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AN ANALYTICAL SOLUTION TO MULTIVARIABLE NONLINEAR MPC FOR SECOND-ORDER LAGUERRE SYSTEMS

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Abstract: An analytical solution to the multivariable (two-input, two-output) nonlinear model predictive control (NMPC) problem was derived for systems represented by (Volterra-)Laguerre models. The standard two-norm squared NMPC objective function was employed, and the minimization was carried out with $m = 1$ and over the prediction horizon, p. The polynomial structure of the system model yielded a polynomial objective function for the NMPC problem. Differentiation of this scalar objective function with respect to the manipulated input variables provided the first-order necessary conditions for optimality: a set of n_u (equal to the number of inputs) coupled polynomials. Via Gröbner basis transformation, this set of polynomials was converted to a structured set of higher-order polynomials solvable via roots calculations and back-substitution. The algorithm was tested using a two-input two-output polymerization case study.

Keywords: nonlinear model predictive control, Volterra models, Laguerre functions, Gröber basis, multivariable systems

1. INTRODUCTION

Model predictive controllers are a class of high-performance control algorithms garnering intense academic and industrial interest (Kouvaritakis and Cannon, 2001). Algorithms based on linear models may effectively control some nonlinear systems (Stack and Doyle III, 1997; Hernjak and Doyle III, 2004). However, a higher degree of performance may be achieved by employing a nonlinear algorithm, and hence a nonlinear model, on the grounds that model quality is correlated with achievable controller performance (Morari and Zafiriou, 1989). Nonlinear models, and hence, nonlinear algorithms are especially necessary for systems displaying input multiplicity (or other even-order polynomial-like behavior), as linear integrating controllers cannot stabilize these processes at the process optimum (Morari, 1983). The construction of nonlinear models, from the fundamental process physics, is an active research area (Doyle III *et al.*, 2002*a*; Daoutidis and Henson, 2002), but these models may take significant time to develop. Furthermore, the structure of these models may make the implicit inversion problem of NMPC more complicated due to their size or by generating performance objectives that are nonconvex. This, in turn, may lead to algorithms that become trapped in local minima (Zheng, 1997). Objective function and model nonlinearities, the root cause of the aforementioned nonconvexity, may inherently limit the ability of an NMPC algorithm to deliver optimal performance.

The ability to analytically solve optimization problems eliminates the above nonconvexity concerns. Careful selection of the objective function, as well as the system model structure, leads to nonlinear optimization problems with desirable properties. This is the nonlinear analog to the original use of step

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response models for linear MPC – the model structure facilitates the solution of the optimization problem. The use of models with polynomial structure, such as the Volterra or Laguerre models, allows the explicit expansion of the optimization objective as a polynomial in the manipulated variables (Dumont and Fu, 1993*a*; Dumont *et al.*, 1994; Parker and Doyle III, 2001; Parker, 2002*b*). In the single-input single-output (SISO) case, NMPC for (Volterra-)Laguerre systems has been previously posed and solved (semi-)analytically in a variety of formulations. Dumont and co-authors employed a steady state tracking objective (minimize error at the end of the prediction horizon) with $m = 1$ (where m is the manipulated variable move horizon in MPC) and did not weight the manipulated variable (1993*a*; 1994). This work was extended by Parker and Doyle III to the case with input penalty and error calculation over the entire prediction horizon (2001). Further extensions to the $m > 1$ case were discussed by Parker using: (i) local linearization for future input moves (2002*a*); and analytical solution for $m = 2$ using Gröbner basis transformation (2002*b*). The present work extends the Gröbner basis NMPC solution methodology for MIMO (Volterra-)Laguerre systems

