GOOD OR BAD – WHEN IS PLANT NONLINEARITY AN OBSTACLE FOR CONTROL?

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Abstract: Virtually every real process is a nonlinear system. Nevertheless, linear system analysis and linear controller design methods have proven to be adequate in many applications. On the other hand, there are nonlinear processes that require or benefit from nonlinear control. Therefore, recognizing a system as being nonlinear does not suffice, but the extent and severity of a system's inherent nonlinearity is the crucial characteristic in order to decide whether linear system analysis and controller synthesis methods are adequate. The introduction of nonlinearity measures is an attempt to systematically approach this problem. In this contribution, we review existing approaches to nonlinearity assessment, we state the most important results and we give a glance ahead to what might be expected from this field in the future.

Keywords: Nonlinear control, robust control, nonlinearity

1. INTRODUCTION

Linear techniques for systems analysis and controller design are well developed. For many control-related engineering problems, methods are available that are theoretically sound as well as practically implementable. Due to the diverse qualitative behaviour of nonlinear systems, tools for nonlinear systems analysis and control will probably never reach the same level of generality. To cope with nonlinear control problems, there are two alternative approaches. For highly nonlinear systems, special methods have to be developed that rely upon certain physical properties of the application or upon mathematical properties of a certain system class, like energy-shaping methods for mechanical control systems or feedback linearization. For mildly nonlinear systems, one can attempt to use a linear model and linear controller design methods, hoping that the nonlinear distortion is not large enough to destabilize destabilize the closed-loop system or to deteriorate closed-loop performance.

However, there are no mathematical definitions of "mildly nonlinear" and "highly nonlinear" process behaviour, and it is often difficult to decide whether a control problem at hand is a candidate for the application of linear or nonlinear controller design techniques. The area of quantitative nonlinearity assessment aims at filling that gap by deriving systematic methods to evaluate the degree of nonlinearity inherent to a plant, and its impact on the control design task. In particular it is of interest to ask the questions

(1) How good can a linear model for a given nonlinear process be?
(2) How good can a linear controller for a given nonlinear process be?
(3) How can a suitable linear controller for a given nonlinear process actually be designed?

The aim of this paper is to introduce the field of nonlinearity assessment and to present recent developments. To this end, we first give a brief overview on existing approaches to nonlinearity assessment in Section 2. Giving a comprehensive treatment of some well-known facts and some recent results, we then study questions (1) and (3) in greater detail. Some approaches to answer question (2) can be found in (Stack and Doyle III, 1997*b*; Schweickhardt *et al.*, 2003; Shastri *et al.*, 2004). In Section 3 we introduce nonlinearity measures as a means to quantify the best achievable quality of a linear model for a nonlinear process. We review the properties of nonlinearity measures and discuss ways to compute the values of nonlinearity measures for practical systems. The material of this Section also provides a basis for the subsequent presentation. In Section 4 an approach is presented that integrates nonlinearity assessment and the design of linear controllers for nonlinear systems based on linear robust control techniques and based on the nonlinearity measures introduced earlier. The paper concludes with Section 5.

2. A LITERATURE REVIEW

Nonlinearity measures appeared for the first time in (Desoer and Wang, 1980), where the induced gain of the difference between a nonlinear system and its best linear model is considered. This viewpoint is also adopted in (Allgöwer, 1995) and will be considered in greater detail in Section 3. The basic idea of this approach is to consider the input/output-behaviour of a system, and how closely it can be reproduced by linear models. In (Desoer and Wang, 1980) also the norm of the error signal itself is proposed as a nonlinearity measure and in (Sourlas and Manousiouthakis, 1992; Sourlas and Manousiouthakis, 1998) a method is given to compute this measure for second order discrete-time Volterra models with any desired precision.

