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DATA-BASED UNCERTAINTY MODELING BY CONVEX OPTIMIZATION TECHNIQUES

Kurt E. Häggblom

Process Control Laboratory, Faculty of Technology,
Åbo Akademi University, FI-20500 Turku, Finland

Abstract: A procedure based on convex optimization techniques for deriving norm-bounded uncertainty models for MIMO systems is presented. The procedure is developed for unstructured additive uncertainty models, but in principle this is no limitation since any uncertainty model of LFT type can be transformed into such a model. The models are determined by matching to process data available in the form of frequency responses of a set of individual models or sets of input-output data. Conditions for the existence of solutions to the data-matching problems are defined by LMIs. Uncertainty models that tightly match the data are obtained by minimizing an ellipsoidal uncertainty region. An application to distillation is included.

Keywords: Uncertainty modeling, linear multivariable systems, convex optimization, linear matrix inequalities, determinant minimization, robust control, distillation columns.

1. INTRODUCTION

Many robust control design methods are based on a linear transfer function model incorporating a weighted norm-bounded uncertainty description. The construction of a non-conservative uncertainty model from process data is a significant problem.

The generation of useful process data for uncertainty modeling is a nontrivial task. This task is especially difficult if the system is multivariable and ill-conditioned or nonlinear. In practice, the dynamics of such a system cannot be captured in a single linear model without some advanced uncertainty modeling.

An appealing approach to uncertainty modeling is first to determine a set of individual models, then to construct an uncertainty model that encompasses all these models. In this way "difficult" dynamics, which may be hard to include in a single model, can be split between several models. Furthermore, the use of several models may facilitate the task of separating noise and dynamics so that noise and bias largely can be excluded from the model set (Häggblom and Böling, 1998).

One way of constructing a norm-bounded uncertainty model from a set of individual models is to employ model-matching techniques. The basis of this method is that the uncertainty model should be capable of reproducing every model in the model set. Such a technique has been used by Hindi *et al*. (2002). However, if the identification has given several models, it is because different input sequences and operating points generate different models. It is then valid to assume that a model only applies to the input sequence used to generate the data from which the model was determined. This suggests derivation of an uncertainty model using input-output matching instead of model matching. It can be shown that model matching cannot produce a model less conservative than input-output matching for a given model structure (Nyström *et al.*, 2003) .

An important aspect in uncertainty modeling is the choice of objective function to be minimized so as to obtain a non-conservative uncertainty model, which "tightly" matches the known data. A natural choice is to minimize the largest possible discrepancy between a nominal model and the uncertainty model. This results in a norm-minimization problem and has been used, e.g., by Hindi *et al.* (2002).

In this paper it is argued that a less conservative uncertainty model is obtained by minimizing the size of the ellipsoidal region that the deviations between the outputs from the uncertainty model and a nominal model cover. This results in a determinant-minimization problem.

A significant task is also the development of a suitable numerical procedure for solving the problem. Generally, convex optimization methods are desired. In this paper it is shown that the determinant-minimization problem can be formulated as a convex optimization problem, where the various data-matching requirements and certain norm-related constraints can be expressed as linear matrix inequalities (LMIs). Both model matching and input-output matching as well as methods based on norm minimization can be handled by essentially the same procedure.

An application to uncertainty modeling of a distillation column is included.

2. PROBLEM FORMULATION

2.1 Uncertainty Description

We consider linear multiple-input multiple-output (MIMO) uncertainty models of the form

$$G(s) = G_0(s) + W_1(s)\Delta(s)W_2(s), \quad \|\Delta\|_\infty \le 1 \qquad (1)$$

where $G_0(s)$ is a stable nominal transfer matrix model, $W_1(s)$ and $W_2(s)$ are stable transfer matrix filters acting as uncertainty weights, and $\Delta(s)$ is a norm-bounded uncertainty matrix. Only uncertainty with an unstructured Δ matrix is considered in this paper.

Structurally, model (1) is an additive uncertainty model, but obviously multiplicative input and output uncertainties can be handled by including G_0 in W_1 or W_2. In fact, general linear fractional transformation (LFT) models

$$\hat{G} = \hat{G}_0 + \hat{W}_1\hat{\Delta}\left(I - \hat{W}_3\hat{\Delta}\right)^{-1}\hat{W}_2, \quad \left\|\hat{\Delta}\right\| \le 1, \left\|\hat{W}_3\right\| < 1 \qquad (2)$$

can be cast in the form of (1) (Chen and Gu, 2000; Hindi *et al*., 2002). Thus, (1) can represent a large class of uncertainty model types.

