RELAXED LYAPUNOV CRITERIA FOR ROBUST GLOBAL STABILIZATION OF NONLINEAR SYSTEMS WITH APPLICATION TO CHEMOSTAT CONTROL

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Paper No. 266b, AIChE Annual Meeting, Philadelphia, PA, Nov. 2008

Keywords: uniform robust global feedback stabilization, chemostat models.

Introduction

The motivation of the present work arises from control problems in continuous stirred microbial bioreactors, often called chemostats. In many applications, the design conditions for the chemostat represent a locally asymptotically stable steady state but the stability region could be too small to allow proper operation in the presence of disturbances. Thus, the need for control arises in the sense of enlargement of the stability region of the design steady state in the presence of disturbances ([8]). Under these circumstances, achieving global stability of the bioreactor in the presence of disturbances will be best possible outcome from the design of a control system.

The problem of designing of a feedback law that achieves robust global stabilization of a nonlinear system is closely related to the existence of a Robust Control Lyapunov Function. However, it may be a big challenge to derive a Control Lyapunov Function satisfying global properties for a given nonlinear system. In the present work, under appropriate hypotheses, we derive relaxed Lyapunov-like sufficient conditions for Uniform Robust Global Asymptotic Stability. The Lyapunov-like conditions will be "relaxed" in the sense that the Lyapunov differential inequality is not required to hold over the entire state space, but only over an appropriate absorbing set, having the property that every trajectory of the system enters the set in finite time.

The theoretical results will be applied to a chemostat stabilization problem, where the dynamics is adequately represented by a two-state model involving the microbial biomass and the limiting organic substrate, with manipulated input the dilution rate. The growth rate of the microorganisms will be assumed follow Haldane kinetics, whereas the death rate of the microorganisms as well as the substrate consumption for cell maintenance will be accounted for assuming first-order kinetics. Moreover, the biomass balance will involve a time-varying uncertainty, accounting for the adjustment of the biomass to changes in the substrate levels. Applying the theoretical results on relaxed Lyapunov crireria, a robust globally stabilizing state feedback control law will be derived. Simulation results will also be used to illustrate the robustness properties of the derived control law.

1. Motivation: Globally stabilizing control of a chemostat

Continuous stirred microbial bioreactors, often called chemostats, cover a wide range of applications. The dynamics of the chemostat is often adequately represented by a simple dynamic model involving two state variables, the microbial biomass X and the limiting organic substrate S (see [13]). For control purposes, the manipulated input is usually the dilution rate D. A commonly used (delay-free) mathematical model for microbial growth on a limiting substrate in a chemostat is of the form:

$$\begin{aligned} \dot{X} &= \mu(S)X - DX \\ \dot{S} &= D(S_i - S) - \frac{1}{Y}\mu(S)X \\ X &\in (0,+\infty)\,, S \in (0,S_i)\,, D \geq 0 \end{aligned} \tag{1.1}$$

where S_i is the feed substrate concentration, $\mu(S)$ is the specific growth rate and $Y > 0$ is a biomass yield factor. Chemostat models with delays can be found in [4,14,15]. In most applications, Monod or Haldane or generalized Haldane models are used for $\mu(S)$.

The literature on control studies of chemostat models of the form (1.1) is extensive. In [3], feedback control of the chemostat by manipulating the dilution rate was studied for the promotion of coexistence. Other interesting control studies of the chemostat can be found in [1,6-10]. The stability and robustness of periodic solutions of the chemostat was studied in [11,12]. The problem of the stabilization of a given non-trivial steady state (X_s, S_s) of the chemostat model (1.1) was considered in [9], where it was shown that the simple feedback law $D = \mu(S)\frac{X}{X_s}$ or equivalently $D = \frac{\mu(S)X}{Y(S_i - S_s)}$ is a globally stabilizing feedback. See also the recent work [8] for the study of the robustness properties of the resulting closed-loop system under this feedback law in the presence of constant or time-varying errors in the inlet substrate concentration S_i.