2. LAGUERRE MODEL STRUCTURE

The present work focuses on second-order multivariable systems of Laguerre structure. This includes Laguerre models that are derived from Volterra series models (Zheng and Zafiriou, 1995), the so-called Volterra-Laguerre models. The multivariable discrete time Laguerre model has the following state space structure (Schetzen, 1980):

$$\ell(k+1) = A(\alpha)\ell(k) + B(\alpha)u(k) \tag{1}$$

$$y(k) = C\ell(k) + L_b^T(k)DL(k) \tag{2}$$

Here $\ell(k)$, $u(k)$, and $y(k)$ are vectors of dimension n_ℓ, n_u, and $n_y \times 1$, respectively. The parameter α is the Laguerre pole, taking on values in the range $[0,1)$, and establishes the dominant time constant of the system. Matrix A is square and of dimension n_ℓ. The input matrix B is of dimension $n_\ell \times n_u$, and the linear output matrix, C^T is of dimension $n_y \times n_\ell$. The second-order output term is composed of matrix $D = blockdiag\{D_1, D_2\}$, where the subscripts denote the output corresponding to the second-order effect. The vector $L(k) = \left[\ell^T(k), \ell^T(k), ..., \ell^T(k)\right]^T$ is composed of n_y $\ell(k)$ terms stacked vertically, thereby having dimension $n_\ell \bullet n_y \times 1$. $L_b(k)$ is the block diagonal version of $L(k)$, where the $\ell(k)$ vectors are along the diagonal, rather than vertically stacked, and $L_b(k)$ has dimension $n_\ell \bullet n_y \times n_y$. It is straightforward to convert (1) to a Δu formulation, as follows:

$$\ell(k+1) = A(\alpha)\ell(k) + B(\alpha)u(k-1) + B(\alpha)\Delta u(k) \tag{3}$$

This structure facilitates controller synthesis as the state equations can be written in terms of two contributions: (i) variables whose values are known at time k; and (ii) the input changes at time k, as follows:

$$\ell(k+1) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \ell(k) \\ u(k-1) \end{bmatrix} + B\Delta u(k) \tag{4}$$

The future state prediction, over a future horizon of length p, is constructed as follows (similar to (Muske and Rawlings, 1993)):

$$\begin{bmatrix} \ell(k+1|k) \\ \ell(k+2|k) \\ \vdots \\ \ell(k+p|k) \end{bmatrix} = \begin{bmatrix} A & \bar{A}_1 B \\ A^2 & \bar{A}_2 B \\ \vdots & \vdots \\ A^p & \bar{A}_p B \end{bmatrix} \begin{bmatrix} \ell(k) \\ u(k-1) \end{bmatrix} + \begin{bmatrix} \bar{A}_1 B \\ \bar{A}_2 B \\ \vdots \\ \bar{A}_p B \end{bmatrix} \Delta u(k) \tag{5}$$

$$\mathscr{L}(k+1|k) = \mathscr{A}H(k) + \mathscr{B}\Delta u(k) \tag{6}$$

$$\bar{A}_n = \sum_{i=1}^{n} A^{i-1} \quad \forall n \geq 1 \tag{7}$$

Similarly, the output prediction over a similar horizon is given as:

$$\mathscr{Y}(k+1|k) = \mathscr{C}^T\mathscr{L}(k+1|k) + \mathscr{L}_b^T(k+1|k)\mathscr{D}\mathscr{L}(k+1|k) \tag{8}$$

Here, $\mathscr{Y}(k+1|k) = [y(k+1)\, y(k+2) \ldots y(k+p)]^T$ and the matrices $\mathscr{C}^T$ and $\mathscr{D}$ are block-diagonal matrices composed of p C^T and D matrices, respectively. The matrix $\mathscr{L}_b^T(k+1|k)$ is a block-diagonal matrix having n_y repetitions of $\ell(k+i|k)$ per sub-block, with p total sub-blocks (*i.e.*, $i \in [1,p]$).