The idea of linear modeling for nonlinear systems is further developed for the discrete-time case in (Partington and Mäkilä, 2002). System gains for nonlinear systems are defined and an upper bound on the modeling error $\|\tilde{G} - G\|_{i,\infty}$ for discrete-time piecewise linear systems is given. In (Mäkilä and Partington, 2003), the best linear models for discrete-time bi-gain systems is given with respect to the l_∞-norm and the existence of a best linear model for nonlinear finite impulse response filters is proven. The relative induced error $\|\tilde{G} - G\|_{i,\infty} / \|G\|_{i,\infty}$ as a measure of nonlinearity is mentioned. It is also of interest to study the achievable approximation quality for one given input only (Mäkilä, 2003; Mäkilä and Partington, 2004; Mäkilä, 2004). It can be shown that there are situations in which the best linear model defined this way is better suited for controller design than the model obtained by linearization around an equilibrium (or trajectory) (Mäkilä and Partington, 2004). A generalization of such a nonlinearity measure to (continuous-time) batch processes can be found in (Helbig *et al.*, 2000). It is also possible to consider not only linear models, but for instance Hammerstein systems, Wiener systems or Volterra series expansions as models for a general nonlinear system, and to quantify the suitability of the respective model class (Menold *et al.*, 1997*a*; Pearson *et al.*, 1997; Menold *et al.*, 1997*b*). A common property of all nonlinearity measures based on system gains is that they are only defined for stable systems.

A rather geometric viewpoint is taken in (Guay *et al.*, 1995; Guay, 1996), where the curvature of the steady state map is introduced as a measure of nonlinearity. A very interesting result is that two components of nonlinearity can be discerned, a tangential component that can be compensated for by input transformations and a normal component that can only be affected by coordinate transformations or feedback. The curvature measure can be extended to dynamic systems using Fréchet derivatives of operators (Guay *et al.*, 1997*b*; Guay, 1996). Again, these measures can only be calculated for stable processes.

A third approach is presented by Hahn and Edgar (2001*a*; 2001*b*), who introduce empirical controllability and observability Gramians in order to quantify the degrees of input-to-state and state-to-output nonlinearity respectively.

All approaches so far consider process nonlinearity as some kind of inherent process property that stands for itself. On the other hand, nonlinearity assessment, like robustness analysis, needs further specification in order to be meaningful. In robustness analysis we want a certain property (to be specified) of the system to be robust under a certain type of uncertainty or disturbance (to be specified). In a similar way, we have to specify the task we want to perform in order to decide whether process nonlinearity is of importance: for the same process, the tasks of process control, process design, process monitoring and model or parameter identification may be affected in very different ways by the nonlinear behaviour.

So far, only identification of nonlinear systems (see e.g. (Haber and Unbehauen, 1990; Dobrowiecki and Schoukens, 2001; Schoukens *et al.*, 2002; Enqvist and Ljung, 2002; Enqvist and Ljung, 2004)) and control-relevant nonlinearity assessment have received considerable attention in the literature. The notion of control-law nonlinearity was introduced by Guay *et al.* (1995; 1996). As the inverse process steady-state map can be used to achieve perfect set-point tracking, it is analyzed by the curvature measure in order to obtain a measure of control law nonlinearity. Similar to the open-loop steady-state map curvature measure, the control-law nonlinearity measure captures only static effects and is defined only for stable systems. In (Guay *et al.*, 1997*a*; Guay, 1996), the approach of the control law nonlinearity measure is extended to a nonlinear interaction measure representing a generalization of the *relative gain array (RGA)* (Bristol, 1966).

Stack and Doyle III (1997*b*) emphasize that not only the plant dynamics and the operating region determine the control-law nonlinearity analysis, but the performance objective plays an important role as well. Therefore they suggest to measure the nonlinearity of the optimal controller given by the classical optimal control theory. The controller structure is not restricted in advance but only the optimization criterion must be specified. In order to circumvent the derivation of the exact solution for the optimal state feedback controller, Stack and Doyle III define the so-called Optimal Control Structure (OCS). By this means, interesting questions can be examined like the dependence of control-relevant nonlinearity on the set-point and region of operation (Stack and Doyle III, 1997*a*), the relation between nonlinearity and controller aggressiveness (Stack and Doyle III, 1997*b*; Hernjak *et al.*, 2002) and the severity of certain classes of nonlinear behaviour (Hernjak *et al.*, 2002). Extensions to measurement feedback have also been made (Stack and Doyle III, 1997*a*; Hernjak *et al.*, 2002). This method is further developed in (Schweickhardt *et al.*, 2003) to the Optimal Control Law (OCL) nonlinearity measure that is better suited for numeric treatment and allows for processes with both stable and unstable modes. A case study of a chemical reactor is treated in (Shastri *et al.*, 2004).

