mathcal{Y}$ such that $\tilde{y} = P\tilde{u}$ and $(\tilde{u}, \tilde{y})|_{[0, T]} = (u, y)$.

The following theorem from [2] will be the basis for this paper.

Theorem 1: Consider $P_1 : \mathcal{U}_e \rightarrow \mathcal{Y}_e$, $P : \mathcal{U}_e \rightarrow \mathcal{Y}_e$ and $C : \mathcal{Y}_e \rightarrow \mathcal{U}_e$ with $P(0) = 0$ and $C(0) = 0$. Suppose $[P, C]$ is gain stable on $\mathcal{W}$, P_1 is stabilisable and $[P_1, C]$ is globally well posed. If

$$\|\Pi_{\mathcal{M}/\mathcal{N}}\|^{-1} > \begin{cases} \inf\{\|\Phi - I|_{\mathcal{M}}\| : \Phi \text{ is a causal,} \\ \text{surjective map from } \mathcal{M} \text{ to } \mathcal{M}_1 \\ \text{with } \Phi = 0\}, \\ \infty \text{ if no such operator } \Phi \text{ exists} \end{cases} \quad (8)$$

then the gain stability of $[P_1, C]$ is assured on $\mathcal{W}$. Here $\mathcal{M}_1$ is the graph of P_1 analogous to $\mathcal{M}$ and $\mathcal{N}$ in (4).

We observe that the quantity on the right hand side of (8) is typically denoted by $\tilde{\delta}(P, P_1)$ and represents the directed nonlinear gap metric introduced by [2]. In the case where P and C are linear and the signal spaces are $L^2(\mathbb{R})$ and the gap is less than one, then the symmetrised version of this quantity reduces to the standard H_2 gap.

The interpretation of this is to let P denote the plant model and P_1 the real plant. The controller C that stabilises P will also stabilise P_1 provided the two plants are close enough in terms of the defined norm.

III. BIASED GAP METRIC

The results of [3] extend those of [2] to the case where systems include some form of bias. In this case the map $\Pi_{\mathcal{M}/\mathcal{N}}$ does not map the zero vector to zero and so its induced norm would be infinite. With the previous robustness theorem this would not lead to any robust stability guarantees as the robust stability margin would be zero.

The proof for the theorem in [3] is similar to that given for the non-biased case and so was not given. It was originally motivated in the 1D case, however has been verified to hold in any signal space provided the signal space's time domain is ordered. This is a requirement for the truncation definition to exist in order for a valid notion of causality to be assigned to operators. A biased norm on the map $A : \mathcal{W} \rightarrow \mathcal{W}$ is defined as

$$\|A\|_{\mathcal{W},x_0} := \sup_{\substack{(x_1-x_0) \in \mathcal{X}_1 \\ \|T_\tau(x_1-x_0)\|_{\mathcal{W}} \neq 0}} \frac{\|T_\tau(Ax_1 - Ax_0)\|_{\mathcal{W}}}{\|T_\tau(x_1 - x_0)\|_{\mathcal{W}}}. \quad (9)$$

where $\|\cdot\|_{\mathcal{W}}$ is the norm on the vector space $\mathcal{W}$. Note: this biased norm relaxes to an induced, non-biased norm if $x_0 = 0$.

A robust stability theorem analogous to theorem 1 is now stated using this biased norm. The theorem holds for normed vector spaces $\mathcal{U}$ and $\mathcal{Y}$ on which a truncation is defined.

Theorem 2: Let $\mathcal{U}$ and $\mathcal{Y}$ be signal spaces and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P : \mathcal{U}_e \rightarrow \mathcal{Y}_e$, $P_1 : \mathcal{U}_e \rightarrow \mathcal{Y}_e$ and $C : \mathcal{Y}_e \rightarrow \mathcal{U}_e$. Suppose that P_1 is stabilisable, $[P_1, C]$ is globally well posed, and that $\|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W},x_0} < \infty$ for some $x_0 \in \mathcal{W}$.