3. CONTROLLER SYNTHESIS

The present work employs the following standard two-norm squared objective function based on the input change, $\Delta\mathscr{U}(k|k)$:

$$\min_{\Delta\mathscr{U}(k|k)} \|\Gamma_y[\mathscr{R}(k+1) - \mathscr{Y}(k+1|k)]\|_2^2 + \|\Gamma_u\Delta\mathscr{U}(k|k)\|_2^2 \tag{9}$$

The vector $\mathscr{R}(k+1)$ is the output reference trajectory, over the horizon p, and the n_u manipulated variables are given by $\Delta\mathscr{U}(k|k)$. Setpoint tracking and manipulated variable movement are weighted by Γ_y and Γ_u, respectively. Substituting (6) into (8), and the result into (9), yields the following optimization problem:

$$\min_{\Delta\mathscr{U}(k|k)} F\left(\Delta\mathscr{U}(k|k), \ell(k), u(k-1), r(k+1), y_m(k)\right) \tag{10}$$

At each time, $y_m(k)$ is measured from the process, $\ell(k)$ is known from the process model, $u(k-1)$ is the

input value implemented at the previous time step, and $r(k+1)$ is specified (thereby specifying $\mathscr{R}(k+1)$). The scalar objective function $F(\bullet)$ is composed of a collection of matrix multiplications dependent on the Laguerre model (6) and (8).

Equation (10) is a nonlinear programming (NLP) problem, which is solvable by standard NLP techniques (Biegler, 1998). In some cases, these NLP solution techniques may fail to converge to the global optimum; this is most easily observed for SISO systems having input multiplicity (Parker and Doyle III, 2001). Hence, an analytical solution to (10) could provide superior performance. Differentiating $F(\bullet)$ with respect to $\Delta\mathscr{U}(k|k)$ and setting the resulting equations equal to zero yields the points at which the objective function has zero slope, *i.e.* its minima, maxima, and saddle points. In the present case, with $m=1$, the differentiation yields a number of equations equal to the number of manipulated variables, each having the following structure:

$$\begin{aligned} f_i = {} & p_{i30}\Delta u_1^3(k|k) + p_{i20}\Delta u_1^2(k|k) + p_{i10}\Delta u_1(k|k) \\ & + p_{i21}\Delta u_1^2(k|k)\Delta u_2(k|k) + p_{i01}\Delta u_2(k|k) + p_{i00} \\ & + p_{i11}\Delta u_1(k|k)\Delta u_2(k|k) + p_{i02}\Delta u_2^2(k|k) \\ & + p_{i12}\Delta u_1(k|k)\Delta u_2^2(k|k) + p_{i03}\Delta u_2^3(k|k) \\ = {} & 0 \quad \forall i \in [1, n_u] \end{aligned} \tag{11}$$

The polynomial coefficients p_{ivw}, where v and w denote the polynomial order of $\Delta u_1(k|k)$ and $\Delta u_2(k|k)$, respectively, are functions of $\ell(k)$, $u(k-1)$, and $r(k+1)$. The i subscript denotes the manipulated variable with respect to which the partial derivative ($f_i = \frac{\partial F}{\partial \Delta u_i}$) was taken. At each sample time the polynomial coefficients are recalculated. The explicit functionality of these coefficients is omitted from the present work due to space constraints. It is interesting to note that the polynomial equations (11) are structurally similar to the corresponding equations resulting from a SISO $m=2$ analysis (Parker, 2002*b*), with $\Delta u_2(k|k)$ corresponding to $\Delta u_1(k+1|k)$, partial derivatives taken with respect to the n_u manipulated variables rather than the m-length horizon of input changes, and different numerical values for the p_{ivw}'s.

To identify the input values leading to process extrema, the set of coupled nonlinear polynomials (11) must be solved simultaneously. While there is no general solution to this set of equations, the Gröbner basis transformation, an analytical geometry technique, can be used to solve the set of polynomials above in a straightforward fashion.