The framework of nonlinear internal model control provides also a basis for control-relevant nonlinearity assessment. This approach is taken e.g. in (Stack and Doyle III, 1999; Eker and Nikolaou, 2002; Hernjak *et al.*, 2003).

Instead of analyzing the nonlinearity numerically in the whole region of operation, one can attempt to quantify the sensitivity of the closed loop performance on process nonlinearity locally at the operating point. First steps in this direction are presented in (Dier *et al.*, 2004; Guay and Forbes, 2004; Guay *et al.*, 2005).

3. NONLINEARITY MEASURES AND LINEAR MODELS FOR NONLINEAR SYSTEMS

In this chapter, we define different nonlinearity measures, specify some of their properties and discuss computational schemes to derive numerical values of the measures.

3.1 Definition and basic properties of nonlinearity measures

The fundamental setup of the input/output-based nonlinearity measures is depicted in Fig. 1. A general nonlinear (i.e. not necessarily linear) stable dynamical system N, described by the transfer operator $N : u \mapsto y = Nu$ is compared to a linear model G described by the linear transfer operator $G : u \mapsto \tilde{y} = Gu$ that approximates the dynamic behaviour of N. The signals u, y and $\tilde{y}$ represent input and output trajectories of the systems N and G respectively. Without loss of generality it is assumed that $N0 = 0$. The error signal e is the difference between the output y of the nonlinear system and the output $\tilde{y}$ of the linear system. In order to quantify this error, a signal norm like $\|e\|_p = \left(\int_0^\infty |e(t)|^p \,\mathrm{d}t\right)^{1/p} < \infty$ for $0 \leq p < \infty$ or $\|e\|_\infty = \operatorname{ess\,sup}_{t\geq 0} |e(t)| < \infty$ for $p = \infty$ (“peak norm”) will be used, where $|\cdot|$ denotes the Euclidean vector norm. In the sequel, we will simply write $\|\cdot\|$, leaving the type of norm unspecified when any choice is possible. The required type of stability for the plant N is always the corresponding (finite-gain) L_p-stability (Vidyasagar, 2002).

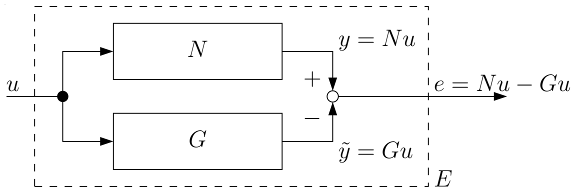


Fig. 1. Setup for comparison of a nonlinear system N with a linear system G.

Yet the norm of the error itself, $\|e\| = \|Nu - Gu\|$ is not the best quantity to look at, as it heavily depends on the magnitude of the input. Therefore, for any causal, L_p-stable system $N : \mathcal{U} \subseteq L_p^m \to L_p^n$ we define the *error-gain nonlinearity measure of N on $\mathcal{U}$* as

$$\gamma_N^{\mathcal{U}} \triangleq \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \frac{\|Nu - Gu\|}{\|u\|}. \qquad (1)$$

This nonlinearity measure gives the gain of the error system E (see Fig. 1), when the worst case input signal $u \in \mathcal{U}$ is considered. The best linear approximation G is chosen among the set of all causal stable linear (convolution) operators $\mathcal{G}$ such that the resulting worst case gain is minimized.

As can be seen from its definition, the error-gain nonlinearity measure $\gamma_N^{\mathcal{U}}$ depends on the system N and on the set of considered inputs $\mathcal{U}$. The set $\mathcal{U}$ usually describes the region of operation in which the nonlinearity of the system N is to be assessed. In this case $\mathcal{U}$ contains e.g. only signals not exceeding a certain maximal amplitude.

