ON OPTIMAL CONTROL PROBLEMS IN FUNDING SYSTEMS

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Abstract: This short paper addresses problems related to the funding of continuous time systems that accumulate capital (during the accumulation phase) for financial obligations that are to be paid sometime in the future (in the decumulation phase). In this situation there exists a fund (henceforth referred to as the "Fund") subject to contributions and drawdowns. The Fund will hold assets and may have liabilities that are contractual or semi-contractual. In the main, such funding systems will involve the interplay between the need to minimize contributions that support the Fund and the need to maintain reasonable solvency in the Fund. This set-up leads to a stochastic optimal control problem that may be solved by making use of methods related to dynamic programming and capital asset pricing modelling (CAPM).

Keywords: Finance; Financial Systems; Optimality; Stochastic Control.

1. INTRODUCTION

The use of stochastic optimal control theory in solving problems related to discrete and continuous time financial systems is an emerging area of research in mathematical finance. For instance, stochastic control problems are often encountered in the context of pricing and hedging in complete and incomplete markets (see El Karoui and Quenez, 1991 and El Karoui and Quenez, 1995); consumption and portfolio selection (see Merton, 1969 and 1971 and Karatzas, *et al.*, 1987) insurance and reinsurance (see, for instance, Hipp and Taksar, 2000), loan and debt management (see, for instance, Hellwig, 1977 and Taksar and Zhou, 1998), interest rate models (see, for instance, Petersen, *et al.*, 2003) and funding systems (see, for instance, Vandebroek, 1990; Boulier, *et al.*, 1995; Boulier, *et al.*, 1996; Cairns and Parker, 1997; Cairns, 2000 and Taylor, 2002). Examples of the latter are, for instance, retirement funds such as pension and provident funds, retirement annuities and preservation funds.

This contribution investigates the use of stochastic control in solving problems in continuous time funding systems like defined benefit pension funds (provide benefits to members that are defined in terms of a member's final salary and the length of membership in the company), shareholder dividend payments by insurers and the maintenance of a prudential margin (usually decided upon by statutes) by a non-life insurer. In this paper we will consider the randomness that arises in such Funds because of the inherent uncertainty in investment returns relative to salary growth. As a result the only sensible thing to do is to cast the Fund in a stochastic framework. The said Fund will hold assets (any item of economic value owned by the Fund, especially that which could be converted to cash) and may have liabilities (financial obligation, debt, claim, or potential loss) that are contractual or semi-contractual. The contributor (usually the employer) to the Fund usually has at least two variables under his control during this process, viz., how the random rate of contribution should be varied and how assets should be disposed (sold, exchanged or retired) per sector. Furthermore, the aforementioned Fund remains solvent only if assets exceed liabilities with the value of these notions largely being uncertain. Optimal control is usually used to ascertain how rates of contribution to the Fund and allocation of its assets per asset sector would react to a change in solvency. In this regard, results are usually obtained from a stochastic differential equation whose solution may be determined by numerical means. Further (stochastic and deterministic) optimal control problems related to those set out above are discussed fairly extensively in the literature. For instance, in the retirement funding framework, we have Haberman, *et al.* (2000); Haberman and Sung (1994); O'Brein (1986) and (1987); Owardally and Haberman (1996); Parlar (1981); Taylor (2002) and Vandebroek (1990).

The motivation for studying the control and structure of Fund dynamics is that it is becoming increasingly important that Fund members become aware of factors affecting defined benefits promised by Funds and sponsoring employers take cognicance of the timing and stability of cashflows. In defined benefit pension funds, for example, pension and other benefits are not dependent on past investment performance. Instead the risk associated with future returns on the fund's assets is borne by the employer. This manifests itself through the contribution rate which must vary through time as the solvency ratio or funding level fluctuates above and below its target level. If these fluctuations are not taken into account, i.e., if the contribution rates remains fixed, then the fund will eventually run out of assets from which to pay the benefits or grow exponentially out of control. In essence the use of optimal control theory in funding systems is motivated by the need to gain a deeper understanding of the solvency of the Fund and the factors that affect it. In particular, sponsors would like to make informed decisions about which contribution rates are optimal in the light of the solvency issue mentioned earlier. The shear monetary size of the retirement fund industry is also a factor that motivates the study of funding systems. In South Africa, for instance, it is estimated that the private and public sectors together support in excess of 15 000 funds with more than 10 million fund members and total assets of approximately 1 trillion South African Rand.