4. GRÖBNER BASIS TRANSFORMATION

It is beyond the scope of the present work to provide a full background on Gröbner basis (GB) transformation methods. Interested readers are referred to (Fröberg, 1997; Cox *et al.*, 1997). Among other problems, Gröbner basis techniques can solve systems of polynomial equations. In fact, solution by Gröbner bases provides all the solutions in k^n of the system of polynomial equations (Cox *et al.*, 1997):

$$g_i(x_1, \ldots, x_n) = 0 \quad \forall i \in [1, n] \tag{12}$$

Here the functions g_i lie on the polynomial ring k^n; the coefficients (such as p_{ivw} in (11)) determine the field, k. General background on fields and polynomial rings can also be found in (Fröberg, 1997; Cox *et al.*, 1997). For the present work, the focus is on the field of rational numbers, ($\mathbb{Q} = \frac{\mathbb{Z}_{num}}{\mathbb{Z}_{den}}$), although integers ($\mathbb{Z}$), reals, and complex numbers are also feasible choices. Under this selection, the NMPC problem equations (11) can be posed as a Gröbner basis problem:

Problem 1. (**NMPC Optimization).** Find all common solutions on the ring $\mathbb{Q}^2[\Delta u_1(k|k), \Delta u_2(k|k)]$ of the system of polynomial equations:

$$f_1(\Delta u_1(k|k), \Delta u_2(k|k)) = f_2(\Delta u_1(k|k), \Delta u_2(k|k)) = 0$$

While the coefficients of most NMPC problems are generally of type real ($p_{ivw} \in \mathbb{R}$), these can be rationalized to place them in $\mathbb{Q}$. The key driving force for this modification is that many Gröbner solution algorithms are more efficient when handling rationals than reals (*e.g.*, groebner::gbasis in MuPAD 3.1.1, ©2005, SciFace Software).

The algorithm that is commonly employed in the solution of the **NMPC Optimization** problem is Buchberger's Algorithm (Buchberger, 1985; Cox *et al.*, 1997). Given a polynomial ideal, such as the equations (11), in $\mathbb{Q}^2$, a Gröbner basis for the ideal can be constructed in a finite number of steps. In practice, a reduced Gröbner basis (rGB) is employed because it is unique for a given ideal where a GB is not necessarily unique (Cox *et al.*, 1997). In employing the GB solution method, an elimination order can be selected for the unknown variables ($\Delta u_i(k|k)$). In the present case, order is not particularly important, aside from the fact that variable ordering alters the polynomial coefficients, p_{ivw}. Hence, $\Delta u_1(k|k) > \Delta u_2(k|k)$ was chosen arbitrarily, where $>$ denotes an order of removal. For this ordering, the variable $\Delta u_1(k|k)$ is removed preferentially from the equations. The resulting set of rGB polynomials contain the following: (i) a set of (one or more) polynomials containing only $\Delta u_2(k|k)$; and (ii) a second set of polynomials with both $\Delta u_2(k|k)$ and $\Delta u_1(k|k)$ appearing. Ordering is more relevant to the $m > 1$ problem, where, for the SISO case, $\Delta u(k+1|k) > \Delta u(k|k)$ was chosen (Parker, 2002*b*) to yield one polynomial in $\Delta u(k|k)$ alone, and a second polynomial in $\Delta u(k|k)$ and $\Delta u(k+1|k)$.

Implementation of the GB solution algorithm has off-line and on-line components. Prior to on-line

execution, the ideal (11) is calculated analytically (*e.g.*, symbolically, using MATLAB ©2005, The Mathworks, Natick, MA). The dynamic variables, $\Delta u(k)$, $\ell(k)$, $u(k-1)$, and $r(k+1)$ are retained within the ideal so that they may be updated at each sample time. The solution of the ideal is accomplished using a specialized GB solution routine within a symbolic or numerical package (*e.g.*, MuPAD). The known variables at a point in time are converted from real to rational form and are passed to the MuPAD subroutine which computes the rGB. For the case study in Section 5.1, the rGB had the following structure:

$$q_{19}\Delta u_2^9(k|k) + \ldots + q_{11}\Delta u_2(k|k) + q_{10} = 0 \quad (13)$$