Let $\mathcal{M} = \mathcal{G}_P$, $\mathcal{M}_1 = \mathcal{G}_{P_1}$. Suppose there exists a causal surjective map $\Phi : \mathcal{M} \rightarrow \mathcal{M}_1$ satisfying $\|(\Phi - I)|_{\mathcal{M}}\|_{\mathcal{W},g_0} < \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W},x_0}^{-1}$, where $g_0 = \Pi_{\mathcal{M}/\mathcal{N}}x_0$. Let $w_0 = (I + (\Phi - I)\Pi_{\mathcal{M}/\mathcal{N}})x_0$. Then

$$\|\Pi_{\mathcal{M}_1/\mathcal{N}}\|_{\mathcal{W},w_0} \leq \frac{\|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W},x_0}}{1 - \frac{\|(\Phi - I)|_{\mathcal{M}}\|_{\mathcal{W},g_0}}{\|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W},x_0}}}. \quad (10)$$

The proof of this theorem is similar to that of [2] and [3] with only a few minor alterations and so is not given here.

IV. ILC ROBUSTNESS

A. 2D Signal Measure

The two dimensional nature of ILC evokes the need for a two dimensional metric in order to adequately measure the signals involved. The following is a discrete version of a 2D norm from [19]. The real line is wrapped up into segments of length T , forming a product space from the $l^p[0, T]$ norm and the natural numbers, $\mathbb{N}$.

$$\|u(\cdot, \cdot)\|_{l^p(\mathbb{N} \times [0, T])} = \begin{cases} \left(\sum_{i=0}^{\infty} \sum_{t=0}^T |u(i, t)|^p \right)^{\frac{1}{p}} & p < \infty \\ \sup_{0 \leq i < \infty} \sup_{0 \leq t \leq T} |u(i, t)| & p = \infty \end{cases}. \quad (11)$$

For truncation and causality definition to hold the space must be totally ordered. This ordering is defined as follows: for

any $\tau_1, \tau_2 \in \mathbb{N} \times [0, T]$, $(i_1, t_1) = \tau_1 < \tau_2 = (i_2, t_2)$ if $i_1 < i_2$, or if $i_1 = i_2$ and $t_1 < t_2$. This is then used as the norm in the biased gap metric.

When this is used with theorem 2 this provides a 2D robust stability tool that can be applied to ILC. The theorem does not transfer the problem into the frequency domain and so no approximations are made through an assumption of infinite trial length; ILC-type 'non-causal' controllers are permitted; and the uncertainty model is fully unstructured. At this stage there is no restriction to linearity or time-invariance, although these are imposed in the next section when the theorem is applied to a set of LTI ILC systems.

B. ILC Framework

Several ILC papers have use what is known as the lifted system or supervector representation to examine discrete-time plants and controllers within ILC [20]. Within this framework the plant and controller are written as matrices allowing the use of linear algebra to find solutions.

$$P : l^p[0, T] \rightarrow l^p[0, T] : x(t+1) = Ax(t) + Bu_1(t) \\ y_1(t) = Cx(t) + Du_1(t). \quad (12)$$

Given a plant obeying (12), if the initial state vector $x(0)$ is set to 0 then the plant can be written as

$$\begin{pmatrix} y_1(0) \\ y_1(1) \\ y_1(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} D & & & \\ CB & D & & \\ CAB & CB & D & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_1(1) \\ u_1(2) \\ \vdots \end{pmatrix}. \quad (13)$$

This matrix is lower-triangular and Toeplitz and will be denoted by P , all the terms within the matrix are the Markov parameters of the system, and terms above the major diagonal are all zero.

For the ILC case this map will be repeated over $\mathbb{N}$. The plant $\bar{P} : l^p(\mathbb{N} \times [0, T]) \rightarrow l^p(\mathbb{N} \times [0, T])$ is therefore defined as

$$\begin{pmatrix} y_1(k, 0) \\ y_1(k, 1) \\ y_1(k, 2) \\ \vdots \end{pmatrix} = \begin{pmatrix} D & & & \\ CB & D & & \\ CAB & CB & D & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} u_1(k, 0) \\ u_1(k, 1) \\ u_1(k, 2) \\ \vdots \end{pmatrix} \quad (14)$$

where $k \in \mathbb{N}$ represents the trial number. For most of the analysis the signals will be examined as vectors along each trial and so the time index shall be dropped. Therefore $y_1(k)$ will denote the entire vector y_1 along trial k :

$$y_1(k) = \begin{pmatrix} y_1(k, 0) \\ y_1(k, 1) \\ y_1(k, 2) \\ \vdots \end{pmatrix} \quad (15)$$

The other signals are written in the same form, so the plant map $\bar{P} : u_1(\cdot) \mapsto y_1(\cdot)$ is defined by $y_1(k) = Pu_1(k) \quad k \in \mathbb{N}$.