In Section 4 the error gain nonlinearity measure will turn out to be useful for controller design. But for analysis purposes, there are two reasons for the error gain nonlinearity measure not being the only quantity we are interested in. Firstly, the measure is not bounded by definition. For different systems, different values indicate a “high degree of nonlinearity”. For example, an additional scalar gain in the I/O-behaviour of a system changes the error gain nonlinearity measure, although the type and qualitative behaviour of the nonlinear system do not change. Secondly, the error gain nonlinearity measure can only be computed for stable systems in general while we might want to quantify the degree of nonlinear distortion also for unstable systems. Thus, for the analysis of nonlinear systems we introduce a second quantity.

Let therefore N be as above, but instead of stability we only require that the system does not exhibit a finite escape time (i.e. $\|(Nu)_T\| < \infty$ for all $T > 0$ and $u \in \mathcal{U}$). We define the *relative nonlinearity measure of N on $\mathcal{U}$* by

$$\varphi_N^{\mathcal{U}} \triangleq \inf_{G\in\mathcal{G}} \sup_{u\in\mathcal{U}} \limsup_{T\to\infty} \frac{\|(Nu - Gu)_T\|}{\|(Nu)_T\|} \qquad (2)$$

where the definitions of $\mathcal{G}$ and $\mathcal{U}$ are as above. An important property of the measure $\varphi_N^{\mathcal{U}}$ is that its value is bounded by one (Allgöwer, 1995), a value close to one corresponding to a highly nonlinear system. We can thus compare the degree of nonlinearity of different systems on a unified scale. The value of $\varphi_N^{\mathcal{U}}$ corresponds to the percent-wise deviation of the output of the best linear approximation G from the output of the nonlinear system N.

There are two more properties that both nonlinearity measures have in common. Firstly, if the measures are zero then the I/O-behaviour of the system N can exactly be reproduced by a linear system for the considered inputs, and N is said to be *linear in $\mathcal{U}$*. Conversely, if N is linear, then the best linear approximation is $G = N$ and thus the nonlinearity measures vanish. Secondly, it has already been said that $\mathcal{U}$ can characterize the region of operation. Note that the nonlinearity measure can not decrease when additional inputs are considered (when $\mathcal{U}$ is made bigger). This fact is mathematically expressed by

$$\mathcal{U}_1 \subseteq \mathcal{U}_2 \Rightarrow \gamma_N^{\mathcal{U}} \leq \gamma_N^{\mathcal{U}} \qquad (3)$$

and the equivalent relationship holds for $\varphi_N^{\mathcal{U}}$. The practical meaning is intuitively clear: if a larger operating regime is considered, the nonlinearity measure will increase or stay constant, but will not decrease. But $\mathcal{U}$ can have other significances as well. In Sec. **??** we discuss how $\mathcal{U}$ can reflect the effect of feedback for control-relevant nonlinearity characterization.

3.2 Nonlinearity measures and steady-state behaviour of nonlinear systems

In this section we consider the evaluation of nonlinearity measures based on the steady-state behaviour of nonlinear dynamic systems. Therefore, we assume that the system under consideration has a unique steady state for all inputs in the considered operating regime. The steady-state locus is a (static) function that maps the steady-state inputs to the steady-state outputs. In order to make statements about the nonlinearity measures of dynamic systems based on their steady-state locus, we first have to obtain results for nonlinearity measures of (static) functions.