The procedure that we follow in order to solve the stochastic optimal control problem for continuous time funding systems is similar to those that are employed for other financial systems like incomplete markets. The main steps are outlined below.

Step 1: After careful consideration of the financial model, select the variables that have uncertainty associated with them and that can therefore be modelled as random variables or stochastic processes. From the deterministic and stochastic variables identify those that can be regarded as control functions. It is preferable that stochastic variables should be able to be represented explicitly in the form of stochastic integral or differential formulas.

Step 2: Consider the relationships between the stochastic variables identified in Step 1 and realize these connections as (systems of) stochastic differential equations.

Step 3: If necessary, make use of some type of model reduction to reduce the (system of) stochastic differential equations determined in Step 2 into standard stochastic systems and control form. At least one of the components of this system should contain the control function identified in Step 1.

Step 4: Deduce an appropriate objective or cost function in stochastic integral form that has to be optimized, i.e., minimized or maximized, from the financial system found in Step 3. Care has to taken in order to ensure that the function is well-defined and appropriate constraints are chosen.

Step 5: Apply one or more of the available methods of approaching stochastic control problems to solve the problem at hand. Some methods that are available are dynamic programming (open loop method), numerical solution of the Bellman equation (based on the discretization of the Hamilton-Jacobi-Bellman Equation), Markov chain approximation methods, regular pertubation methods, Monte Carlo techniques (probabilistic), binomial/trinomial trees (probabilistic), stochastic approximation, stochastic programming (closed loop method), stochastic maximum principle (Pontryagin), convex duality methods, martingale approaches (like the martingale transformation method) and stochastic linear quadratic theory.

Step 6: Test that the solution obtained in Step 5 makes sense in a real-world situation. This will in all likelihood involve producing a numerical example to illustrate the theory.

2. THE FUNDING SYSTEM

In this section we describe aspects of the Fund that are important for the ensuing analysis. Throughout we assume that the fund membership structure remains stable although it is not uncommon for the company contributing to the

Fund to restructure and evolve during both the accumulation and decumulation phases. We show that concepts related to the Fund such as rates of drawdowns, investment returns and increase of liabilities before drawdowns may be modelled as random variables that are driven by an associated Wiener process and may be expressed explicitly in terms of their respective expected values and a diffusion term. Furthermore, we are able to produce a system of stochastic differential equations that provide information about **Fund assets at time** t denoted by $\mathbb{A}_t$ and **Fund liabilities at time** t denoted by $\mathbb{L}_t$ and their relationship. Responsibility for the assets of a pension fund is usually borne by a group of Fund managers and/or trustees who must have the best interests of the Fund members at heart. They choose how the Fund assets are invested with the particular investment strategy depending on such factors as tax status, maximization of returns, minimization of risk, diversification, security, avoidance of self-investment and cashflow specifications. $\mathbb{A}_t$ is stochastic in nature because it depends in part on the stochastic rates of return of the investments of the Fund. Also, $\mathbb{L}_t$ is stochastic because its value has a reliance on the liability cash flows and asset values that both have randomness associated with them. The Fund's assets $\mathbb{A}_t$ may be described as the amount that remains when past drawdowns are deducted from contributions and investment returns. Furthermore, we use the notation $\mathbb{X}_t$ for

$$\mathbb{X}_t = \begin{bmatrix} \mathbb{A}_t \\ \mathbb{L}_t \end{bmatrix} \tag{1}$$

and represent the **funding level** or **solvency ratio**, $\mathbb{S}_t$, of the Fund as

$$\mathbb{S}_t = \mathbb{A}_t / \mathbb{L}_t \,. \tag{2}$$

It is important for Fund solvency that $\mathbb{S}_t$ has to maintain a high value. Obviously, low values of $\mathbb{S}_t$ indicate that the fund is struggling to stay solvent.

