Rayleigh-Taylor and Richtmyer-Meshkov instabilities of flat and curved interfaces

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Abstract

In this work we uncover a non-trivial effect of the interfacial curvature on the stability of uniformly and suddenly accelerated interfaces, such as liquid rims. The new stability analysis is based on operator and boundary perturbation theories, and allows us to treat the Rayleigh-Taylor and Richtmyer-Meshkov instabilities as a single phenomenon and thus to understand the interrelation between these two fundamental instabilities. This leads, in particular, to clarification of the validity of the original Richtmyer growth rate equation and its crucial dependence on the frame of reference. The main finding of this work is the revealed and quantified influence of the interfacial curvature on the growth rates and on the wavenumber selection of both types of instabilities. The key results are summarized in this extended abstract.

1 Introduction

Interfaces either uniformly [1, 2] or impulsively [3, 4] accelerated are ubiquitous in Nature, and usually exhibit long-wave instabilities, named after Rayleigh-Taylor (RT) and Richtmyer-Meshkov (RM), respectively. In the first case the instability occurs if the light fluid is accelerating into the heavy one, while in the second case the instability takes place when an interface between fluids of different density is impulsively accelerated, e.g. by the passage of a shock wave. It is believed, after the work of [3], that the occurrence of instability in the latter case does not depend upon the direction of impulsive acceleration. The known facts on RT and RM instability phenomena were reviewed in [5] and [6], accordingly. The physical situations where the RM instability appears span from combustion, *cf.* [7] to astrophysics, *cf.* [8]. The RT instability also occurs in various phenomena: e.g. inertial confinement fusion [9], astrophysics [10, 11, 12], geophysics [13, 14], and many others. Because of this wide fundamental impact, these classical RT and RM instabilities still attract attention. Despite numerous studies and refinements of both the RT and the RM instabilities, the two key aspects – influence of the frame of reference on the expression for the growth rate and the interfacial curvature on the development of these instabilities – have never been pointed out in the literature. These are the subjects of this work.

The behavior in the RT case can be described by the time-evolution equation [1] for interfacial perturbations $f(t)$, i.e. deviations from the flat interface, of wavenumber k under constant acceleration g in the coordinate system fixed in the interface

$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = |k|\, g\, f(t), \tag{1}$$

which is given for the case when density of one of the fluids can be neglected. Apparently, if $g > 0$ then the initially non-zero perturbations will grow exponentially in time. Richtmyer [3] applied the above Taylor's analysis to the case of impulsive acceleration $g(t) = V_0 \delta_D(t)$, which implies that the interface attains a jump in velocity equal to V_0 at the time instant $t = 0$, i.e. $V(t) = V_0 H(t)$, where $H(t)$ is the Heaviside step function. Integrating (1) for such an impulsive acceleration yields the famous Richtmyer growth rate relation

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = |k|\, V_0\, f(0), \tag{2}$$

which predicts linear growth in time, proportional to the initial amplitude of perturbation f_0 and the velocity jump V_0. It should be stressed that the Richtmyer argument was based on the ingenious generalization of Taylor's result, as developed in the reference frame moving with the interface.

The main results of physical significance are summarized in Assertions 1 - 4 throughout the text, and in brief can be stated as follows:

The interpretation and the growth rate of the Richtmyer-Meshkov instability depend on the frame of reference. The interfacial curvature and its sign influence the growth rates of both the Rayleigh-Taylor and Richtmyer-Meshkov instabilities, as well as the most unstable wavenumber range selection in the transverse direction in the 3D case.

2 2D flat interfaces

First, consider the two-dimensional configuration of an interface between two fluids – phase 1 is of density ρ and phase 2 is inertialess – in a gravity field $\mathbf{g}$, as in figure 1. In the analysis of the RT and RM instabilities we adopt Kelvin's restrictive assumption [15], i.e. we consider an inviscid and incompressible approximation of irrotational fluids:

$$\text{bulk (velocity) } y \le f(t,x): \qquad \begin{cases} \Delta\phi = 0, \\ \nabla\phi \to 0,\ y \to -\infty, \end{cases} \tag{3a}$$

$$\text{bulk (pressure) } y \le f(t,x): \qquad \frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) = -\frac{1}{\rho} p - g \cdot y + C(t), \tag{3b}$$

$$\text{interface (dynamic) } y = f(t,x): \qquad p = -\frac{\sigma f_{xx}}{(1+f_x^2)^{3/2}}, \tag{3c}$$

$$\text{interface (kinematic) } y = f(t,x): \qquad \frac{\partial f}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial y}, \tag{3d}$$

where ϕ is the velocity potential, $\mathbf{u} = \nabla\phi$, p – pressure, σ – surface tension, $\Delta = \partial_x^2 + \partial_y^2$ – the Laplacian, and $C(t)$ – the time-dependent constant in the Lagrange-Cauchy integral (3b).