To this end, we consider memoryless systems of the form

$$N_f : u \mapsto y : \ y(t) = f(u(t)) \ \forall t \qquad (4)$$

where $f : \mathcal{V} \to \mathbb{R}^n$ is a function satisfying $|f(v)| < \infty$ for all $v \in \mathcal{V}$. Here, the set $\mathcal{V} \subseteq \mathbb{R}^m$ determines the set of allowed input values (i.e. the region of operation), that is we define $\mathcal{U} = \{u \in L_p^m | u(t) \in \mathcal{V}\ \forall t\}$. We then have the following equivalences

$$\gamma_{N_f}^{\mathcal{U}} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|v|} \qquad (5)$$

$$\varphi_{N_f}^{\mathcal{U}} = \inf_{K\in\mathbb{R}^{n\times m}} \sup_{v\in\mathcal{V}} \frac{|f(v) - Kv|}{|f(v)|}. \qquad (6)$$

Note that this equality holds regardless of the norm used, i.e. for all $p \in [1, \infty]$, and the value of the nonlinearity gain of N_f does not depend on p for our definition of the L_p-norms. The above equalities have the following significance:

(1) When we want a linear model for a memoryless nonlinear system, a dynamic linear model has no advantage over a memoryless linear system (which is a gain matrix).
(2) Instead of a signal set $\mathcal{U}$, we only need to consider the maximum over a set of real numbers (or vectors) $\mathcal{V}$. We thus end up with a much easier optimization problem.

The computation of the measure for static functions can be done analogously to the procedure described in (Allgöwer, 1995). We therefore discretize the set $\mathcal{V}$ that describes the operating regime in terms of admissible values for the input signals. We then calculate the corresponding steady-state responses. This way, a finite number of points $(u_{SS,i}, y_{SS,i})$ on the steady state locus are obtained. Then we have to solve the optimization problem

$$\gamma_{N_f}^{\mathcal{U}} = \min_{z\in\mathbb{R}, K\in\mathbb{R}^{n\times m}} z \quad \text{s.t.} \ \frac{\left|y_{SS,i} - K u_{SS,i}\right|}{\left|u_{SS,i}\right|} - z \leq 0 \ \forall i$$

where $u_{SS,i}$ in the denominator must be replaced by $y_{SS,i}$ for the calculation of $\varphi_{N_f}^{\mathcal{U}}$.

In the case of a scalar function $f : \mathbb{R} \to \mathbb{R}$ the nonlinearity measures of memoryless systems can be obtained even simpler by using the sector bounds on f. Consider therefore a function f that lies in the sector $[k^-, k^+]$ for all $v \in \mathcal{V}$, but may lie outside for $v \neq \mathcal{V}$ (see Fig. 2 with $\mathcal{V}$ an interval on the v-axis). It can be seen that the slopes of the straight lines that bound the sector are given by

$$k^+ = \sup_{v\in\mathcal{V}\setminus\{0\}} \frac{f(v)}{v} \ \text{ and } \ k^- = \inf_{v\in\mathcal{V}\setminus\{0\}} \frac{f(v)}{v}.$$

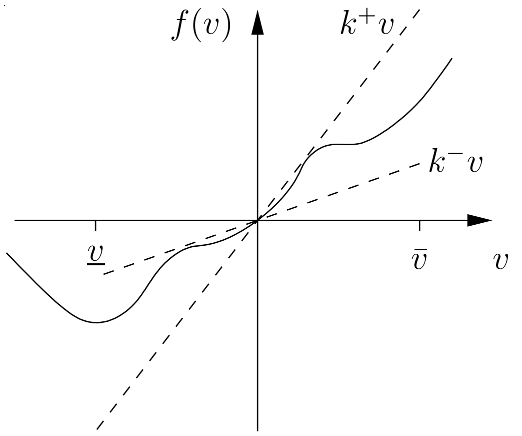


Fig. 2. A nonlinear function and its sector bounds for $\mathcal{V} = [\underline{v}, \bar{v}]$.