2.2 Data Matching in the Frequency Domain

We wish to determine W_1 and/or W_2 so that the uncertainty model (1) can reproduce sets of known frequency-response data with a minimum amount of conservatism. This means that any reduction of the region of uncertainty covered by the model would result in some data being irreproducible.

We shall use data matching as the technique for determining a non-conservative uncertainty model. It is assumed that frequency-response data are known at a number of relevant frequencies $\omega \in \Omega$ for a number of data sets k, $k = 1,\ldots,N$, either as (a) smoothed (noise-free) input-output data $\{u_k(\mathrm{j}\omega), y_k(\mathrm{j}\omega) : \omega \in \Omega\}$ or (b) transfer matrix data $\{G_k(\mathrm{j}\omega) : \omega \in \Omega\}$. In addition, a nominal model $G_0(\mathrm{j}\omega)$ may be known in either case. In the sequel, the argument "$\mathrm{j}\omega$" is omitted for convenience.

In practice, the data may be obtained from a number of identification experiments k, $k = 1,\ldots,N$. In case (a), the smoothing of output data can be accomplished by fitting a model G_k to data and using the output from this model as the output y_k, i.e.,

$$y_k = G_k u_k, \ \forall k \qquad (3)$$

This is a convenient way of excluding noise and retaining "difficult" dynamics, which cannot easily be included in a single model (e.g., due to nonlinearity), in the model set (Böling *et al*., 2004; Häggblom *et al*., 2003). The modeling technique to be described does not require a nominal model G_0 to be known initially, but if such a model is used, it can be obtained, e.g., by fitting a single model to all available input-output data.

Since the uncertainty modeling is based on data obtain, e.g., through identification, it is important that the identification experiments are thoroughly exciting and that they adequately cover the relevant operating region. We think that this requirement is of greater concern than, e.g., the possible uncertainty associated with the individual models G_k. Therefore, we do not take such uncertainty into account.

2.3 Input-Output Matching vs Model Matching

Case (a), mentioned above, can be considered an input-output matching problem and case (b) a model matching problem. Insofar as these kinds of techniques have been used for uncertainty modeling, model matching seems to be the predominant choice (see Hindi *et al*., 2002; Farag and Werner, 2004). However, model matching generally results in an uncertainty model having a larger region of uncertainty than a model determined by input-output matching (Nyström *et al*., 2003).

This can be explained by the fact that in model matching, every model G_k, $k = 1,\ldots,N$, is assumed to apply for all possible inputs, whereas only a single input-output pair (u_k, y_k) is associated with G_k in input-output matching. If the identification experiments have resulted in a set of models, it is because different input sequences (as well as different operating points and conditions) give different models. Thus, it is realistic to assume that a model applies only to the input sequence used for generating the data, from which the model was determined.

Because of this, we propose input-output matching as the main method for uncertainty modeling. However,

because both methods have merit, and only model data might be available in a given case, we do consider both methods in this paper. Furthermore, the techniques for input-output matching and model matching are quite similar.

3. UNCERTAINTY MODELING BY CONVEX OPTIMIZATION

3.1 Minimizing the Region of Uncertainty

Calculations for the uncertainty modeling are performed "frequency-by-frequency" for a set of relevant frequencies $\omega \in \Omega$. At each frequency, we desire a non-conservative norm-bounded uncertainty description

$$G = G_0 + W_1 \Delta W_2, \quad \bar{\sigma}(\Delta) \le 1 \tag{4}$$

where $\bar{\sigma}(\Delta)$ is the maximum singular value of Δ.

As a result of the optimization, we will obtain frequency responses of W_1 and W_2, and possibly of a new G_0, at all frequencies considered. A model of the form (1) requires that transfer function matrices are fitted to these frequency responses. Although the frequency responses are determined subject to appropriate data-matching conditions, the same data-matching conditions should again be enforced during the fitting so as not to introduce unnecessary conservatism or violation of constraints. However, we shall not consider this part of the problem in this paper.

