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Central Suboptimal H_∞ Control Design for Nonlinear Polynomial Systems

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***Abstract*— This paper presents the central finite-dimensional H_∞ regulator for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the previously obtained results, the paper reduces the original H_∞ control problem to the corresponding optimal H_2 control problem, using the technique proposed in [1]. The paper yields the central suboptimal H_∞ regulator for nonlinear polynomial systems in a closed finite-dimensional form, based on the optimal H_2 regulator obtained in [2]. Numerical simulations are conducted to verify performance of the designed central suboptimal regulator for nonlinear polynomial systems against the central suboptimal H_∞ regulator available for the corresponding linearized system.**

I. INTRODUCTION

Over the past two decades, the considerable attention has been paid to the H_∞ control (regulator) problems for linear and nonlinear systems. The seminal H_∞ control paper [1] established a background for consistent treatment of regulator design problems in the H_∞ framework. The H_∞ regulator design implies that the resulting closed-loop control system is robustly stable and achieves a prescribed level of attenuation from the disturbance input to the output in L_2/l_2-norm. A large number of results on this subject have been reported for systems in the general situation, linear or nonlinear (see ([3]–[16]). The sufficient conditions for existence of an H_∞ regulator, where the control gain matrices satisfy Riccati equations, were obtained for linear systems in [1]. However, the criteria of existence and suboptimality of solution for the central H_∞ control problems based on the reduction of the original H_∞ problem to the induced H_2 one, similar to those obtained in [1] for linear systems, remain yet unknown for nonlinear polynomial systems.

Although the optimal LQR problem for linear systems was solved in 1960s [17], [18], the optimal regulator for nonlinear systems has to be determined using the general principles of maximum [18] or dynamic programming [19], which do not provide an explicit form for the optimal control in most cases. Thus, there is a long tradition of the optimal control design for various classes of nonlinear systems (see, for example, [20]–[25]), in particular, polynomial [2] systems.

This paper presents the central (see [1] for definition) finite-dimensional H_∞ regulator for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to the results previously obtained for nonlinear systems [3]–[8], the paper reduces the original H_∞ control problem to the corresponding optimal H_2 control problem, using the technique proposed in [1]. To the best authors' knowledge, this is the first paper which applies the reduction technique of [1] to certain classes of nonlinear systems. Indeed, application of the reduction technique makes sense, since the optimal regulator equations solving the H_2 control problems have been obtained for nonlinear polynomial systems [2]. Designing the central suboptimal H_∞ regulator for nonlinear polynomial systems presents a significant advantage in the control theory and practice, since (1) it enables one to address H_∞ control problems for non-autonomous nonlinear polynomial systems, where the LMI technique is hardly applicable and the HJB equation-based methods fail to provide a closed-form solution, (2) the obtained H_∞ regulator is suboptimal, that is, optimal for any fixed γ with respect to the H_∞ noise attenuation criterion, and (3) the obtained H_∞ regulator is finite-dimensional and has the same structure of the controlled state and gain matrix equations as the corresponding optimal H_2 regulator.

It should be commented that the proposed design of the central suboptimal H_∞ regulator for nonlinear polynomial systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal H_2 regulator (see [2]) for nonlinear polynomial systems with unbounded disturbances (white noises). The entire design approach creates a complete control algorithm of handling the nonlinear polynomial systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds γ uniformly or for any fixed γ separately. A similar algorithm for linear systems was developed in [1].

Numerical simulations are conducted to verify performance of the designed central suboptimal regulator for nonlinear polynomial systems against the central suboptimal H_∞ regulator available for the corresponding linearized system. The simulation results show a definite advantage in the values of the noise-output transfer function H_∞ norm in favor of the designed regulator.

The paper is organized as follows. Section 2 presents the H_∞ control problem statement for nonlinear polynomial

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systems. The central suboptimal H_∞ regulator for nonlinear polynomial systems is designed in Section 3. An example verifying performance of the H_∞ regulator designed in Section 3 against the central suboptimal H_∞ regulator available for the corresponding linearized system is given in Section 4. Section 5 presents conclusions to this study.

II. H_∞ Control Problem Statement for Polynomial Systems

Consider the following continuous-time polynomial system:

$$\begin{aligned} \mathscr{S}_1 : \dot{x}(t) &= f(x,t) + B(t)u(t) + G(t)\omega(t), \quad (1)\\ x(t_0) &= x_0,\\ z(t) &= L(t)x(t) + D(t)u(t), \quad (2) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^q$ is the output, $\omega(t) \in \mathscr{L}_2^k[0,\infty)$ is the disturbance input, $B(\cdot)$, $D(\cdot)$, $G(\cdot)$, and $L(\cdot)$ are known continuous functions.