We now examine an ILC controller of the form

$$\begin{aligned} C : l^p(\mathbb{N} \times [0, T]) &\rightarrow l^p(\mathbb{N} \times [0, T]) \\ u_2(k+1, \cdot) &= Q(u_2(k, \cdot) - Ly_2(k, \cdot)) \quad k \in \mathbb{N} \\ u_2(0, \cdot) &= 0 \end{aligned} \quad (16)$$

Here $L : l^p[0, T] \rightarrow l^p[0, T]$ is a matrix of learning gains and $Q : l^p[0, T] \rightarrow l^p[0, T]$ is usually a filter used to tune the asymptotic properties of the system.

Similarly to (15), since the reference signal $y_{ref}(k)$ is identical for all k , it will be written simply as y_{ref} ; and is inserted alongside the disturbance y_0 as per fig. 2. This

Fig. 2. Feedback configuration

leaves the closed loop system defined by (1b), (1c), (1d) and $y_0 + y_{ref} = y_1 + y_2$. This results in the controller input being the error between plant output and reference signal.

C. 2D Gain

Recall the biased norm, which with the 2D norm is given by

$$\begin{aligned} \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{l^p(\mathbb{N} \times [0, T]), x_0} &:= \sup_{\substack{(x_1(\cdot, \cdot) - x_0(\cdot, \cdot)) \in l^p(\mathbb{N} \times [0, T]) \\ \|T_\tau(x_1(\cdot, \cdot) - x_0(\cdot, \cdot))\|_{l^p(\mathbb{N} \times [0, T])} \neq 0}} \tau > 0 \\ \frac{\|T_\tau(\Pi_{\mathcal{M}/\mathcal{N}}x_1(\cdot, \cdot) - \Pi_{\mathcal{M}/\mathcal{N}}x_0(\cdot, \cdot))\|_{l^p(\mathbb{N} \times [0, T])}}{\|T_\tau(x_1(\cdot, \cdot) - x_0(\cdot, \cdot))\|_{l^p(\mathbb{N} \times [0, T])}}. \end{aligned} \quad (17)$$

and also that $\mathcal{M}$ and $\mathcal{N}$ are the graphs of $\bar{P}$ and C respectively.

Theorem 3: Let $p = 1$. Consider the closed loop system given by fig. 2, and (14) and (16). Suppose $\|Q(I - LP)\| < 1$. Then with the biased norm of (17), the gain of $\Pi_{\mathcal{M}/\mathcal{N}}$ is bounded by

$$\begin{aligned} \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{l^1(\mathbb{N} \times [0, T]), \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix}} &\leq \left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\| \left(1 + \frac{\|QLP\| + \|QL\|}{1 - \|Q(I - LP)\|} \right). \end{aligned} \quad (18)$$

Note that the norms on the right hand side are induced norms in $l^1[0, T]$, and are always finite.

Proof: We can derive an expression for $u_2(k+1)$ in terms of the disturbances and the previous control signal $u_2(k)$.

$$\begin{aligned} u_2(k+1) &= Q(u_2(k) - L(y_0(k) + y_{ref} - P(u_0(k) - u_2(k)))) \\ &= Q(I - LP)u_2(k) - QL(y_0(k) + y_{ref} - Pu_0(k)) \end{aligned} \quad (19)$$

With $u_2(0) = 0$ this can then be written as a recurrence relation to obtain

$$u_2(k) = \sum_{i=1}^k [Q(I - LP)]^{i-1} (-QL(y_0(k-i) + y_{ref} - Pu_0(k-i))). \quad (20)$$