Although the uncertainty modeling is based on certain frequency-response data, a model relevant for arbitrary (norm-bounded) inputs u, $\|u\| \le 1$, is desired. Here, $\|u\|$ denotes the Euclidean 2-norm of u. For such an input, the deviation e between the output from the uncertainty model and the nominal model is a measure of the uncertainty. This deviation is given by

$$e := (G - G_0)u = W_1 \Delta W_2 u \tag{5}$$

Here, $W_2 u$ is an input to the uncertainty block Δ. An unstructured uncertainty Δ, $\bar{\sigma}(\Delta) \le 1$, can then produce any output x bounded by $\|x\| \le \|W_2 u\|$. When x varies over its range of possible values, the deviation $e = W_1 x$ covers an ellipsoidal region. This interpretation may not be immediately obvious when the entities are complex valued, but it can be justified (Häggblom, 2005a). When W_1 has full row rank (which is necessary for arbitrary data matching), the size (volume) of this ellipsoid is proportional to $\det(W_1 W_1^*)^{1/2} \|x\|^n$, where n is the size of the matrix $W_1 W_1^*$ (equal to the number of outputs) and superscript $*$ denotes complex-conjugate transpose. We are interested in minimizing the size of this ellipsoid for the worst input u. This gives the objective function

$$J = \det(W_1 W_1^*)^{1/2} \|W_2\|^n \tag{6}$$

to be minimized subject to appropriate data-matching and other constraints.

The scaling factor $\|W_2\|$ introduces a potential problem in the minimization of J. However, since the values of the right-hand sides of (4) and (6) do not change if one of the weights is multiplied by a positive scalar and the other weight is divided by the same scalar, W_2 can, without loss of generality, be required to have a given norm and can thus be excluded from (6).

If possible, we require the objective function to be convex. The determinant in (6) is the determinant of a positive definite matrix, but it is not a convex function. Fortunately, it can be transformed to an equivalent convex function and one way of doing it is to take the logarithm of the determinant (Vandenberghe *et al.*, 1998). However, this results in a convex objective function for a maximization problem. Since we want to minimize the determinant, we need to use the inverse of the positive definite matrix. We thus want to

$$\text{minimize} \ \log\det Y^{-1} \tag{7}$$

where

$$Y := (W_1 W_1^*)^{-1/2} \succ 0 \tag{8}$$

Here, "$\succ$" denotes "positive definite".

An obvious alternative to the minimization of $\det Y^{-1}$ is minimization of $\|W_1\|$. This would minimize the largest possible deviation of e, but the size of the resulting uncertainty region would generally be larger than that obtained by determinant minimization. In norm minimization, a scalar weight would in fact be sufficient (Böling *et al.*, 2004).

For robustness reasons in controller design, it might be desirable to restrict $\|W_1\|$ even when $\det Y^{-1}$ is minimized. Let us introduce the restriction

$$\|W_1\| \le \gamma_1 \tag{9}$$

This is equivalent with the matrix inequalities

$$(W_1 W_1^*)^{1/2} = Y^{-1} \preccurlyeq \gamma_1 I \quad \Leftrightarrow \quad \gamma_1 I - I Y^{-1} I \succcurlyeq 0 \tag{10}$$

or

$$\begin{bmatrix} Y & I \\ I & \gamma_1 I \end{bmatrix} \succcurlyeq 0 \tag{11}$$

which can be used as a constraint when $\det Y^{-1}$ is minimized.

We note that although we do not try to minimize the largest deviation e, we do consider the most harmful input u, $\|u\| \le 1$. Since the input $W_2 u$ to Δ also covers an ellipsoidal region when u, $\|u\| \le 1$, varies over its admissible range of values, a reasonable alternative would be to minimize the product of the sizes of the two ellipsoids (Häggblom, 2005b).

Next we shall derive constraints imposed by data matching requirements. For ease of presentation, we start with model matching.

3.2 Model Matching

In model matching it is required that every $G = G_k$ can be reproduced by some allowed perturbation $\Delta = \Delta_k$ in accordance with (4). We thus require a condition which guarantees that

$$E_k := G_k - G_0 = W_1 \Delta_k W_2 , \quad \bar{\sigma}(\Delta_k) \le 1 \tag{12}$$

can be satisfied exactly. In principle, we can construct such a Δ_k explicitly, but we shall here use a result from the literature.