The nonlinear function $f(x,t) \in R^n$ is considered a polynomial of n variables, components of the state vector $x(t) \in R^n$, with time-dependent coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [26], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = \alpha_0(t) + \alpha_1(t)x + \alpha_2(t)xx^T + \ldots + \alpha_p(t)x \ldots_{p\ times} \ldots x^T,$$

where $\alpha_0(t)$ is a vector of dimension n, α_1 is a matrix of dimension $n \times n$, α_2 is a 3D tensor of dimension $n \times n \times n$, α_p is an $(p+1)$D tensor of dimension $n \times \ldots_{(p+1)\ times} \ldots \times n$, and $x \times \ldots_{p\ times} \ldots \times x$ is a pD tensor of dimension $n \times \ldots_{p\ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself (see [26] for more definition). Such a polynomial can also be expressed in the summation form

$$f_k(x,t) = \alpha_{0\ k}(t) + \sum_i \alpha_{1\ ki}(t)x_i(t) + \sum_{ij} \alpha_{2\ kij}(t)x_i(t)x_j(t) + \ldots \quad (3)$$
$$+ \sum_{i_1 \ldots i_p} \alpha_{p\ ki_1 \ldots i_p}(t)x_{i_1}(t) \ldots x_{i_p}(t), \quad k,i,j,i_1,\ldots,i_p = 1,\ldots,n.$$

For the system (1),(2), the following standard condition (see [1] for linear systems) are assumed:

- the state $x(t)$ governed by (1) is uniformly stabilizable; $(\mathscr{C}_1)$
- $D^T(t)L(t) = 0$ and $D^T(t)D(t)$ is a positive definite matrix. $(\mathscr{C}_2)$

Here, I_m is the identity matrix of dimension $m \times m$. The definitions of the uniform stabilizability for nonlinear systems can be found in [27].

The $\mathscr{H}_\infty$ control problem to be addressed is as follows: develop a robust $\mathscr{H}_\infty$ regulator for the polynomial system $(\mathscr{S}_1)$, such that the following two requirements are satisfied.

1) The resulting controlled system dynamics $(\mathscr{S}_1)$ is robustly asymptotically stable in the absence of disturbances, $\omega(t) \equiv 0$;
2) The controlled system dynamics $(\mathscr{S}_1)$ ensures a noise attenuation level γ in an $\mathscr{H}_\infty$ sense. More specifically, for zero state initial conditions and any nonzero disturbance input $\omega(t) \in \mathscr{L}_2^p[0,\infty)$, the inequality
$$\|z(t)\|_2^2 < \gamma^2 \|\omega(t)\|_2^2 \quad (4)$$
holds, where $\|f(t)\|_2^2 := \int_{t_0}^{T_1} f^T(t)f(t)dt$, T_1 is the rightmost point of a time interval where the solution of (1) exists and is bounded, and γ is a given real positive scalar.

Remark 1. Hereinafter, the formulated $\mathscr{H}_\infty$ control problem is considered in a time interval $[t_0, T_1]$, $T_1 < T^*$, T^* is an escape time for the system (1). Thus, the solution of the state equation (1) still exists and is bounded in $[t_0, T_1]$.

III. Design of Central H_∞ Regulator for Polynomial Systems

The proposed design of the central H_∞ regulator (see Theorem 4 in [1] for definition) for polynomial systems is based on the general result (see Theorem 3 in [1]) reducing the H_∞ controller problem to the corresponding optimal H_2 controller problem. In this paper, only the control (regulator) part of this result, valid for the entire controller problem, is used. Then, the optimal H_2 polynomial-quadratic regulator for polynomial systems ([2]) is employed to obtain the desired result, which is given by the following theorem.

Theorem 1. *The central H_∞ regulator for the polynomial state with linear control input (1), ensuring the H_∞ noise attenuation condition (4) for the output $z(t)$, is given by the control law*

$$u(t) = (D^T(t)D(t))^{-1}B^T(t)[Q(t)x(t) + p(t)], \quad (5)$$

where the matrix function $Q(t)$ is the solution of the quadratic equation

$$\begin{aligned} \dot{Q}(t) = {} & L^T(t)L(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \\ & \ldots + pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q(t) - \quad (6)\\ & Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \ldots \\ & + a_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)] - \\ & Q(t)[B(t)(D^T(t)D(t))^{-1}B^T(t) - \gamma^{-2}G(t)G^T(t)]Q(t), \end{aligned}$$