In this work we consider the robust global feedback stabilization problem for the more general uncertain chemostat model

$$\begin{aligned} \dot{X} &= \left(\mu(S) + \Delta(S,t)\right)X - DX - bX \\ \dot{S} &= D(S_i - S) - \frac{1}{Y}\mu(S)X - mX \\ X &\in (0,+\infty), S \in (0,S_i), D \geq 0 \end{aligned} \tag{1.2}$$

In the above,

- The term $-bX$ in the biomass balance represents the death rate of the cells in the chemostat. The parameter $b \geq 0$ is the cell mortality rate.
- The term $-mX$ in the substrate balance accounts for the rate of substrate consumption for cell maintenance ([2]) as well as the rate of release of substrate due to the death of the cells in the chemostat (which is proportional to bX). The parameter m is either positive or assumes a small negative value. The parameter m is related to the presence of variable apparent yield coefficient, which has been studied recently in [16].
- The term $\Delta(S,t)$ represents possible deviations of the specific growth rate of the biomass, primarily accounting for the adjustment of the biomass to changes in the substrate levels. It is assumed to be of the form $\Delta(S,t) = d_1(t)|S - S_s| - d_2(t)\max\{0, S_s - S\}$ with $d_1(t) \in [0,a]$, $d_2(t) \in [0,a]$, $a \geq 0$. Notice that at design steady state conditions $S_s \in (0, S_i)$, the uncertainty $\Delta(S,t)$ is assumed to vanish. See Figure 1 below for a sketch of the shape of the uncertainty range as a function of S_s.

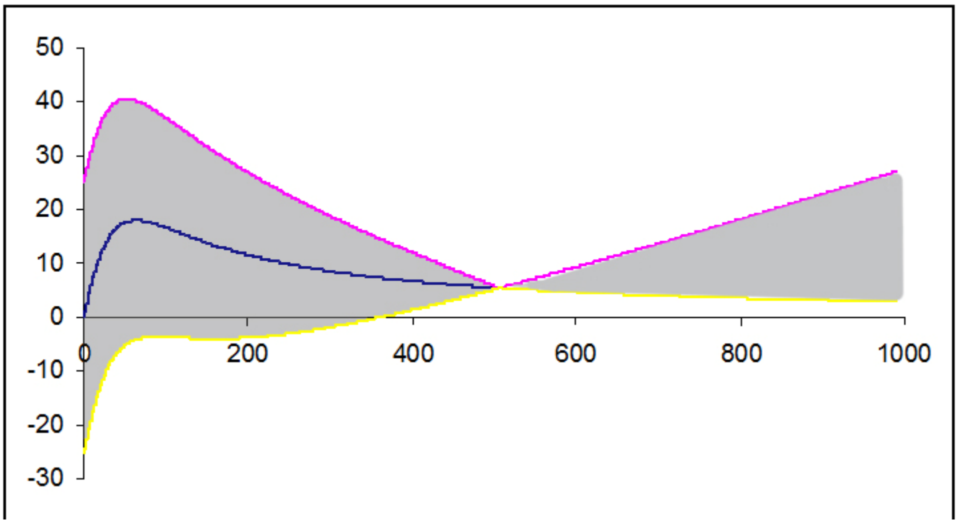

Figure 1: Indicative uncertainty range for the specific growth rate of the biomass, $\mu(S) + \Delta(S,t)$

(here $\mu(S) = \frac{75S}{100 + S + 0.025S^2}$, $a = 0.05$, $S_s = 506.72$)

Notice that model (1.2) becomes (1.1) if we set $a = b = m = 0$.

It is also important to notice that even in the case of zero uncertainty ($a=0$), zero mortality rate b and for positive values for the constant m, the application of the feedback law $D=\mu(S)\frac{X}{X_s}$ does not necessarily lead to global stability. For example, for the Haldane model $\mu(S)=\frac{\mu_{\max}S}{K_S+S+\frac{S^2}{K_I}}$, it is easy to verify that for arbitrarily small positive values for the constant m, the resulting closed-loop system under $D=\mu(S)\frac{X}{X_s}$ and $a=b=0$ has two equilibrium points in the first quadrant $(S_s^{[1]},X_s)$ and $(S_s^{[2]},X_s)$, with $0<S_s^{[1]}<S_s^{[2]}$. The equilibrium point $(S_s^{[2]},X_s)$, is locally asymptotically stable with region of attraction the set $\{(S,X):S>S_s^{[1]},X>0\}$. The stable manifold of the unstable equilibrium $(S_s^{[1]},X_s)$ is the straight line $S=S_s^{[1]}$ and if the initial condition for the substrate is less than $S_s^{[1]}$ then the system is led to shut-down in finite time (i.e., there exists $T\geq 0$ such that $\lim_{t\to T^-} S(t)=0$). Therefore, the feedback law needs to be modified in order to be able to guarantee global asymptotic stability for the desired equilibrium point.