$$\Delta u_1(k|k) = \sum_{z=0}^{8} q_{2z}\Delta u_2^z(k|k) \quad (14)$$

The original problem, (11), which had no easily identified solution has been transformed to an rGB which is straightforward to solve. The univariate polynomial (13) is solved using a command such as *roots* in MATLAB. For each real root, a candidate for $\Delta u_2(k|k)$, the corresponding $\Delta u_1(k|k)$ candidate is calculated from (14). Roots with imaginary component are discarded because the manipulated variables are physical quantities, and hence, the solutions must be real-valued. Each pair of candidate solutions is simulated over the prediction horizon, which is a straightforward matrix multiplication for the Laguerre system using (1) and (2). The objective function value is calculated for each simulation profile, and the input move combination resulting in the lowest objective function value is implemented as $\Delta\mathscr{U}(k|k)$. This process repeats at each sample time with the dynamic updating of the ideal, rGB solution, candidate move identification, and move selection.

5. RESULTS

5.1 Polymerization Case Study

The case study employed in this work is the multivariable polymerization reactor of Hidalgo and Brosilow (1990), including the addition of number average molecular weight (NAMW) as a controlled variable and operation about the low-conversion steady state point, as made by Doyle *et al.* (2002*b*). Monomer (Q_i) and cooling water (Q_c) flow rates are manipulated to control polymer NAMW and temperature. It is assumed that (Doyle III *et al.*, 2002*b*): (i) monomer disappearance is driven primarily by propagation, rather than chain transfer to another monomer; (ii) that the rate constant of overall polymer chain termination is the sum of combination and disproportionation contributions (Schmidt and Ray, 1981); and a constant volume fraction of solvent is maintained such that the gel effect can be neglected (Choi, 1986).The full process description, including state space formulation and model parameters, can be found in (Doyle III *et al.*, 2002*b*). The inputs and outputs were scaled as follows:

$$W_u = \begin{bmatrix} 108 & 0 \\ 0 & 471.6 \end{bmatrix} \qquad W_y = \begin{bmatrix} 2500 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Doyle *et al.* (2002*b*) developed a MIMO second-order Volterra model for this process via Carlemann linearization (Rugh, 1981). This model was projected onto the Laguerre basis using a least-squares approach similar to (Parker and Doyle III, 1998). The projection was accomplished over a range of α values ($\alpha \in [0.01, 0.99]$), and the C and D matrices of the Volterra-Laguerre model were calculated for each α. The α, C, D combination that provided the best fit to the original Volterra model was employed in the model. This resulted in each input channel having a different Laguerre pole value, such that the monomer flow rate used $\alpha = 0.76$ and the cooling water flow rate had $\alpha = 0.81$. The state space matrices for this model are as follows:

$$A = \begin{bmatrix} 0.76 & 0 & 0 & 0 & 0 & 0 \\ 0.422 & 0.76 & 0 & 0 & 0 & 0 \\ -0.321 & 0.422 & 0.76 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.81 & 0 & 0 \\ 0 & 0 & 0 & 0.344 & 0.81 & 0 \\ 0 & 0 & 0 & -0.279 & 0.344 & 0.81 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.65 & 0 \\ -0.494 & 0 \\ 0.375 & 0 \\ 0 & 0.586 \\ 0 & -0.475 \\ 0 & 0.385 \end{bmatrix} \qquad C^T = \begin{bmatrix} -2.997 & 1.571 \\ -3.01 & 1.44 \\ -0.108 & 0.04 \\ 0.602 & -2.012 \\ 1.095 & -1.884 \\ 0.438 & -0.063 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.412 & 0.508 & 0.072 \\ 0.508 & 0.470 & 0.010 \\ 0.072 & 0.010 & -0.072 \\ -0.019 & 0.041 & 0.037 \\ 0.041 & 0.056 & 0.036 \\ 0.037 & 0.036 & 0.001 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0.043 & 0.078 & 0.032 \\ 0.078 & 0.108 & 0.016 \\ 0.032 & 0.016 & -0.058 \\ 0.244 & 0.223 & 0.001 \\ 0.223 & 0.345 & 0.086 \\ 0.001 & 0.086 & 0.131 \end{bmatrix}$$