Given this information, we can determine the error gain nonlinearity measure by

$$\gamma_{N_f}^{\mathcal{U}} = \begin{cases} \frac{1}{2}(k^+ - k^-) & \text{if } \left|k^+\right|, \left|k^-\right| < \infty \\ \infty & \text{else} \end{cases}$$

and the best linear model is the straight line with slope $k^* = \frac{1}{2}(k^+ + k^-)$ if $\gamma_{N_f}^{\mathcal{U}}$ is finite. In a similar way we get

$$\varphi_{N_f}^{\mathcal{V}} = \begin{cases} \left|\frac{k^+ - k^-}{k^+ + k^-}\right| & \text{if } 0 < k^+k^-, \left|k^+\right|, \left|k^-\right| < \infty \\ 0 & \text{if } k^+ = k^- = 0 \\ 1 & \text{else} \end{cases}$$

for the relative nonlinearity measure and the best linear model with respect to the latter is given by

$$\begin{cases} \frac{1}{k^*} = \frac{1}{2}\left(\frac{1}{k^+} + \frac{1}{k^-}\right) & \text{if } 0 < k^+k^-, \left|k^+\right|, \left|k^-\right| < \infty \\ k^* = 0 & \text{else.} \end{cases}$$

Now let us assume we have calculated the nonlinearity measure of the steady-state locus. Recall that we assumed that the steady-state response is unique, i.e. $y(t) = (Nu)(t) \to f(v)$ whenever $u(t) \to v$ for $t \to \infty$. Then, the nonlinearity measures of the (dynamic) plants are bounded below by the respective quantities of the steady-state locus:

$$\gamma_N^{\mathcal{U}(\mathcal{V})} \geq \gamma_{N_f}^{\mathcal{V}} \text{ and } \varphi_N^{\mathcal{U}(\mathcal{V})} \geq \varphi_{N_f}^{\mathcal{V}}.$$

Proofs of the facts given in this section can be found in (Schweickhardt and Allgöwer, 2005). Next, we turn our attention to the computation of the nonlinearity measure for general dynamic systems.

3.3 Computation of nonlinearity measures

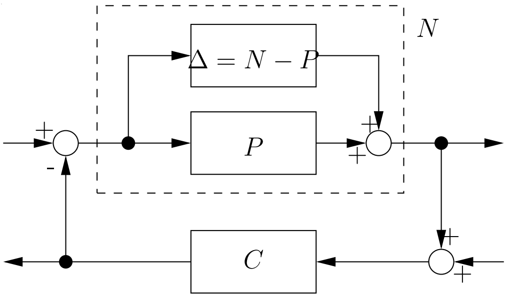
In the literature, different schemes can be found to approximately compute the nonlinearity measure. A lower bound can be calculated by considering only harmonic input signals (Allgöwer, 1995). Approximate computational scheme are given in (Allgöwer, 1996) for the general case, and in (Sourlas and Manousiouthakis, 1992; Sourlas and Manousiouthakis, 1998) an approach to derive a value for the absolute measure from (Desoer and Wang, 1980) is developed for a class of discrete-time systems. In (Kihas and Marquez, 2004) a quantity very similar to nonlinearity measures is considered. A procedure is proposed to approximately compute the L_2-gain of the error system defined as the difference between a nonlinear system and its Jacobi-linearization (both in continuous time). To this end, a Hamilton-Jacobi inequality is approximated at a finite number of points in a given region of the state space. Then the input-to-state- and L_∞-gain of the nonlinear process and its linearization respectively are calculated in order to estimate an upper bound on $\|u\|$ that guarantees that the system remains in the given region of the state-space. For all approaches, the most difficult part in obtaining accurate values for the nonlinearity measure is the computation of gains (Nikolaou and Manousiouthakis, 1989; Choi and Manousiouthakis, 2000; van der Schaft, 2000). Nonetheless, the mentioned approximation procedures often give sufficiently accurate results. We will not go into further detail but concentrate on the main ideas of nonlinearity assessment.