In the main, the sponsoring employer is free to choose how the contribution rate can be varied. The underlying principle governing this decision is that the amount of surplus or deficit has to be taken into account. Roughly speaking, the rate of contribution can be reduced during times of surplus and should be increased beyond the normal rate when the Fund is in deficit. In the sequel, the deterministic variable $\chi(t)$ is the **normal rate of contributions per unit of the Fund's liabilities**. In this case $\chi(t)dt$ turns out to be the value of contributions per unit of fund liabilities over the time period $(t, t+dt)$. A notion related to this is the **adjustment to the rate of contributions per unit of the Fund's liabilities for surplus or deficit**, $\alpha(t, \mathbb{S}_t)$ that depends on the solvency ratio. Here the amount of surplus or deficit is reliant on the excess of assets over liabilities. We denote the sum of χ and α by the contribution rate κ, i.e.,

$$\kappa(t) = \chi(t) + \alpha(t, \mathbb{S}_t). \tag{3}$$

The contribution rate κ is a predictable process and, as we shall see in the sequel, provides us with a means of controlling the dynamics of the Fund. The **rate of drawdown per unit of the Fund's liabilities**, $\delta(t)$, is given by

$$\delta(t)dt = \mathbf{E}\delta(t)dt + \sigma_\delta dW_\delta(t), \tag{4}$$

where $\delta(t)$ is a random variable, σ_δ is the volatility in the drawdown per unit of the Fund's liabilities, $W_\delta(t)$ is a standard Wiener process and $\delta(t)dt$ is the value of drawdowns per unit of fund liabilities over the interval $(t, t+dt)$. Furthermore, we consider

$$\rho(t)dt = \mathbf{E}\rho(t)dt + \sigma_\rho dW_\rho(t), \tag{5}$$

where the random variable $\rho(t)$ in (5) is the **rate of investment return on Fund assets**, σ_ρ is the volatility in the rate of investment return on Fund assets, $W_\rho(t)$ is a standard Wiener process and $\rho(t)dt$ is the value of investment return on Fund assets over the time period $(t, t+dt)$. We suppose from the outset that the Fund invests in a financial market with $n+1$ financial assets. One of these assets is risk free and will be called a money market. Assets $1,\ 2,\ \ldots,\ n$ are risky and will be known as stocks. These assets evolve continuously in time and are usually driven by a d-dimensional Wiener process $(1 \leq d \leq n+1)$. In this multidimensional context, the **rate of investment return on Fund assets in the k-th asset** is denoted by $\rho_k(t)$ and we assume that the expectation of $\rho_k(t)$ may be represented by

$$\mathbf{E}\rho_k(t) = \mathbf{E}\rho_0(t) + K\beta_k, \tag{6}$$

where

$$\beta_k = \kappa_{kn}(t)/\kappa_{nn}(t) \text{ independent of } t, \tag{7}$$

and $K > 0$ is a constant.

We can choose from two approaches when modelling our Fund in a stochastic setting. The first is a realistic model that incorporates all the aspects of the Fund like salary growth, individual mortality and individual members. Alternatively, we can develop a simple model which acts as a proxy for something more realistic and which emphasizes features that are specific to our particular study. In our situation we choose the latter option, with the notions of Fund assets $\mathbb{A}_t$ and liabilities $\mathbb{L}_t$ at time t and their relationship being modelled by the stochastic differential equations

$$d\mathbb{A}_t = \rho(t)\mathbb{A}_t dt + [\kappa(t) - \delta(t)]\mathbb{L}_t dt; \quad (8)$$
$$d\mathbb{L}_t = [\iota(t) - \delta(t)]\mathbb{L}_t dt, \quad (9)$$

where, for the **rate of increase of liabilities before drawdowns**, $\iota(t)$, the volatility in the increase of liabilities before drawdowns, σ_ι, and corresponding standard Wiener process $W_\iota(t)$, we have

$$\iota(t)dt = \mathbf{E}\iota(t)dt + \sigma_\iota dW_\iota(t). \quad (10)$$

The random variable $\iota(t)$ in (10) may typically originate from liabilities that have recently been accrued or instability in the value of pre-existing liabilities that may result from factors such as, for example, inflation. The stochastic differential equations (8) and (9) may be rewritten by substituting appropriate terms from (4), (5) and (10). This procedure yields an alternative system of stochastic differential equations that may be expressed as

$$\begin{aligned} d\mathbb{A}_t &= \mathbf{E}\rho(t)\mathbb{A}_t dt + [\kappa(t) - \delta(t)]\mathbb{L}_t dt \quad (11)\\ &\quad + \sigma_\rho \mathbb{A}_t dW_\rho(t) - \sigma_\delta \mathbb{L}_t dW_\delta(t);\\ d\mathbb{L}_t &= [\mathbf{E}\iota(t) - \mathbf{E}\delta(t)]\mathbb{L}_t dt \quad (12)\\ &\quad + \sigma_\iota \mathbb{L}_t dW_\iota(t) - \sigma_\delta \mathbb{L}_t dW_\delta(t). \end{aligned}$$