Let $E_k \in \mathbb{C}^{n\times m}$, $W_1 \in \mathbb{C}^{n\times p}$, $W_2 \in \mathbb{C}^{q\times m}$, $n \le p$, $q \ge m$. Then there is a $\Delta_k \in \mathbb{C}^{p\times q}$, $\bar{\sigma}(\Delta_k) \le 1$, if and only if (Poolla *et al.*, 1994; Chen and Gu, 2000)

$$\begin{bmatrix} W_1 W_1^* & E_k \\ E_k^* & W_2^* W_2 \end{bmatrix} \succcurlyeq 0 \tag{13}$$

We note that this condition does not require any of the matrices to be square; it is only required that the dimensions of Δ_k are not less than those of E_k.

In order to be useful for our purposes, the data matching condition should be linear with respect to the optimization variables. As given, (13) is not linear with respect to Y, defined in (8). However, (13) can be reformulated in various ways (see, e.g., VanAntwerp and Braatz, 2000). The condition is equivalent with

$$W_1 W_1^* \succ 0 , \quad W_2^* W_2 - E_k^* (W_1 W_1^*)^{-1} E_k \succcurlyeq 0 \tag{14}$$

If we introduce Y according to (8) and define

$$X := W_2^* W_2 \succcurlyeq 0 \tag{15}$$

the latter part of (14) can be written

$$X - E_k^* Y^* Y E_k \succcurlyeq 0 \tag{16}$$

This, in turn, can be written as the linear matrix inequality (LMI)

$$\begin{bmatrix} I & YE_k \\ (YE_k)^* & X \end{bmatrix} \succcurlyeq 0 \ , \quad \forall k \tag{17}$$

which is linear in Y and X.

If we want to optimize also with respect to G_0, (17) has to be modified slightly. By defining

$$Z := YG_0 \tag{18}$$

(17) can be written as

$$\begin{bmatrix} I & YG_k - Z \\ (YG_k - Z)^* & X \end{bmatrix} \succcurlyeq 0 \ , \quad \forall k \tag{19}$$

which is linear in Y, Z and X.

As discussed in the previous section, $\|W_2\|$ needs to be restricted. Otherwise, (16) can always be satisfied by a sufficiently large X regardless of Y. Let us thus introduce the restriction

$$\|W_2\| \le \gamma_2 \tag{20}$$

This is equivalent with the matrix inequalities

$$W_2^* W_2 = X \preccurlyeq \gamma_2^2 I \quad \Leftrightarrow \quad \gamma_2^2 I - X X^{-1} X \succcurlyeq 0 \tag{21}$$

or

$$\begin{bmatrix} X & X \\ X & \gamma_2^2 I \end{bmatrix} \succcurlyeq 0 \tag{22}$$

In principle, arbitrary structures can be imposed on W_1 and W_2, resulting in corresponding structures for Y and X. These structures would most commonly be block-diagonal ones. If desired, the structure of Z could also be constrained, thus affecting the structure of the estimated G_0.

More generally, Y, X and Z may belong to certain sets $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Z}$, respectively. The general optimization problem based on model matching can then be formulated as follows:

$$\begin{aligned} &\underset{Y\in\mathcal{Y}, X\in\mathcal{X}, Z\in\mathcal{Z}}{\text{minimize}} \ \det Y^{-1} , \forall \omega \in \Omega \\ &\text{subject to (19), (22), (11)} \end{aligned} \tag{23}$$

3.2 Input-Output Matching

In the case of input-output matching we require

$$e_k := y_k - G_0 u_k = W_1 \Delta_k W_2 u_k , \quad \bar{\sigma}(\Delta_k) \le 1 \tag{24}$$

Similarly as above, we can derive the LMIs

$$\begin{bmatrix} I & Ye_k \\ (Ye_k)^* & u_k^* X u_k \end{bmatrix} \succcurlyeq 0 , \quad \forall k \tag{25}$$

and

$$\begin{bmatrix} I & Yy_k - Zu_k \\ (Yy_k - Zu_k)^* & u_k^* X u_k \end{bmatrix} \succcurlyeq 0 , \quad \forall k \tag{26}$$

where the latter is used if G_0 is to be updated.

The general optimization problem based on input-output matching can now be formulated as follows:

$$\begin{aligned} &\underset{Y\in\mathcal{Y}, X\in\mathcal{X}, Z\in\mathcal{Z}}{\text{minimize}} \ \det Y^{-1} , \forall \omega \in \Omega \\ &\text{subject to (26), (22), (11)} \end{aligned} \tag{27}$$

3. APPLICATION TO DISTILLATION

A distillation column is a multivariable system usually characterized by a strong directionality, which means that the transfer matrix is ill-conditioned and nearly singular. In order to be useful for controller design, a model must provide a good description of the directionality properties.