4. LINEAR CONTROL OF NONLINEAR SYSTEMS – A SMALL GAIN APPROACH

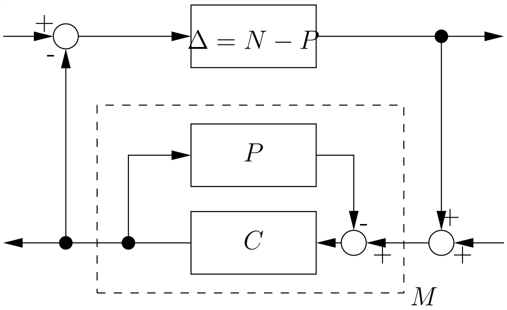
Once we have decided that linear controller design is adequate for the nonlinear control problem at hand, we turn the attention to the design of a suitable controller. The usual approach to the design of linear controllers for nonlinear systems is to use the linearization around the operating point as a linear model, design a linear controller and analyze (e.g. by simulation) the stability and performance of the closed loop with the nonlinear plant. By this method, no stability and performance guarantees can be made and the degree of nonlinearity of the plant is not taken into account in the controller design step. In this Section, a novel approach is presented that integrates nonlinearity analysis and linear controller design for nonlinear systems in order to

(1) decide for a given control problem whether linear controller design is adequate,
(2) derive a suitable linear model (not necessarily equivalent to the local linearization)
(3) describe a linear controller design procedure that guarantees stability (and possibly performance) of the closed loop containing the nonlinear process.

The suggested method in this Section will deal with stable nonlinear systems exhibiting a finite gain $\gamma(N) := \sup_u \frac{\|Nu\|}{\|u\|}$. The idea is to use the small gain theorem in order to maintain stability despite a non-



(a) Linear closed loop with nonlinear uncertainty



(b) Reformulation for use with small gain theorem

Fig. 3. Formulation of the nonlinear control problem as linear control problem with norm bounded nonlinear uncertainty.

linear uncertainty (for the small gain theorem, see e.g. (Vidyasagar, 2002)). With the help of the small gain theorem, a controller C that satisfies $\gamma(N)\gamma(C) < 1$, where N is the plant, will achieve closed loop stability, see Fig. 3 (a). While stability can be achieved this way, it remains unclear how performance requirements can be met and how the controller should look like apart from the requirement $\gamma(C) < \gamma(N)^{-1}$. In particular, this approach excludes controllers with integral action as the controller has to be stable itself. To circumvent these problems, a different approach is taken. We therefore split the nonlinear system N into a (stable) linear part P and a (stable) nonlinear part $\Delta = N - P$ and use linear techniques to design a (not necessarily stable) linear controller C for the linear part P.

With the help of the loop transformation theorem (Vidyasagar, 2002, Ch. 6), we can reorder the closed loop as depicted in Fig. 3 (b). Now, let the linear closed loop transfer operator be $M = CS = C(I + PC)^{-1}$. The small gain theorem then states that if the gains of M and Δ satisfy

$$\gamma(\Delta)\gamma(M) < 1$$

the closed loop system is stable. Moreover, the linear controller can be designed such that the performance requirements are met at least for the linear model P and integral action is possible. In order to get the best results, the linear model must be chosen such that the gain of the model uncertainty $\gamma(\Delta)$ takes the least possible value. This is achieved by choosing P such that $\gamma(\Delta) = \gamma(N - P)$ is minimized. But this procedure exactly corresponds to the definition of the error gain nonlinearity measure $\gamma_N^{\mathcal{U}} = \inf_G \gamma(N - P)$. We call a model P^* with $\gamma(N - P^*) = \gamma_N^{\mathcal{U}}$ the best linear model. Whenever no such P^* exists we choose a $\tilde{P}^*$ for which $\gamma(N - \tilde{P}^*) \approx \gamma_N^{\mathcal{U}}$. The nonlinearity assessment then consists of (i) computing $\gamma_N^{\mathcal{U}}$ and P^* and (ii) check with linear H_∞-techniques whether $\gamma(M) < 1/\gamma_N^{\mathcal{U}}$ is achievable. If so, P^* gives a suitable linear model and linear controller design can be done to optimize customized performance objectives. In principal, any controller design method can be utilized as long as the constraint $\gamma_N^{\mathcal{U}}\gamma(M) < 1$ is guaranteed to be satisfied.