To calculate the stability margin using the biased norm defined previously, with the bias as the reference signal y_{ref} we require $u_1(k)$ and $y_1(k)$. These are given by

$$\begin{aligned} u_1(k) &= \sum_{i=1}^k [Q(I - LP)]^{i-1} (QL(y_0(k-i) + y_{ref} - Pu_0(k-i))) + u_0(k) \\ y_1(k) &= Pu_1(k). \end{aligned} \quad (21)$$

Let $u_0, y_0 \in l^1(\mathbb{N} \times [0, T])$. By substituting in the appropriate terms:

$$\begin{aligned} x_0(k) &= \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix} \\ x_1(k) &= \begin{pmatrix} u_0(k) \\ y_0(k) + y_{ref} \end{pmatrix} \\ \Pi_{\mathcal{M}/\mathcal{N}}x_0(k) &= \begin{pmatrix} I \\ P \end{pmatrix} \sum_{i=1}^k [Q(I - LP)]^{i-1} (QLy_{ref}) \\ \Pi_{\mathcal{M}/\mathcal{N}}x_1(k) &= \begin{pmatrix} I \\ P \end{pmatrix} \sum_{i=1}^k [Q(I - LP)]^{i-1} (QL(y_0(k-i) + y_{ref} - Pu_0(k-i))) + u_0(k), \end{aligned} \quad (22)$$

we arrive at

$$\begin{aligned} \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{l^1(\mathbb{N} \times [0, T]), \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix}} &= \sup_{\substack{\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \in l^1(\mathbb{N} \times [0, T]) \\ \left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\|_{l^1(\mathbb{N} \times [0, T])} \neq 0}} \\ \frac{\sum_{k=0}^{\infty} \left\| \begin{pmatrix} I \\ P \end{pmatrix} \sum_{i=1}^k [Q(I - LP)]^{i-1} (QL(y_0(k-i) - Pu_0(k-i))) + u_0(k) \right\|_{l^1[0, T]}}{\sum_{k=0}^{\infty} \left\| \begin{pmatrix} u_0(k) \\ y_0(k) \end{pmatrix} \right\|_{l^1[0, T]}}. \end{aligned} \quad (23)$$

As the system has two disturbances, u_0 and y_0 , it has two gains. The gain of $\Pi_{\mathcal{M}/\mathcal{N}}$ can therefore be bounded by examining the gains due to u_0 and y_0 separately and using the following property.

$$\frac{\|\Pi_{\mathcal{M}/\mathcal{N}} \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}\|}{\left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\|} \leq \|\Pi_{\mathcal{M}/\mathcal{N}}|_{y_0=0}\| + \|\Pi_{\mathcal{M}/\mathcal{N}}|_{u_0=0}\| \quad (24)$$

since $\left\| \begin{pmatrix} u_0 \\ 0 \end{pmatrix} \right\|, \left\| \begin{pmatrix} 0 \\ y_0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\|$.

Firstly, let us examine y_0 and so set $u_0 = 0$. By various substitutions; triangle inequalities; and rearrangement

of summations [21, Definition 8.21 (due to all terms being normed and therefore positive)] we obtain:

$$\begin{aligned} & \|\Pi_{\mathcal{M}/\mathcal{N}} \begin{pmatrix} 0 \\ y_0 \end{pmatrix}\|_{l^1(\mathbb{N} \times [0, T])} \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \left\| \begin{pmatrix} I \\ P \end{pmatrix} \sum_{i=1}^k [Q(I - LP)]^{i-1} QL y_0(k-i) \right\|_{l^1[0, T]} \\ &\leq \left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\| \|QL\| \sum_{j=0}^{\infty} \left\| [Q(I - LP)]^j \right\| \sum_{n=0}^{\infty} \|y_0(n)\|_{l^1[0, T]} \end{aligned} \quad (25)$$

Hence, as $\|Q(I - LP)\| < 1$ and $\|Q(I - LP)^j\| \leq \|Q(I - LP)\|^j$ $0 \leq j \leq \infty$,

$$\|\Pi_{\mathcal{M}/\mathcal{N}}|_{u_0=0}\|_{l^1(\mathbb{N} \times [0, T])} \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix} \leq \frac{\left\| \begin{pmatrix} 1 \\ P \end{pmatrix} \right\| \|QL\|}{1 - \|Q(I - LP)\|} \quad (26)$$