It tends to be almost impossible to capture these properties with sufficient accuracy in a single linear model determined through system identification. The nonlinearity of the plant further complicates the

matter. Therefore, an appealing approach in the modeling of a distillation column is to determine a set of linear models. Such a set has been determined by Häggblom and Böling (1998) and will be used in this application.

The distillation column is a pilot-scale two-product column, which was identified by applying a series of step changes in the high- and low-gain input directions. From these experiments, a nominal model as well as six additional models were determined as transfer matrix models composed of second-order transfer functions with deadtime (Häggblom and Böling, 1998). The models have two outputs (distillate and bottoms composition) and two inputs (reflux and vapor to the reboiler).

Various types of "simple" uncertainty models have been considered for this column and a multiplicative output uncertainty model of the form

$$G = (I + W_1\Delta)G_0 \tag{28}$$

was found adequate in previous studies (Nyström *et al.*, 2003; Böling *et al.*, 2004; Häggblom, 2005b). In these studies, the 2×2 weight matrix W_1 was determined frequency by frequency by matching to input-output data using determinant minimization. Transfer function filters were also determined by fitting to the calculated weights. The possibility of adjusting the nominal model so as to reduce the conservatism of the uncertainty model was not considered.

The convex optimization formulation presented in this paper was not available in the previous works. Thus, the optimizations were non-convex with various numerical problems. By the present formulation it is straightforward to solve the problem. The calculation of a nominal model and other types of uncertainty models can be handled within the same framework.

In this paper, we shall compare the uncertainty models obtained by determinant minimization with models obtained by minimization of the maximum singular value of the weight matrix. We shall consider full weight matrices, diagonal weight matrices and scalar weights. We shall also illustrate how the uncertainty model can be improved by optimizing the nominal model. For simplicity, we only illustrate the results by steady-state data.

Table I shows the nominal model obtained by fitting to all input-output data in Table II. Experiments 1–3 are step changes in the low-gain direction and experiments 4–6 step changes in the high-gain direction of the distillation column.

Figure 1 shows the experimental data points as the coordinates of the components of the deviation

$$e_k = y_k - G_0 u_k \tag{29}$$

normalized by $\|u_k\|$. Because the data points farthest away from the origin are close to the coordinate axes, it is sufficient to use a diagonal uncertainty weight

TABLE I
Nominal Steady-State Model

$$G_0(0) = \begin{bmatrix} -0.04229 & 0.09349 \\ 0.11733 & -0.27858 \end{bmatrix}$$

TABLE II
Steady-State Data of Individual Experiments

Exp. #	u_1	u_2	y_1	y_2
1	10.0	5.0	0.06180	–0.23315
2	–20.0	–10.0	–0.09280	0.42640
3	10.0	5.0	0.04135	–0.20590
4	0.5	–1.0	–0.11513	0.50204
5	–1.0	2.0	0.22997	–0.76869
6	0.5	–1.0	–0.17393	0.33254

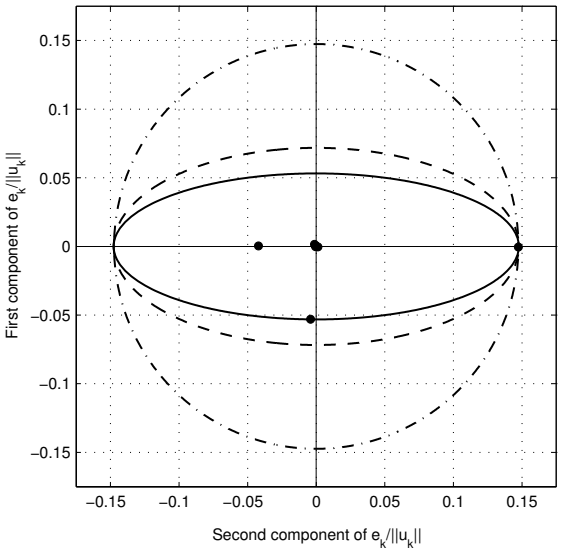

Fig. 1. Normalized output deviations of experimental points with uncertainty regions at steady state: smallest region (——), region with smallest norm (— — —), smallest region with scalar weight (— · —).

matrix W_1 in the uncertainty model (28). The smallest ellipse in Fig. 1 illustrates the uncertainty region of model (28) obtained when the size (i.e., area) of this region is minimized. The uncertainty model can generate any point inside the ellipse. As can be seen, all experimental points are in the region and it cannot be made smaller, using a diagonal weight and the given nominal model, without excluding some experimental point.