We will illustrate the presented approach with a small example. Consider the system

$$\dot{y} = -y - y^3 + u$$

in the operating range $|u(t)| < 2\,\forall t$. Using the steady-state locus, which is easy to compute, and using the formulas given in Section 3, we can immediately give the lower bound of $\gamma_N^{\mathcal{U}} \geq 0.25$ for the nonlinearity measure. Indeed, the numeric computations of the dynamic nonlinearity measure results in a value of 0.271. The corresponding best linear approximation is of order 10 and can be reduced to obtain the model

$$P^*(s) = \frac{0.040s + 0.796}{s + 1.025}$$

as opposed to the linearization around the steady state which yields

$$P_{lin}(s) = \frac{1}{s+1}.$$

For both models we design a controller such that the closed loop (with the model) has a first order delay behaviour with a bandwidth of $3rad/s$. As discussed above the error between the best linear model P^* and the true nonlinear process can be taken into consideration and it is guaranteed that the nonlinear closed loop is stable. This is accomplished by verifying that for the controller based on the best model P^* we have $\gamma_N^{\mathcal{U}}\gamma(M) = 0.271 \cdot 3.298 = 0.892 < 1$. For P_{lin} the size of the error is not known and therefore no guarantees can be given. In fact P^* is the linear model that makes the error smallest and thus will give rise to the lowest conservativeness which, with all due caution, is related to a better performance. This can also be seen from the closed loop simulations. In Fig. 4, step responses of the closed loops containing the nonlinear plant and the controllers based on the best linear model and based on the local linearization respectively are plottet.

Due to integral action, both controllers achieve vanishing steady state errors. But it can be seen that for this simple system already the proposed procedure not only guarantees stability, but also leads to a better performance when compared to the conventional approach. More details on the approach and

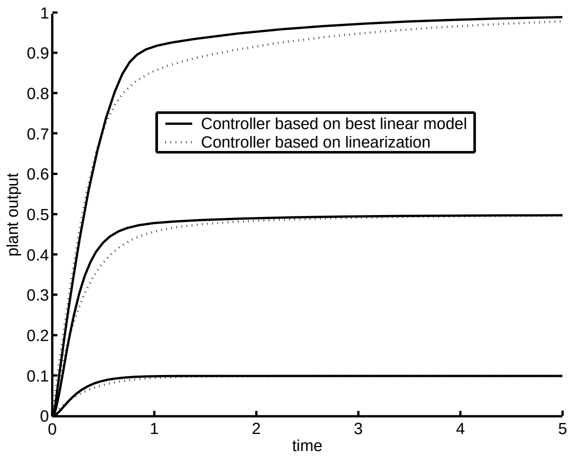


Fig. 4. Responses of the closed loops to steps in the reference signal of different height. The solid line corresponds to the controller based on the best linear model, the dotted line to the controller based on the local linearization.

more complex worked out examples can be found in (Schweickhardt and Allgöwer, 2006).

5. CONCLUSIONS

The development of advanced controller design techniques for nonlinear processes requires much effort. Linear controller design can be implemented much more easily and fortunately leads to satisfactory results in many practical situations. Thus, one requires to have tools in order to determine prior to controller design whether linear controller design is adequate or whether nonlinear techniques have to be used. It is interesting to observe that similar studies are taking place in the area of empirical modeling, where process nonlinearity can obstruct identification of linear models, or render the identified models obsolete.

In this note, we presented an introduction to recent results on the input/output-based approach to nonlinearity assessment. Useful formulas to quickly determine lower bounds of nonlinearity measures based on the steady-state process behaviour were given. We motivated the necessity of control-relevant nonlinearity assessment and introduced the corresponding optimal control law nonlinearity measure. We then presented a novel approach that combines nonlinearity assessment and controller design by using linear robust control methods and nonlinearity measures. An example was given that showed the usefulness of the proposed approach.

In the future, two directions of further investigations are expected to play an important role. Firstly, it is desirable to get methods that more precisely quantify how much performance can be gained by using nonlinear controller design techniques instead of linear ones. Secondly, the proposed method for linear controller design of nonlinear systems is only a first step. Many other approaches can be imagined, and once easily realizable and reliable methods are developed that also guarantee a certain level of performance they are expected to have a tremendous impact on process control practice.

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