Now moving on to u_0 and so setting $y_0 = 0$ we are able to follow a similar process to obtain

$$\begin{aligned} & \|\Pi_{\mathcal{M}/\mathcal{N}}|_{y_0=0}\|_{l^1(\mathbb{N} \times [0, T])} \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix} \\ &\leq \frac{\left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\| \|QLP\|}{1 - \|Q(I - LP)\|} + \left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\|. \end{aligned} \quad (27)$$

By (26), (27) and the property given by (24) we can conclude that

$$\begin{aligned} & \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{l^1(\mathbb{N} \times [0, T])} \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix} \\ &\leq \left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\| \left(1 + \frac{\|QLP\| + \|QL\|}{1 - \|Q(I - LP)\|} \right). \quad \blacksquare \end{aligned}$$

In order to bound the summation in (26) we required that $\|Q(I - LP)\| < 1$. This condition arises in a host of papers including [22], [23], [24], [25] and [10]; where this is supplied as the sufficient condition for monotonic convergence of this algorithm.

D. 2D Robust Stability

We now have a bound on $\|\Pi_{\mathcal{M}/\mathcal{N}}\|^{-1}$ for $\mathcal{W} = l^1(\mathbb{N} \times [0, T])$ so we can apply the robust stability theorem:

Theorem 4: Consider the closed loop system given by fig. 1, and (14) and (16). Suppose $\|Q(I - LP)\| < 1$. Suppose P_1 is stabilisable, $[P_1, C]$ is globally well posed, let $\mathcal{W} = l^1(\mathbb{N} \times [0, T])$ and $x_0 = \begin{pmatrix} 0 \\ y_{ref} \end{pmatrix}$.

Define

$$b_{P,C} = \left(\left\| \begin{pmatrix} I \\ P \end{pmatrix} \right\| \left(1 + \frac{\|QLP\| + \|QL\|}{1 - \|Q(I - LP)\|} \right) \right)^{-1} \quad (28)$$

(with all norms on the right in $l^1[0, T]$), and

$$\vec{\delta}_{\mathcal{W}, x_0}(P, P_1) = \inf_{\Phi \in \mathcal{Q}_{P, P_1}^{\mathcal{W}, x_0}} \|(\Phi - I)|_{\mathcal{M}}\|_{\mathcal{W}, g_0} \quad (29)$$

where $g_0 = \Pi_{\mathcal{M}/\mathcal{N}} x_0$ and

$$\mathcal{Q}_{P, P_1}^{\mathcal{W}} := \{\Phi : \mathcal{G}_P^{\mathcal{W}} \rightarrow \mathcal{G}_{P_1}^{\mathcal{W}} : \Phi \text{ is causal and surjective}\}. \quad (30)$$

If

$$\vec{\delta}_{\mathcal{W}, x_0}(P, P_1) < b_{P,C} \quad (31)$$

then stability of $[P_1, C]$ is assured on $\mathcal{W}$ and

$$\begin{aligned} & \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W}, w_0} \\ &\leq \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W}, x_0} \frac{1 + \vec{\delta}_{\mathcal{W}, x_0}(P, P_1)}{1 - \|\Pi_{\mathcal{M}/\mathcal{N}}\|_{\mathcal{W}, x_0} \vec{\delta}_{\mathcal{W}, x_0}(P, P_1)} \end{aligned} \quad (32)$$

with $w_0 = (I + (\Phi - I)\Pi_{\mathcal{M}/\mathcal{N}})x_0$.

Proof: The proof follows directly from theorems 2 and 3. $\blacksquare$

This theorem provides a 2D robust stability guarantee for a set of closed loop linear plants engaged in trajectory tracking within the lifted system framework, providing a set of plants for which the controller is sufficient to guarantee stability.

V. INTERPRETATION

On closer inspection of the result it is clear that the smallest gain and hence the largest stability margin can be achieved by setting the filter Q to zero. From a robust stability view this would provide the best possible stability margin. Clearly however, changing Q alters the asymptotic properties of the system.