with the terminal condition $Q(T_1) = 0$, and the vector function $p(t)$ is the solution of the linear equation

$$\begin{aligned} \dot{p}(t) = {} & -Q(t)a_0(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \\ & \ldots + pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T p(t) - \quad (7)\\ & Q(t)[B(t)(D^T(t)D(t))^{-1}B^T(t) - \gamma^{-2}G(t)G^T(t)]p(t), \end{aligned}$$

with the terminal condition $p(T_1) = 0$. The optimally controlled system $(\mathscr{S}_1)$ takes the form

$$\begin{aligned} \dot{x}(t) = {} & f(x,t) + B(t)(D^T(t)D(t))^{-1} \times \quad (8)\\ & B^T(t)[Q(t)x(t) + p(t)] + G(t)\omega(t), \quad x(t_0) = x_0. \\ z(t) = {} & L(t)x(t) + D(t)(D^T(t)D(t))^{-1}B^T(t)[Q(t)x(t) + p(t)]. \end{aligned}$$

Proof. According to Theorem 3 from [1], the central H_∞ control (regulator) problem is equivalent to the H_2 optimal control problem, where the H_2 quadratic cost function is in the Bolza-Meyer form

$$J = \frac{1}{2}\int_{t_0}^{T_1} [u^T(s)(D^T(t)D(t))u(s) + x^T(s)(L^T(s)L(s))x(s)]ds, \quad (9)$$

and the matrix $B(t)(D^T(t)D(t))^{-1}B^T(t)$ should be changed to $B(t)(D^T(t)D(t))^{-1}B^T(t) - \gamma^{-2}G(t)G^T(t)$ in the gain matrix equation. Therefore, as follows from Theorem 1 in [2], the solution of the optimal H_2 (linear-quadratic) control problem for the polynomial state (1) with linear control input with respect to the quadratic criterion (9) is given by the equation for the optimally controlled state

$$\dot{x}(t) = f(x,t) + B(t)(D^T(t)D(t))^{-1}(t)\times \quad (10)$$
$$B^T(t)[Q(t)x(t) + p(t)] + G(t)\omega(t), \quad x(t_0) = x_0.$$

where the matrix function $Q(t)$ is the solution of the quadratic equation

$$\dot{Q}(t) = L^T(t)L(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \dots$$
$$+pa_p(t)x(t)\dots_{p-1\ times}\dots x(t)]^T Q(t) -$$
$$Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \dots$$
$$+a_p(t)x(t)\dots_{p-1\ times}\dots x(t)] -$$
$$Q(t)B(t)(D^T(t)D(t))^{-1}B^T(t)Q(t),$$

with the terminal condition $Q(T_1) = 0$, and the vector function $p(t)$ is the solution of the linear equation

$$\dot{p}(t) = -Q(t)a_0(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \dots$$
$$+pa_p(t)x(t)\dots_{p-1\ times}\dots x(t)]^T p(t) - \quad (12)$$
$$Q(t)B(t)(D^T(t)D(t))^{-1}B^T(t)p(t),$$

with the terminal condition $p(T_1) = 0$. Taking into account the correspondence between the matrix $B(t)(D^T(t)D(t))^{-1}B^T(t)$ in the H_2 optimal control problem and the matrix $B(t)(D^T(t)D(t))^{-1}B^T(t) - \gamma^{-2}G(t)G^T(t)$ in the central H_∞ control (regulator) problem yields the result of Theorem 1, i.e., the central H_∞ regulator equations (5)-(8). ■

Remark 2. The boundedness of the controlled system state $x(t)$, as well as the regulator gain matrix $Q(t)$ and the vector $p(t)$, is determined by the definiteness of the most superior polynomial term in the right-hand sides of (6),(7). If this term is stable, then $x(t)$, $Q(t)$, and $p(t)$ remain bounded for all $t \in [t_0, T_1]$, where $T_1 < \infty$ is any finite time moment, and the regulator gain matrix $P(t)$ also remains negative definite. In the latter case, it makes sense to consider the H_∞ noise-output attenuation problem with a certain level γ in the infinite interval $[t_0, \infty)$. Otherwise, if the most superior polynomial terms in (6),(7) are unstable, then $x(t)$, $Q(t)$, and $p(t)$ diverge to infinity for a finite time and the designed regulator does not work properly for all $t \in [t_0, \infty)$. However, even in this case, the designed central suboptimal H_∞ regulator for polynomial systems yields the least possible value of the output H_∞ norm in those finite time intervals $[t_0, T_1]$, where the solution of (8) exists and is bounded.