This observation is illustrated by simulation in Figures 2 and 3. Figure 2 depicts the response of system (1.1) under the feedback law $D=\mu(S)\frac{X}{X_s}$ for the following parameter values, design steady state and initial conditions:

$$S_i=10000\ mg/l,\ \frac{1}{Y}=20\ mg/mg,\ \mu_{\max}=0.5\ d^{-1},\ K_S=100\ mg/l, K_I=4000\ mg/l$$
$$X_s=474.66\,mg/l,\ S_s=506.72\,mg/l,\ X(0)=10\,mg/l,\ S(0)=10\,mg/l$$

It is observed that the controller is very effective in bringing the system to its design steady state.

If, however, a term $-mX$ is present in the substrate balance, with $m=0.1\ d^{-1}\cdot mg/mg$ and all the other parameters remain the same, the control law $D=\mu(S)\frac{X}{X_s}$ completely fails to bring the system to its design steady state; instead, it leads the system to shut down in finite time, as shown in Figure 3.

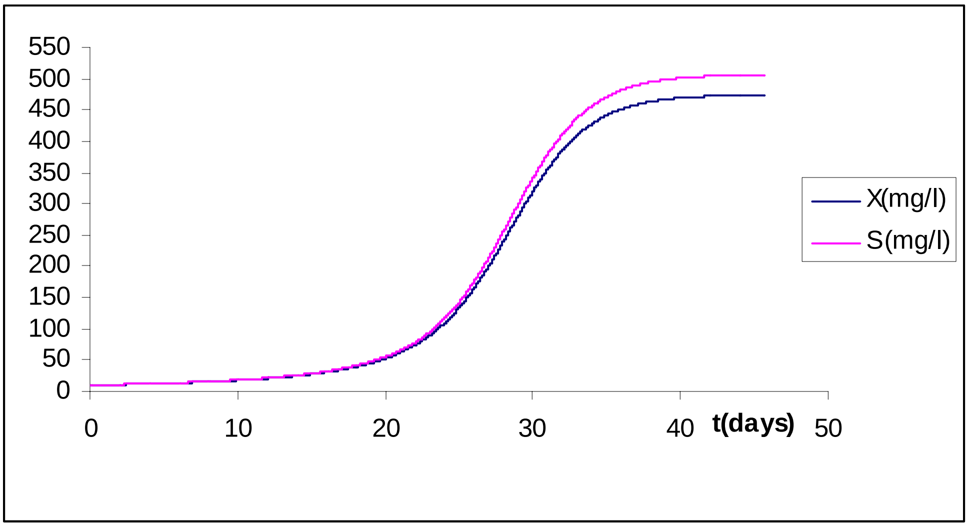

Figure 2: Evolution of states for system (1.2) with $a=b=m=0$ under the control law $D=\mu(S)\frac{X}{X_s}$