5.2 Controller Evaluation

An NMPC controller was synthesized using the method in Section 3. Controller tuning parameters were $m = 1$ and $p = 20$, with the weighting matrices as follows (Doyle III *et al.*, 2002*b*):

$$\Gamma_y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Gamma_u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (15)$$

For comparison, a gradient-based NMPC algorithm was synthesized and solved using the *fmincon*

function in MATLAB. The objective function surface, calculated from equation (9), using a setpoint value of $r(k+1) = [0 \quad 0]^T$ for the mismatch case is shown in Figure 1. This objective function surface has a

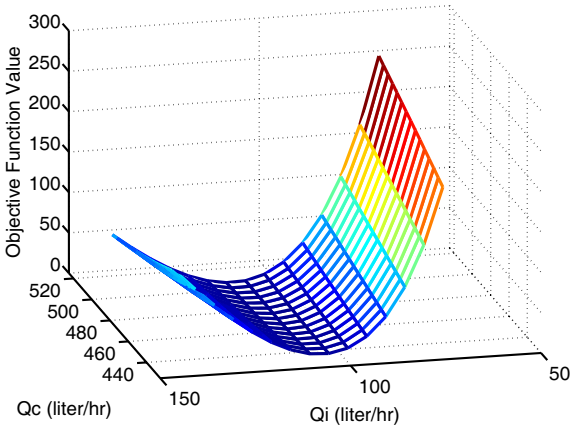

Fig. 1. Objective function surface for the polymerization case study at the nominal conditions and the reference trajectory $R(k+1) = [0 \quad 0]^T$.

global minimum, with no local minima. Given the smooth convex nature of the surface, the gradient-based algorithm should track the desired setpoint. The performance of the analytical NMPC algorithm in response to setpoint changes, therefore, should be within optimizer numerical accuracy to that of the gradient-based solution.

In the nominal case, where the plant and model are both described by the Volterra-Laguerre equations, the system was subjected to a reference change of $[+21519 \quad 0]^T$ (deviation from nominal) [results not shown]. The gradient-based algorithm tracks the desired reference trajectory, and the performance of the Gröbner NMPC algorithm is nearly identical. The output profiles overlay; as such, differentiation between the responses is not significant with respect to measurement noise associated with NAMW or temperature sensing devices.

Simulations using the nonlinear ODEs (Hidalgo and Brosilow, 1990; Doyle III *et al.*, 2002*a*) as the process description, with the controller synthesized from the Volterra-Laguerre model, return similar results to those above. As an additional test in the mismatch case, a normally-distributed noise signal ($\mu = 0, \sigma = \begin{bmatrix} 2400 & 0 \\ 0 & 0.5 \end{bmatrix}$, corresponding to 5% measurement noise on average for the NAMW channel) was added to the plant output. The closed-loop performance under both NMPC algorithms is shown in Figure 2. The gradient-based and analytical controllers again provide similar levels of closed-loop performance.

6. DISCUSSION AND SUMMARY

An analytical solution to the multivariable nonlinear MPC problem was developed for two-