By considering the vector form $\mathbb{X}_t$ in (1) the equations above can in turn be rewritten into standard stochastic systems form as

$$\begin{aligned} d\mathbb{X}_t &= M(t)'\mathbb{X}_t dt + N(t)'\kappa(t)dt\\ &\quad + G(t)'dW(t)', \quad (13) \end{aligned}$$

where the various terms here are defined as follows.

$$M(t)' = \begin{bmatrix} \mathbf{E}\rho(t) & -\mathbf{E}\delta(t) \\ 0 & \mathbf{E}\iota(t) - \mathbf{E}\delta(t) \end{bmatrix}; \quad (14)$$

$$N(t)' = \begin{bmatrix} \mathbb{L}_t \\ 0 \end{bmatrix}; \quad (15)$$

$$G(t)' = \begin{bmatrix} \sigma_\rho \mathbb{A}_t & -\sigma_\delta \mathbb{L}_t & 0 \\ 0 & -\sigma_\delta \mathbb{L}_t & \sigma_\iota \mathbb{L}_t \end{bmatrix}; \quad (16)$$

$$W(t)' = \begin{bmatrix} W_\rho(t) \\ W_\delta(t) \\ W_\iota(t) \end{bmatrix}, \quad (17)$$

where $W_\rho(t)$, $W_\delta(t)$ and $W_\iota(t)$ are mutually (stochastically) independent Wiener processes.

Next, we introduce a **diffusion process**, ϕ, that may be represented as

$$\phi(t)dt = \mathbf{E}\phi(t)dt + H(t)dW_\phi(t), \quad (18)$$

where $W_\phi(t)$ is a d-dimensional Wiener process that is mutually (stochastically) independent of W_δ and W_ι and $H(t)$ is a matrix of dimension $(n \times 1) \times d$ such that

$$L[\phi(t)dt] = H(t)H(t)^T dt = C(t)dt \quad (19)$$

where ratios of pairs of $C(t)$ are independent of t and

$$\phi(t) = [\rho_0(t),\ \rho_1(t),\ \ldots,\ \rho_n(t)]^T. \quad (20)$$

In the sequel we will make use of a Capital Asset Pricing Model (CAPM) structure to analyze (18).

Various concepts related to the expected value of the rate of return per sector, $\mathbf{E}\rho_k(t)$, in (6) are presented next. Firstly, we observe that if we put $k = 0$ in (6) and (7) we have

$$\kappa_{0n} = 0 \quad (21)$$

Also, we can set $k = n$ in (6) and (7) to obtain the **share market rate of return**

$$\mathbf{E}\rho_n(t) = \mathbf{E}\rho_0(t) + K. \quad (22)$$

Furthermore, if we suppose that $\kappa_{00}(t) = 0$ then

$$\kappa_{0k}(t) = \kappa_{k0}(t) = 0,\ k = 0,\ 1,\ \ldots,\ n. \quad (23)$$

This enables us to rewrite (6) in vector form

$$\begin{aligned} \phi(t) &= [\rho_0(t),\ \rho_1(t),\ \ldots,\ \rho_n(t)]^T\\ &= \mathbf{E}\rho_0(t)1 + K\beta, \quad (24) \end{aligned}$$

where the $(n+1)$-th vector is 1 with every component unity and

$$\begin{aligned} \beta &= [\beta_0,\ \beta_1,\ \ldots,\ \beta_n]^T. \quad (25)\\ &= \begin{bmatrix} \beta_0 \\ \widetilde{\beta} \end{bmatrix}. \quad (26) \end{aligned}$$