Remark 3. According to the comments in Subsection V.G in [1], the obtained central H_∞ regulator (5)–(8) presents a natural choice for H_∞ regulator design among all admissible H_∞ regulators satisfying the inequality (4) for a given threshold γ, since it does not involve any additional actuator loop (i.e., any additional external state variable) in constructing the regulator gain matrix. Moreover, the obtained central H_∞ regulator (5)–(8) has the suboptimality property, i.e., it minimizes the criterion

$$J = \|\tilde{z}(t)\|_2^2 - \gamma^2 \|\omega(t)\|_2^2$$

for any such positive $\gamma > 0$ that the the controlled system is stable.

Remark 4. Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained H_∞ regulator (5)–(8), since the the regulator gain matrix $Q(t)$ is negative definite as the solution of (6). Therefore, the stability failure is the only reason why the designed regulator can stop working.

IV. Example: Central H_∞ Regulator for Polynomial System

This section presents an example of designing the central H_∞ regulator for a second degree polynomial system with linear control input and comparing it to the central H_∞ regulator available for the corresponding linearized system, that is the regulator obtained in Theorems 3 and 4 from [1].

Let the system state $x(t) = [x_1(t), x_2(t)] \in R^2$ be given by

$$\dot{x}_1(t) = x_2(t), \quad (13)$$
$$\dot{x}_2(t) = 0.1x_2^2(t) + u_1(t) + w(t),$$

with the initial condition $x(0) = x_0 = [x_{10}, x_{20}]$, and the output $z(t) \in R$ be represented as

$$z(t) = x_1(t) + u_2(t), \quad (14)$$

Here, $w(t)$ is an L_2^2 disturbance input. It can be readily verified that the condition $(\mathscr{C}_2)$ (see Section 2) holds for the system (13),(14).

The H_∞ control problem to be addressed is the same as described at the end of Section 2. Note that the second degree coefficient in (13) is positive, i.e., the superior polynomial term is unstable (see Remark 1 in Section 3). The control horizon is set to $t_1 = 2.1$, prior to the escape time for the system state (13).

The H_∞ regulator equations (5)–(8) take the following particular form for the system (13),(14). The control law (5) is given by

$$u^*(t) = Q(t)x(t) + p(t), \quad (15)$$

where the entries of the gain matrix $Q(t)$ and the vector $p(t)$ satisfy the equations

$$\dot{Q}_{11}(t) = 1 - (1 - \gamma^{-2})Q_{12}^2(t), \quad (16)$$

$$\dot{Q}_{12}(t) = Q_{11}(t) + 0.2x_2(t)Q_{12}(t) - (1-\gamma^{-2})Q_{12}(t)Q_{22}(t),$$
$$\dot{Q}_{22}(t) = 2Q_{12}(t) + 0.4x_2(t)Q_{22}(t) - (1-\gamma^{-2})Q_{22}^2(t),$$

with the zero terminal conditions, $Q(2.1) = 0$, for all entries of the matrix $Q(t)$, and

$$p(t) \equiv 0. \tag{17}$$

The obtained system (13),(15),(16) can be solved using simple numerical methods, such as "shooting." This method consists in varying initial conditions of (16) until the given terminal condition is satisfied.

Upon substituting the control (15),(17) into (13),(14) the optimally controlled state equations take the form

$$\dot{x}_1(t) = x_2(t), \tag{18}$$
$$\dot{x}_2(t) = 0.1x_2^2(t) + Q_{12}(t)x_1(t) + Q_{22}(t)x_2(t) + w(t).$$

The designed regulator (15)–(18) is compared to the central H_∞ linear regulator given by Theorems 3 and 4 in [1]. The central H_∞ linear regulator, applied to the linearized system (13),(14) yields the control law is given by

$$u(t) = Q(t)x(t), \tag{19}$$

where the entries of the gain matrix $Q(t)$ satisfy the equations

$$\dot{Q}_{11}(t) = 1 - (1-\gamma^{-2})Q_{12}^2(t), \tag{20}$$
$$\dot{Q}_{12}(t) = Q_{11}(t) + 0.2Q_{12}(t) - (1-\gamma^{-2})Q_{12}(t)Q_{22}(t),$$
$$\dot{Q}_{22}(t) = 2Q_{12}(t) + 0.4Q_{22}(t) - (1-\gamma^{-2})Q_{22}^2(t),$$

with the zero terminal conditions, $Q(2.1) = 0$, for all entries of the matrix $Q(t)$.