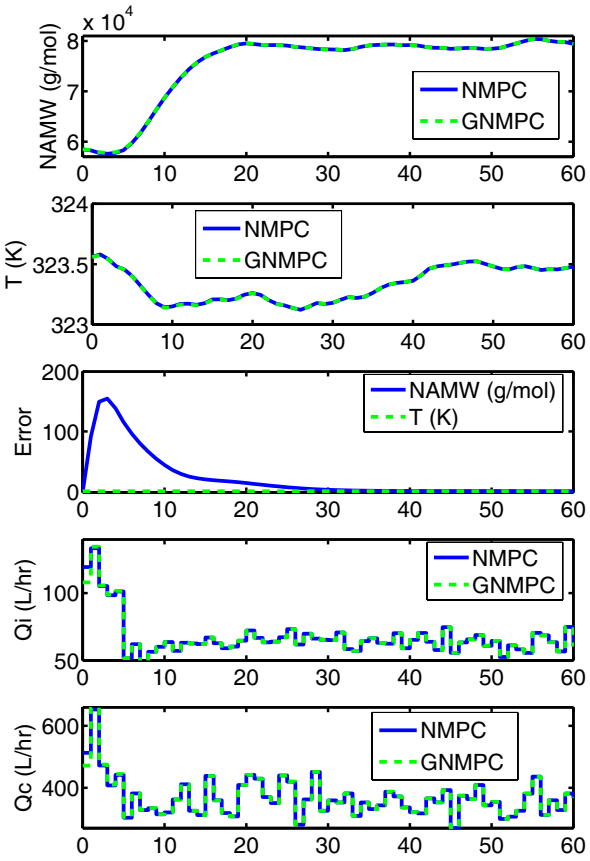

Fig. 2. Closed-loop NMPC on the mismatch system in response to a setpoint of $\mathscr{R}(k+1) = [80{,}000 \quad 0]^T$ (absolute variables) at time = 5 hr in the presence of measurement noise. Controller response curves generally overlay in panes 1, 2, 4, and 5. Note: GNMPC = Gröbner NMPC.

input two-output systems modeled with second-order (Volterra-)Laguerre equations. The future model predictions were posed in terms of information available at the current time. The NMPC objective function, and its derivatives with respect to the input vector, were precalculated symbolically off-line using the model prediction equations. The matrix polynomial corresponding to the first-order necessary condition for an optimum, which was previously limited to numerical solution in the multivariable case, was transformed using Gröbner bases to an easily solvable form. The resulting equations provided candidate input move combinations calculated by a combination of polynomial roots calculations and back-substitution. For systems with smooth convex objective functions, the closed-loop performance was shown to be equivalent to that obtained using gradient-based NMPC solution algorithms.

While performance advantages were not observed for the multivariable polymerization case study above, this was not surprising. Previous work with analytical NMPC solutions on SISO systems (Dumont and Fu, 1993*b*; Parker and Doyle III, 2001; Parker, 2002*b*) has demonstrated the advantage exists primarily when the system displays a nonconvex objective function surface. Bioreactor and chip refiner case studies

that have "unreachable" setpoints (*i.e.*, second-order systems having setpoints above the steady state locus) or input multiplicity are problematic for gradient-based algorithms. In the latter case, the gradient-based routine may become trapped on one side of the optimum, which may result in suboptimal performance (especially in the presence of input constraints, as shown in (Parker, 2002*b*)).

The present work complements the SISO $m > 1$ analytical NMPC work of (Parker, 2002*b*) in demonstrating the utility of Gröbner basis transformation for the solution of MIMO NMPC problems. In fact, the $m = 2$ SISO solution has identical structure to that of equations (13) and (14). While these studies have focused on second-order problems, the extension to third-order (Volterra-)Laguerre systems is ongoing in our laboratory. Also of interest is the coupling of the MIMO and $m > 1$ analytical solution problems in a unified algorithm construction. It is worth noting that there is no theoretical limit on either the size of m or the input-output dimension of the multivariable system that can be formulated using the Gröbner basis technique. The limitation of this solution method lies in the Gröbner transformation routine, where the total number of manipulated variable moves ($m \bullet n_u$) characterizes the complexity of the problem and its possible solution. Increasing m, n_u, or polynomial order will lead to: (i) an increased number (due to m or n_u) of higher-order (due to polynomial order) polynomial equations (11); (ii) a larger number of variables (due to m or n_u) in the Gröbner problem (12); and (iii) potential roots calculation issues, such as identifiability and numerical stability, in the larger number of potentially higher-order solution equations, akin to (13) and (14). Items (i) and (ii) can likely be addressed with sufficient computational power (CPU speed, memory/swap, etc.). More theoretical issues, which remain open at present, include limitation (iii) as well as a proof of Gröbner NMPC algorithm stability.

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