System (3) allows us to determine the base state corresponding to the interface, which starts moving with velocity $\mathbf{j} \cdot V(t)$ (i.e. in the positive direction of y-axis) at the initial time instant $t = 0$. Introducing linearization around the base state, denoted by 0-superscript, $p = p^0(t,y) + p'(t,x,y)$, $\phi = \phi^0(t,y) + \phi'(t,x,y)$, and the undisturbed interfacial position $f = f^0(t) + f'(t,x)$, we get the

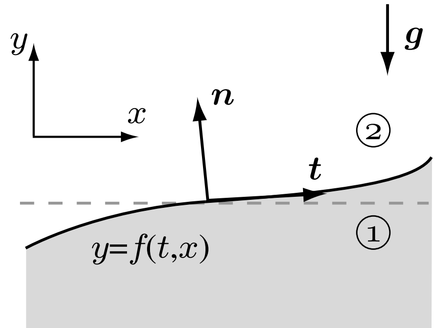

Figure 1: Interface between two fluids.

following equations for the evolution of disturbances

$$\frac{\partial \phi'}{\partial t} + V\phi'_y = -\frac{1}{\rho}p',\ y \leq f^0, \tag{4a}$$

$$-\rho\left(g + \frac{\mathrm{d}V}{\mathrm{d}t}\right) f' + p'|_{y=f^0} = -\sigma f'_{xx},\ y = f^0, \tag{4b}$$

$$\frac{\partial f'}{\partial t} = \frac{\partial \phi'}{\partial y},\ y = f^0, \tag{4c}$$

while the velocity potential is determined from the free-boundary problem for the Laplace equation:

$$\Delta\phi' = 0,\ y \leq f^0, \tag{5a}$$

$$\nabla\phi' \to 0,\ y \to -\infty. \tag{5b}$$

The analysis of (4-5) in the laboratory and moving frames of reference was constructed with the use of Fourier and Laplace transforms and leads to the following conclusion.

Assertion 1 *While the growth rate in the Rayleigh-Taylor instability does not depend upon whether the phenomena are considered in the laboratory or accelerating with the interface frames of reference, the growth rate of the Richtmyer-Meshkov instability is algebraic in the frame moving with the interface and exponential in the laboratory frame of reference. The anisotropy of the Richtmyer-Meshkov instability with respect to the direction of the impulsive acceleration reveals itself only in the laboratory frame as opposed to the moving frame. In the pure Richtmyer-Meshkov case (no constant acceleration and surface tension effects), the system is exponentially unstable when the impulsive acceleration is directed towards lighter fluid, and requires nonlinear stability analysis in the reversed situation.*

It is worth mentioning that this effect has been masked by the natural limitations of current experimental accuracy and the $O(t^2)$-difference between linear $y(t) = y(0)(t+1)$ and exponential $y(t) = y(0)\,e^t$ growths. In fact, there are just a few experimental measurements for small times, i.e. when one can expect the linear mechanisms to govern the dynamics. For example, the data of [16] for low-Mach numbers (figure 5 in that reference) can be attributed to the linear regime (vs. nonlinear regime) of the perturbation evolution at most up to the times $t \sim 2\,\mathrm{ms}$ and does not allow one to distinguish linear algebraic from exponential growth due to the experimental scatter. The reason for the above paradoxical conclusions is that *an observer moving with an interface does not discern whether the interface is accelerated or decelerated*, while in the laboratory frame of reference the difference is obvious, which leads to the anisotropy.

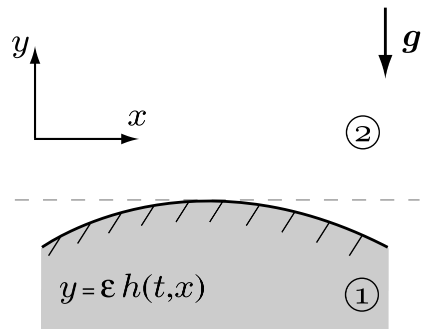

Figure 2: Curved interface as an $O(\epsilon)$-perturbation.

3 2D curved interfaces

Since true base state interfaces are frequently not flat and sometimes are highly curved, it is natural to explore possible deviations of the stability characteristics from those in the flat interface case. The key idea is to consider the curved interface *locally*, as depicted in figure 2, with small deviation from flatness, i.e. $\widetilde{f}(\tau,\xi) = \epsilon h(\tau,\xi)$ with $\epsilon \ll 1$. Using the linear operator [17] and the boundary perturbation [18] theories, in the RT case we get the following results on the eigenvalues governing the time evolution of disturbances (real parts of eigenvalues are growth rates):