We suppose that $\pi_k(t)$ is the notation used for the **proportion of assets invested in asset sector k at time** t. We express $\pi(t)$ as

$$\begin{aligned} \pi(t) &= [\pi_0(t),\ \pi_1(t),\ \ldots,\ \pi_n(t)]^T \quad (27)\\ &= \begin{bmatrix} \pi_0(t) \\ \widetilde{\pi}(t) \end{bmatrix}, \quad (28) \end{aligned}$$

where

$$\sum_{k=0}^{n} \pi_k(t) = 1. \quad (29)$$

Given the disposition of assets described in the above we can rewrite $\rho(t)dt$ in (5) by using (24) and (29) as

$$\begin{aligned}\rho(t)dt &= \pi(t)^T\phi(t)dt && (30)\\ &= \pi(t)^T[\mathbf{E}\phi(t)dt + H(t)dW_\phi(t)]\\ &= \pi(t)^T[\mathbf{E}\rho_0(t)1dt + K\beta dt + H(t)dW_\phi(t)]\\ &= [\mathbf{E}\rho_0(t) + \pi(t)^TK\beta]dt + p(t)^TdW_\phi(t)] && (31)\end{aligned}$$

From (22) and (31) it follows that

$$\mathbf{E}\rho(t) = \mathbf{E}\rho_0(t) + \pi(t)^TK\beta. \quad (32)$$

and by (19)

$$\begin{aligned}L[\rho(t)dt] &= \sigma_\rho^2 dt && (33)\\ &= \pi(t)^TH(t)H(t)^T\pi(t)dt\\ &= \pi(t)^TC(t)\pi(t)dt && (34)\end{aligned}$$

which implies that

$$\sigma_r^2 = \pi(t)^TC(t)\pi(t). \quad (35)$$

(29) results in there only being n degrees of freedom in the choice of $\pi(t)$. If we substitute (28) and (26) in (32) we obtain

$$\mathbf{E}\rho(t) = \mathbf{E}\rho_0(t) + \widetilde{\pi}(t)^TK\widetilde{\beta}. \quad (36)$$

Next, by applying (23) and (24) we obtain

$$\sigma_r^2 = \widetilde{\pi}(t)^T\widetilde{C}(t)\widetilde{\pi}(t), \quad (37)$$

where we obtain the $n \times n$ full-rank (poisitive definite) matrix $\widetilde{C}(t)$ by deleting the first row and column of $C(t)$ that are all zeros.

The funding system's evolution may be described as follows. Firstly, we represent the proposed control function u in the form

$$u(t) = \begin{bmatrix}\kappa(t)\\ \widetilde{\pi}(t)\end{bmatrix} \quad (38)$$

This enables us to reduce (13) to

$$d\mathbb{X}_t = M(t)\mathbb{X}_tdt + N(t)u(t)dt + G(t)dW(t) \quad (39)$$

with the various terms in this stochastic differential equation being

$$M(t) = \begin{bmatrix}\mathbf{E}\rho_0(t) & -\mathbf{E}\delta(t)\\ 0 & \mathbf{E}\iota(t) - \mathbf{E}\delta(t)\end{bmatrix}; \quad (40)$$

$$N(t) = \begin{bmatrix}\mathbb{L}_t & K\widetilde{\beta}^T\mathbb{A}_t\\ 0 & 0\end{bmatrix}; \quad (41)$$

$$G(t) = \begin{bmatrix}[\widetilde{\pi}(t)^T\widetilde{C}_t\widetilde{\pi}(t)]^{1/2}\mathbb{A}_t & -\sigma_\delta\mathbb{L}_t & 0\\ 0 & -\sigma_\delta\mathbb{L}_t & \sigma_\iota\mathbb{L}_t\end{bmatrix}; \quad (42)$$

$$W(t) = \begin{bmatrix}W_R(t)\\ W_\delta(t)\\ W_\iota(t)\end{bmatrix}, \quad (43)$$

where $W_R(t)$ is a new standard Wiener process and $W_R(t)$, $W_\delta(t)$ and $W_\iota(t)$ are mutually (stochastically) independent.

3. STOCHASTIC CONTROL OF FUNDING SYSTEMS

We are now in a position to state the stochastic optimal control problem for continuous time funding systems that we solve. The said problem may be formulated as follows.