By examination of the third line in (22) (which shows the plant's input and output trajectories on trial k for zero disturbances) we can derive an expression for the limit of the plant's output trajectory as the iteration number tends to infinity in the face of zero disturbances¹:

$$y_{\infty} = \lim_{k \rightarrow \infty} P \sum_{i=0}^k [Q(I - LP)]^i (QL y_{ref}) \quad (33)$$

Since $\|Q(I - LP)\| < 1$, it follows that $[Q(I - LP)]^{k+1} \rightarrow 0$ as $k \rightarrow \infty$ leaving

$$y_{\infty} = P(I - Q(I - LP))^{-1} (QL y_{ref}). \quad (34)$$

It can be clearly seen from this that setting Q to I allows the plant's output to converge to the reference signal ($y_{\infty} = y_{ref}$). This however requires the plant and learning matrix to obey $\|(I - LP)\| < 1$ in order to fulfil the stability conditions. If $Q = 0$ then no learning takes place; apparent in the above equation by $y_{\infty} = 0$.

This is the reasoning behind Q as a filter; it makes it possible to set the asymptotic convergence properties [23, Theorem 9]. An example of this could be to set Q to attenuate high frequency error components. In this case Q would approximate I at low frequencies and fall-off at higher frequencies.

From a robust stability point of view, if a relative degree 1 plant is controlled by a P-type controller the resulting LP matrix would not be full row rank and so, in order for the stability condition to be satisfied, Q could be reduced such that $\|(Q(I - LP))\| < 1$. If the closed-loop system is regarded as the controller and a relative degree 0 plant

¹Since $u_0, y_0 \in l^1(\mathbb{N} \times [0, T])$ it follows that $u_0(k), y_0(k) \rightarrow 0$ as $k \rightarrow \infty$.

with some form of perturbation, reducing Q could be seen as increasing the stability margin around this system to stabilise the relative degree 1 perturbed plant.

Another point to consider is the case of inverse-model based ILC. From a convergence perspective it is apparent that if the learning matrix L is given by P^{-1} , and Q by the identity, then very good results can be achieved as $Q(I - LP)$ would be zero. A major weakness in this approach is demonstrated explicitly by the definition of $b_{P,C}$ in (28). The numerator of the fraction contains two terms; the left-most term would be zero, however it can be assumed that if L were the inverse of P , then $\|L\|$ would likely result in a very large gain and hence small stability margin. Clearly, matching the controller too tightly to the plant can result in a severe degradation of robustness to plant uncertainty.

VI. CONCLUSIONS

This paper has demonstrated the use of the gap metric to examine the robust stability of ILC systems engaged in tracking tasks. The biased robust stability theorem from [3] was adapted to fit with the 2D structure of ILC using a norm adapted from [19], retaining the bias as the reference trajectory. The analysis here has generalised some of the work in [19] to take account of tracking trajectories and the gain bound has been given for the case of linear ILC algorithms.

The method employs a general unstructured uncertainty model and so is able to account for a large range of possible uncertainty representations. In contrast, more structured uncertainty models have strong limitations, e.g. additive uncertainty does not allow stable and unstable models to be compared; and parametric does not permit changes on model order [2]. The results in theorem 2 are also by no means restricted to linear systems.

The robust stability margin depends on the gain from external to internal signals and so was calculated for systems expressed in the supervector formulation. This can be used for design purposes to find a balance between an appropriate stability margin and perfect tracking.

The problem of actually calculating the gap in 2D has not been approached; the analysis here is more qualitative in nature, trying to find the effects that different ILC methods have on a system's robustness properties. It is believed that this is the first step into gap metric results for ILC. It is expected that the results given are reasonably conservative; however it is hoped that they will lead to some more concrete conditions in future work, provide some conditions that would enable an optimisation over performance and robustness for ILC, and perhaps lead to a benchmarking tool for evaluating the robustness of different ILC algorithms.