Figure 1 also shows the uncertainty region obtained by minimizing the maximum singular value of the diagonal weight W_1 as well as the uncertainty region for an optimal scalar weight. Clearly, these are larger than the uncertainty obtained by minimizing the size of the uncertainty region. It can also be mentioned that the uncertainty region obtained by model matching is an order of a magnitude larger than the one obtained by data matching.

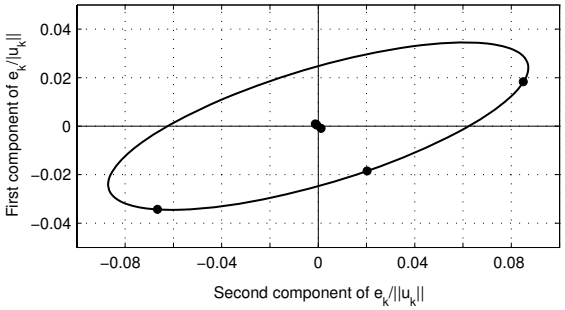

Fig. 2. Experimental points and uncertainty region for full optimal weight and nominal model.

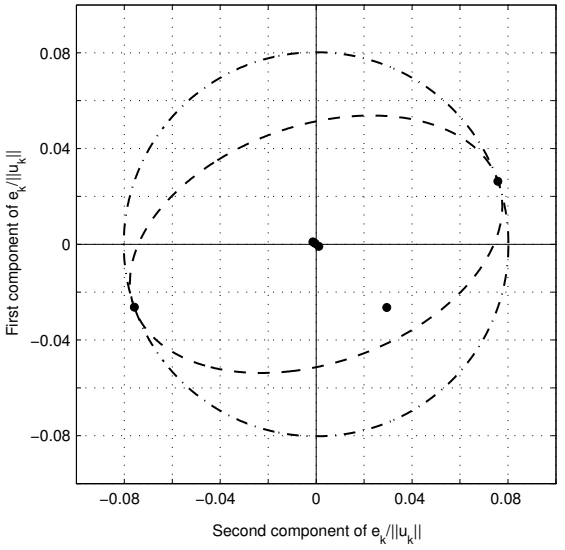

Fig. 3. Experimental points and uncertainty regions for full weight with smallest norm (— —) and smallest scalar weight (— · —) with optimal nominal model.

The experimental points in Fig. 1 indicate that the uncertainty regions could be reduced by adjusting the nominal model. Figure 2 shows the result of such an adjustment obtained by solving the problem defined in Eq. (27). Note that the positions of the experimental points are changed because the nominal model affects the deviations e_k. Note also the differences between the scales in Fig. 1 and Fig. 2.

Figure 3 shows the corresponding result for the cases when the maximum singular value of a full weight matrix W_1 is minimized and when an optimal scalar weight is used. Even though the nominal model is adjusted so as to minimize these weights, the resulting models have uncertainty regions significantly larger than that of the model obtained by solving Eq. (27).

4. CONCLUSIONS

A procedure based on convex optimization techniques for deriving norm-bounded uncertainty models for MIMO systems has been presented. The procedure applies for uncertainty models with a norm-bounded unstructured uncertainty, but otherwise quite general model types (additive, multiplicative, LFT uncertainty) as well as placement and structures of weights can be handled. Data for the uncertainty modeling may be available as sets of input-output data or a number of deterministic models. Generally, input-output data is preferable since it gives a less conservative uncertainty model. The uncertainty modeling is based on data (or model) matching in the frequency domain, for which necessary and sufficient conditions are expressed by LMIs. The size (area, volume) of an ellipsoidal uncertainty region, or its norm (largest distance from the origin), may be minimized. As indicated by an application to distillation modeling, minimization of the size of the uncertainty region tends to be the superior approach. The uncertainty models contain a nominal model, which strongly affects the size of the uncertainty region, and which may be adjusted so as to minimize this region.

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