Upon substituting the control (19) into (13),(14) the optimally controlled state equations take the same form as the equations (18).

For numerical simulation of the system (13),(14) and the regulator equations (15)–(18) and (19),(20), the initial values $x_1(0) = 1$ and $x_2(0) = 0$ are assigned. The L_2 disturbance $w(t)$ is realized as $w_1(t) = 1/(1+t)^2$. The attenuation level value is set to $\gamma = 1.05$.

The following graphs are obtained: graphs of the H_∞ controlled output $z(t)$ corresponding to the regulator (15)–(18); graphs of the H_∞ controlled output $z(t)$ corresponding to the regulator (19),(20) (Fig. 1). The graphs of the controlled outputs are shown in the entire simulation interval from $t_0 = 0$ to $t_1 = 2.1$. Figure 1 also demonstrates the dynamics of the noise-output H_∞ norms corresponding to the shown H_∞ controlled outputs in each case.

The following values of the noise-output H_∞ norm $T = \|z(t)\|/\|\omega(t)\|$ are obtained for the simulated disturbance $w(t)$ at the final simulation time $t_1 = 2.1$: $T = 0.05$ for the H_∞ controlled output $z(t)$ corresponding to the regulator (15)–(18); $T = 0.48$ for the H_∞ controlled output $z(t)$ corresponding to the regulator (19),(20).

It can be concluded that the central suboptimal H_∞ regulator (15)–(18) provides reliably convergent behavior of the output, yielding very small values of the corresponding H_∞ norm, even in comparison to the assigned threshold value $\gamma = 1.05$, and almost zero output values in the final time. In contrast, the conventional central H_∞ regulator (19),(20) provides divergent behavior of the output, yielding a larger value of the corresponding H_∞ norm. Thus, the simulation results show definite advantages of the designed central suboptimal H_∞ regulator for polynomial systems, in comparison to the previously known conventional H_∞ linear regulator.

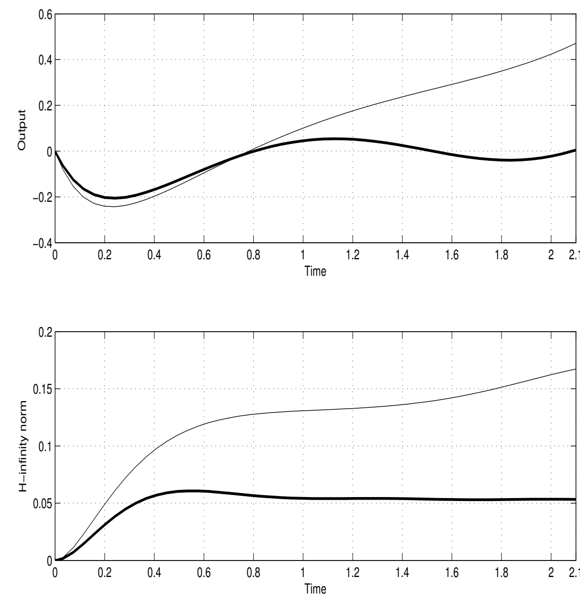

Fig. 1. **Above.** Graphs of the H_∞ controlled outputs corresponding to the regulator (15)–(18) (thick line) and the regulator (19),(20) (thin line), in the simulation interval [0,2.1]. **Below.** Graph of the noise-output H_∞ norm T for the shown H_∞ controlled outputs corresponding to the regulator (15)–(18) (thick line) and the regulator (19),(20) (thin line), in the simulation interval [0,2.1].

V. CONCLUSIONS

This paper designs the central finite-dimensional H_∞ regulator for nonlinear polynomial systems, that is suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In the example, the numerical simulations are conducted to verify performance of the designed central suboptimal regulator for a second order polynomial system against the central suboptimal H_∞ regulator available for the corresponding linearized system. The simulation results show a definite advantage in the values of the noise-output transfer function H_∞ norm in favor of the designed regulator. In particular, the estimation error given by the obtained regulator converges to zero, whereas the estimation error of the conventional regulator diverges. This significant improvement in the estimate behavior is obtained due to the more careful selection of the control gain matrix in the designed regulator. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration.

The proposed design of the central suboptimal H_∞ regulators for nonlinear polynomial systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal H_2 regulators for nonlinear polynomial systems with unbounded disturbances (white noises). The entire design approach creates a complete control algorithm of handling the nonlinear polynomial systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds γ uniformly or for any fixed γ separately. The presented approach would be applied in the future to obtain the central suboptimal H_∞ regulators for nonlinear polynomial time-delay systems.

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