For continuous time funding systems subject to control, how do we select the values of the contribution rate and the disposition of assets per sector at any time in some optimal way ?

In order for a Fund manager or trustee to determine an optimal contribution rate and asset allocation strategy it is imperative that a well-defined objective function (loss function in our case) with appropriate constraints is considered. The choice has to be carefully made in order to avoid ambiguous solutions to our stochastic control problem. In this particular contribution, we choose to determine control functions $\kappa(t, \mathbb{X}_t)$ and $\pi(t, \mathbb{X}_t)$ that minimize the exponential loss function

$$\begin{aligned}&L(s, \mathbb{X}_s) = && (44)\\ &\mathbf{E}\Big[\int_s^T f(t, \mathbb{X}_t, \kappa(t, \mathbb{X}_t))dt + g(T, \mathbb{X}_T)\Big],\end{aligned}$$

where f and g are appropriate real-valued, non-negative functions. Here we adopt the convention that the symbol $L(s, \mathbb{X}_s)$ denotes the value of the loss function over the interval $[s, T]$. Also, $\kappa(t)$ is taken to be a control variable and therefore depends on $\mathbb{X}_t$. Next, we make appropriate choices for f and g that will lead to a loss function that is exponential. Firstly, we define f as

$$f(t, \mathbb{X}_t, \kappa) = e^{-\alpha t}[1/2\kappa^2(t) + s_v(\mathbb{S}_t)], \quad (45)$$

where the **solvency value function**, s_v, is non-negative, strictly decreasing, $\alpha > 0$ is a **discount rate** and $e^{-\alpha t}$ is a **discount function** reflecting future time weighting. In (45) it is clear that the desire for high solvency against high contribution rates is balanced. Also, as $\kappa(t)$ increases so does $f(t, \cdot, \cdot)$ but $f(s, \cdot, \cdot)$ for $s < t$ decreases because the higher value of $\kappa(t)$ increases solvency at these times s. Furthermore, the solvency value function s_v has the properties that

$$s_v \geq 0, \quad s_v^{'} < 0, \quad s_v^{''} > 0. \quad (46)$$

Furthermore, g in (44) is chosen to be

$$g(T, \mathbb{X}_T) = e^{-\alpha T} U(\mathbb{S}_T), \tag{47}$$

where $L(s, \mathbb{X}_s)$ in (44) is factorized as

$$L(s, \mathbb{X}_s) = e^{-\alpha s} U(\mathbb{S}_s). \tag{48}$$

It is clear that we cannot choose g freely and that the choice made in (47) is a reasonable one.

We are now in a position to state and prove the main result related to the stochastic control problem that we solve in this paper.

Theorem 1. **(Control of Funding Systems)** Suppose that the stochastic system (39) is described by $u(t)$, $M(t)$, $N(t)$, $G(t)$ and $W(t)$ that are given explicitly by (38), (40), (41), (42) and (43), respectively, and the summation (29) holds. Then the loss function $L(s, \mathbb{X}_s)$ represented by (44) is minimized by

$$u(\mathbb{S}_t) = u^*(\mathbb{S}_t) = \begin{bmatrix} \kappa^*(\mathbb{S}_t) \\ \widetilde{\pi}^*(\mathbb{S}_t) \end{bmatrix} \tag{49}$$
$$= \begin{bmatrix} -U_{\mathbb{S}_t} \\ -\mathbb{S}_t^{-1} U_{\mathbb{S}_t} U_{\mathbb{S}_t \mathbb{S}_t}^{-1} K \widetilde{C}^{-1} \widetilde{\beta} \end{bmatrix},$$

where we assume that the optimal control strategy u^* (actually κ^* and $\widetilde{\pi}^*$) exists and U is the solution of the ordinary differential equation

$$\alpha U + a(\mathbb{S}_t) U_{\mathbb{S}_t} + b(\mathbb{S}_t) U_{\mathbb{S}_t \mathbb{S}_t} + \gamma (U_{\mathbb{S}_t}^2 / U_{\mathbb{S}_t \mathbb{S}_t})$$
$$+ U_{\mathbb{S}_t}^2 - s_v = 0 \tag{50}$$

with $U_{\mathbb{S}_t} = dU/d\mathbb{S}_t$, $U_{\mathbb{S}_t \mathbb{S}_t} = d^2U/d\mathbb{S}_t^2$ and