Assertion 2 *If the flat interface is unstable in the RT case, i.e. there exists real* $\lambda_{+0}^{(n)} > 0$*, then the addition of a positive curvature (concave interface: cf figure 3(a)) makes the physical system more unstable, while the addition of negative curvature (convex interface: cf figure 3(b)) makes the system less unstable. The eigenvalues obey*

$$\lambda_{\pm}^{(n)} = \lambda_{\pm 0}^{(n)} + \epsilon\lambda_{\pm 1}^{(n)} + o(\epsilon), \tag{6}$$

where

$$\lambda_{\pm 0}^{2} = -(g+a)|n| - \frac{\sigma}{\rho}|n|^3, \tag{7}$$

which is the dispersion relation governing the stability of a two-dimensional column with a flat top and

$$\lambda_{\pm 1}^{(n)} \simeq \frac{\sigma}{\rho}\frac{n^2}{\lambda_{\pm 0}^{(n)}}\int_{-\pi}^{\pi} h_{\xi\xi}^{0}\mathrm{d}\xi. \tag{8}$$

In order to appreciate these results, let us make the following few corollary type clarifications. First, the flat interface results, given by the growth rates (7), are trivially recovered in the limit of vanishing curvature $\kappa \to 0$, i.e. $\epsilon \to 0$. Second, the interpretation of these curvature effects is not as trivial as one might think, i.e. that the presence of surface tension tends to flatten the interface, since the curved interface base state is truly an equilibrium base state. Third, because of the curvature effect, the Rayleigh-Taylor instability can be reversed, i.e. the sign of the growth rate can change as a function of base state curvature! Indeed, if the heavy phase 1 accelerates the light phase 2 and the interface is flat, then there should be no instability according to the RT criterion; however, if the interface is concave (cf. figure 3(a)), then the instability may appear. In fact, the above

(a) Concave interface: $\kappa > 0$.   (b) Convex interface: $\kappa < 0$.

Figure 3: Two generic curved interfaces; phase 1 is the (heaviest) liquid phase.

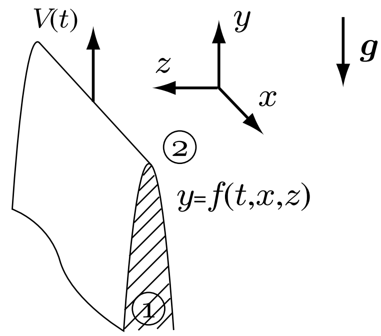

Figure 4: Three-dimensional rim.

two points can be illustrated with the well-known phenomena of vapor-filled underwater collapsing bubbles [19, 20], which are unstable despite that the denser liquid is accelerated towards the lighter vapor. Similar effects were found in the problems of radially imploding/exploding spherical [21] and cylindrical [22] shells. Moreover, as commented above, this instability result is unaffected by surface tension, which just allows for the existence of a base state (spherical bubble, in this case). This problem of underwater collapsing bubbles has been studied exactly because of its spherical symmetry, but to the author's knowledge the conclusion that this is a particular case of the more general effect of interfacial curvature has never been established.

Similar analysis in the RM case yields:

Assertion 3 *Infinitesimal perturbations of the neutrally stable interface under the conditions of impulsive acceleration lead either to exponential stabilization or to destabilization in the laboratory frame of reference depending upon the sign of the interfacial curvature.*

4 3D curved interfaces

With the above understanding of the stability of two-dimensional (2D) flat and weakly curved interfaces, one can easily address the stability of three-dimensional (3D) rims. Naturally, the main question of interest is the rim instability along x-axis, as shown schematically in figure 4. The idea of the stability analysis is the same as in §3, i.e. to analyze the structure of the solution near the rim tip, so that *locally* it is almost flat with $O(\epsilon)$ curvature. Then, the translation of the results of the previous section onto the 3D case turns out to be straightforward, as suggested by the structure

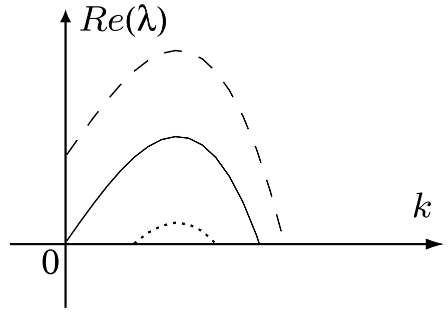

Figure 5: Effect of the interfacial curvature on the eigenvalues in the 3D case. Solid curve corresponds to zero curvature, dashed line to concave interface (positive curvature), and dotted line to convex interface (negative curvature).

of the velocity: potential solution

Assertion 4 *The stability of 3D rims, as that shown in figure 4, is affected by the transverse curvature: concave interfaces are less stable than flat ones, while convex interfaces are more stable. The range of lengthwise-unstable wavenumbers (i.e. along-the-rim wavenumbers) is narrowed in the case of convex interfaces and widened in the case of concave interfaces (cf. figure 5).*

5 Conclusions

In this work, increased understanding of the Richtmyer-Meshkov instability was gained, which uncovered important gaps in the Richtmyer's original treatment. The second contribution of this study is the clarification of the interfacial curvature effects in the two-dimensional and three-dimensional cases on the development of the Rayleigh-Taylor and Richtmyer-Meshkov instabilities, as well as on the wavenumber selection. All the major results are summarized in Assertions 1-4. The analysis of the stability of curved interfaces led to the rigorous generalization of the classical idea due to [23] on approximating the potential function in free-boundary problems with curved base state interfaces.

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