REFERENCES

- [1] Y. Huang and R. W. Longman, "The source of the often observed property of initial convergence followed by divergence in learning and repetitive control," *Advances in the Astronautical Sciences*, vol. 90, pp. 555–572, 1996.
- [2] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: an input output approach," *IEEE Transactions on Automatic Control*, vol. 42, no. 9, pp. 1200–1221, September 1997.
- [3] T. Georgiou and M. Smith, "Biased norms and robustness analysis for nonlinear feedback systems," in *Proceedings of the 36th IEEE Conference on Decision and Control*, 1997, pp. 642–643.
- [4] J. Ghosh and B. Paden, "Pseudo-inverse based iterative learning control for plants with unmodelled dynamics," in *Proceedings of the American Control Conference*, Chicago, Illinois, USA, June 2000, pp. 472–476.
- [5] D. de Roover and O. H. Bosgra, "Synthesis of robust multivariable iterative learning controllers with applications to a wafer stage motion system," *International Journal of Control*, vol. 73, no. 10, pp. 968–979, 2000.
- [6] P. B. Goldsmith, "On the equivalence of causal LTI iterative learning control and feedback control," *Automatica*, vol. 38, pp. 703–708, 2002.
- [7] —, "The fallacy of causal iterative learning control," in *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida, USA, December 2001, pp. 4475–4480.
- [8] M. H. A. Verwoerd, G. Meinsma, and T. J. A. de Vries, "One the use of noncausal LTI operators in iterative learning control," in *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, December 2002, pp. 3362–3366.
- [9] T. Sugie and T. Ono, "An iterative control law for dynamical systems," *Automatica*, vol. 27, no. 4, pp. 729–732, 1991.
- [10] T. J. Harte, J. Hätonen, and D. H. Owens, "Discrete-time inverse model-based iterative learning control: stability, monotonicity and robustness," *International Journal of Control*, vol. 78, no. 8, pp. 577–586, May 2005.
- [11] A. Tayebi and M. B. Zeremba, "Robust iterative learning control design is straightforward for uncertain LTI systems satisfying the robust performance condition," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 101–106, 2003.
- [12] J.-H. Moon, T.-Y. Doh, and M. J. Chung, "A robust approach to iterative learning control design for uncertain systems," *Automatica*, vol. 34, no. 8, pp. 1001–1004, 1998.
- [13] T. Donkers, J. van de Wijdeven, and O. Bosgra, "A design approach for noncausal robust iterative learning control using worst case disturbance optimisation," in *Proceedings of the American Control Conference*, Seattle, Washington, USA, 2008, pp. 4567–4572.
- [14] —, "Robustness against model uncertainties of norm optimal iterative learning control," in *Proceedings of the American Control Conference*, Seattle, Washington, USA, 2008, pp. 4561–4566.
- [15] J. van de Wijdeven and O. Bosgra, "Noncausal finite-time robust iterative learning control," in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, December 2007, pp. 258–263.
- [16] H.-S. Ahn, K. L. Moore, and Y. Chen, *Iterative Learning Control: Robustness and Monotonic Convergence for Interval Systems*. Springer, 2007.
- [17] —, "Monotonic convergent iterative learning controller design based on interval model conversion," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 366–371, February 2006.
- [18] —, "Schur stability radius bounds for robust iterative learning controller design," in *Proceedings of the American Control Conference*, Portland, OR, USA, June 2005, pp. 178–183.
- [19] M. French, "Robust stability of iterative learning control schemes," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 10, pp. 1018–1033, July 2008.
- [20] K. L. Moore, *Iterative Learning Control for Deterministic Systems*. Springer-Verlag, 1993.
- [21] T. M. Apostol, *Mathematical Analysis*. Addison-Wesley, 1974.
- [22] J. Hätonen, "Issues of algebra and optimality in iterative learning control," Ph.D. dissertation, Department of Process and Environmental Engineering, University of Oulu, 2004.
- [23] M. Norlöf and S. Gunnarsson, "Time and frequency domain convergence properties in iterative learning control," *International Journal of Control*, vol. 75, no. 14, pp. 1114–1126, 2002.
- [24] J. J. Hätonen, D. H. Owens, and K. L. Moore, "An algebraic approach to iterative learning control," *International Journal of Control*, vol. 77, no. 1, pp. 45–54, January 2004.
- [25] M. Norlöf, "Iterative learning control analysis, design, and experiments," Ph.D. dissertation, Linköping Studies in Science and Technology, 2000.