$$a(\mathbb{S}_t) = (\mathbf{E}\delta(t) + \sigma_\delta^2) - (\nu + \sigma^2)\mathbb{S}_t; \tag{51}$$

$$b(\mathbb{S}_t) = -1/2\sigma_\delta^2 + \sigma_\delta^2 \mathbb{S}_t - 1/2\sigma^2 \mathbb{S}_t^2; \tag{52}$$

$$\nu = \mathbf{E}\rho_0(t) + \mathbf{E}\delta - \mathbf{E}\iota; \tag{53}$$

$$\sigma^2 = \sigma_\delta^2 + \sigma_\iota^2; \tag{54}$$

$$\gamma = K^2 (\widetilde{\beta}^T \widetilde{C}^{-1} \widetilde{\beta}). \tag{55}$$

Proof. The result is proved by using any standard text for stochastic control theory and stochastic differential equations such as Fleming and Rishel (1975), Krylov (1980), Merton (1990) and Oksendal (1998).

We have another method of determining an optimal solution for our stochastic control problem that is directly related to a Hamilton-Jacobi-Bellman equation from dynamic programming. The highlights of this procedure is outlined below. For a controlled pension fund we define the value function

$$W(t, \mathbb{X}_s)(\kappa, \pi) = \tag{56}$$
$$\mathbf{E}\big[\int_t^\infty e^{-\alpha s} L'(s, X_s, \kappa(s, \mathbb{X}_s)) ds | \mathbb{X}_s\big].$$

As in the discussion preceding our main result, $e^{-\alpha s}$ is a discount function and $L'(s, X_s, \kappa(s, \mathbb{X}_s))$ is a quadratic loss function of the form

$$L'(s, X_s, \kappa(s, \mathbb{X}_s)) = \tag{57}$$
$$1/2\kappa^2(s) + s_v(\mathbb{S}_s) + U(\mathbb{S}_s),$$

where $s_v(\mathbb{S}_s)$ and $U(\mathbb{S}_s)$ are as defined earlier. The value function, W, in (56) is also a function of the chosen contribution strategy $\kappa(s, \mathbb{X}_s)$ and $\pi(s, \mathbb{X}_s)$. In order to investigate the minimization of $W(t, \mathbb{X}_s)$ we have to consider

$$Y(t, \mathbf{X}_s) = \tag{58}$$
$$\inf_{(\kappa,\pi)} W(t, \mathbb{X}_s)(\kappa, \pi) = W(t, \mathbb{X}_s)(\kappa^*, \pi^*)$$

under the assumption that the optimal control strategies κ^* and π^* exist. From a consideration of Fleming and Rishel (1975), Krylov (1980), Merton (1990) and Oksendal (1998) it is clear that $Y(t, \mathbb{X}_s)$ satisfies the Hamilton-Jacobi-Bellman equation

$$0 = \inf_{(\kappa,\pi)} \big(e^{-\alpha t} L'(t, \kappa, \mathbb{X}_s) + Y_t$$
$$+ \big[(\rho_0 + \pi^T \lambda)\mathbb{X}_s + \kappa - \delta\big] Y_{\mathbb{X}_s}$$
$$+ 1/2 Y_{\mathbb{X}_s \mathbb{X}_s} \big(\mathbb{X}_s^2 \pi^T D \pi + \sigma_\delta^2\big), \tag{59}$$

where $Y_t = \dfrac{\partial Y}{\partial t}$, $Y_{\mathbb{X}_s} = \dfrac{\partial Y}{\partial \mathbb{X}_s}$ and $Y_{\mathbb{X}_s \mathbb{X}_s} = \dfrac{\partial^2 Y}{\partial \mathbb{X}_s}^2$. Also, D is the **instantaneous covariance matrix** and

$$\lambda = [\lambda_1, \ \ldots, \ \lambda_n]^T \text{ where } \lambda_i = \rho_i - \rho_0, \tag{60}$$

is the **risk premium** associated with the i-th asset. In order to determine the optimal contribution rate κ and the asset allocation π we differentiate the bracketed expression. For the contribution rate κ we obtain

$$\frac{\partial}{\partial \kappa}(\cdot) = e^{-\alpha t} L'_\kappa + Y_{\mathbb{X}_s}, \ L'_\kappa = \frac{\partial L'}{\partial \kappa} \tag{61}$$
$$\kappa^*(t, \mathbb{X}_s) = L_\kappa^{'-1}(e^{-\alpha t} Y_{\mathbb{X}_s}). \tag{62}$$

Furthermore, for the asset allocation π we have

$$\frac{\partial}{\partial \pi}(\cdot) = \lambda \mathbb{X}_s Y_{\mathbb{X}_s} + D\pi \mathbb{X}_s^2 Y_{\mathbb{X}_s \mathbb{X}_s} = 0, \tag{63}$$
$$\pi^*(t, \mathbb{X}_s) = -\big(\frac{Y_{\mathbb{X}_s}}{\mathbb{X}_s Y_{\mathbb{X}_s \mathbb{X}_s}}\big) D^{-1} \lambda. \tag{64}$$

4. CONCLUSIONS AND ONGOING INVESTIGATIONS

In this paper we have applied stochastic optimal control theory in order to establish how Fund solvency would be affected by member contribution and asset allocation strategies.

As regards future research on this subject we can identify the following main concerns.

(1) In order to assess the applicability of our stochastic analysis we should construct suitable numerical examples.

(2) We have to investigate further which optimal control strategies exist and are appropriate for different types of funding systems.

(3) We would like to apply the procedure suggested in this paper to other funding systems like defined contribution pension funds, retirement annuities, provident funds and preservation funds. Defined contribution pension funds are gaining in importance. By contrast with defined benefit pension plans the benefits are no longer dependent on the final salary but on past contribution levels and past returns on investments. In this process investment risk is passed from the employer to the individual fund members.

(4) Another possibility for further study is to seek alternative models for the evolution of assets and liabilities than that constituted by (8) and (9). These may not be restricted to only considering contribution rates and asset allocation strategies. Other factors that are under the control of the Fund manager and his advisers are the method and period of amortization, the intervaluation period, the funding method and the valuation basis.

(5) The choice of appropriate objective (or loss) functions for pension funds (if at all possible) is another current area of interest. In our contribution we have made use of an exponential loss function. Alternative objective functions that may be considered for future use will in all likelihood include power, quadratic functions or other exponential functions. Past experience has shown that working with power and exponential loss functions result in solutions to the optimal control problem that may be non-stationary. Also, the optimal asset allocation strategy derived during the use of exponential and quadratic functions have been found to be contrary to current practice. These experiences lead us to believe that future research must concentrate on finding alternative objective functions that give rise to stationary solutions and asset-allocation strategies that make sense for real-world situations.

(6) An analysis which involves the optimization of the objective function as the central theme does not usually rely on a calculation of the actuarial liability. However, the latter may result in a solution of the control problem that may be suboptimal. Therefore, future investigations may have to take the interplay between the actuarial liability and the optimized objective function into account.

(7) Other areas of possible research may focus on finding suitable constraints on contributions and investments.

(8) In our contribution we have assumed that the membership structure of the Fund is stable. In some instances there may be changes in this structure. This would mean that we would have to re-think an analysis that involves the objective function and solvency ratio (funding level) as its key components. In this regard, problems may only be solveable when using pre-existing actuarial techniques in combination with optimal control methods.

(9) Some of the other open problems related to our study arise out of the following objectives that are related to Fund design. For instance, a future study should be on how risks that Fund providers face during the accumulation and decumulation phases can be optimally hedged (see Blake, 2003; Blake and Burrows, 2001; Blake, *et al.*, 2001 and Blake, *et al.*, 2003 for the case of pension plans). Another investigation is to show that for a continuous-time Fund containing risky assets and risk-free assets in the presence of randomness on the level of benefit outflow certain Markov control strategies optimize over the contribution rate and over the range of possible asset allocations (see Cairns, 2000 for a continuous-time stochastic pension fund model). Also, we could investigate how to utilise stochastic lifestyling which takes into account both the degree of risk aversion of the Fund member and the correlation between the Fund member's salary progression and asset returns over time is more reliable than deterministic lifestyling which involves the gradual switch from equities to bonds according to preset rules (compare Blake, *et al.*, 2003).

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