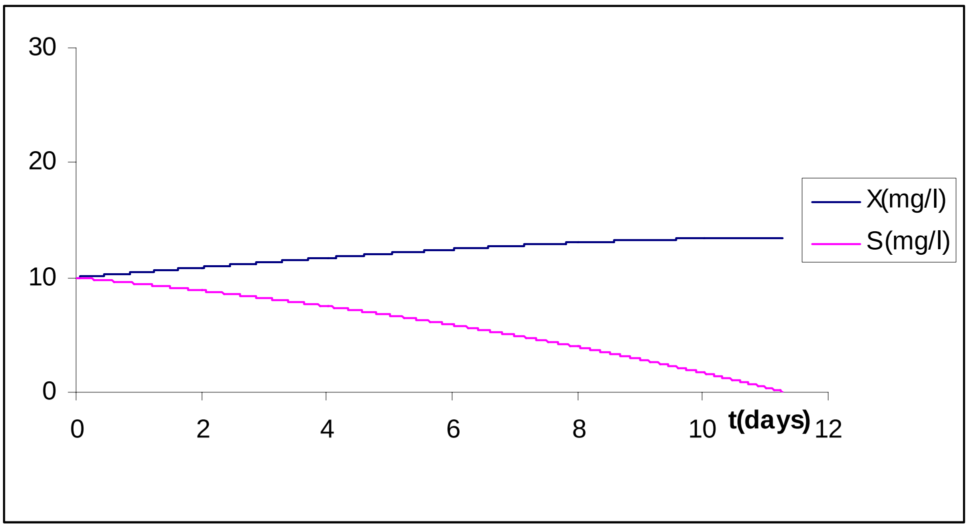

Figure 3: Evolution of states for system (1.2) with $a = b = 0$, $m = 0.1\ d^{-1} \cdot mg / mg$ under the control law $D = \mu(S)\dfrac{X}{X_s}$

2. Theory: Relaxed Lyapunov criteria for robust global stabilization of nonlinear systems

In this section, the main theoretical results of the present work are presented. We start with systems without manipulated input that are subject to disturbances *d*, and we review the notion of Uniform Robust Global Asymptotic Stability (URGAS). Consider a dynamical system of the form:

$$\begin{aligned} &\dot{x} = F(d, x) \\ &x \in \Re^n \,, d \in D \end{aligned} \tag{2.1}$$

We assume throughout this section that system (2.1) satisfies the following hypotheses:

(H1) $D \subset \Re^l$ is closed and bounded.

(H2) The mapping $D \times \Re^n \ni (d, x) \to F(d, x) \in \Re^n$ is continuous.

(H3) The vector field $F(d, x)$ is locally Lipschitz continuous with respect to $x \in \Re^n$.

Hypothesis (H3) guarantees that for every $(x_0, d) \in \Re^n \times M_D$, there exists a unique solution $x(t)$ of (2.1) with initial condition $x(0) = x_0$ corresponding to input $d \in M_D$. (Here by M_D we denote the class of all measurable and locally essentially bounded mappings $d : \Re^+ \to D$).

Definition 1 ([5])**:** *Let* $x(t; x_0, d)$ *denote the unique solution of (2.1) with initial condition* $x(0) = x_0 \in \Re^n$ *corresponding to input* $d \in M_D$. *Assume that hypotheses (H1-3) hold, with* $F(d,0) = 0$ *for all* $d \in D$.

We say that $0 \in \Re^n$ *is Uniformly Robustly Globally Asymptotically Stable (URGAS) for (2.1) if the following properties hold:*

- *for every* $s > 0$, *it holds that*
  $\sup\left\{ |x(t; x_0, d)| \,;\, t \ge 0 \,,\ |x_0| \le s \ ,\, d \in M_D \right\} < +\infty$ *(Uniform Robust Lagrange Stability)*
- *for every* $\varepsilon > 0$ *there exists a* $\delta := \delta(\varepsilon) > 0$ *such that:*
  $\sup\left\{ |x(t; x_0, d)| \,;\, t \ge 0 \,,\ |x_0| \le \delta \ ,\, d \in M_D \right\} \le \varepsilon$ *(Uniform Robust Lyapunov Stability)*
- *for every* $\varepsilon > 0$ *and* $s \ge 0$, *there exists a* $\tau := \tau(\varepsilon, s) \ge 0$, *such that:*
  $\sup\left\{ |x(t; x_0, d)| \,;\, t \ge \tau \,,\ |x_0| \le s \ ,\, d \in M_D \right\} \le \varepsilon$ *(Uniform Attractivity for bounded sets of initial states)*

Next we present relaxed Lyapunov-like sufficient conditions for URGAS. The Lyapunov-like conditions of the following theorem are "relaxed" in the sense that the Lyapunov differential inequality is not required to hold over the entire state space, but only for states that belong to an appropriate set of the state space, such that every trajectory of the system enters the set in finite time.

Theorem 1: *Consider system (2.1) under hypotheses (H1-3) with* $F(d,0)=0$ *for all* $d \in D$ *and suppose that there exists a set* $\Omega \subseteq \Re^n$ *with* $0 \in \Omega$, *functions* $V \in C^1(\Omega;\Re^+)$ *being positive definite and radially unbounded,* $T \in C^0(\Re^n;\Re^+)$, $G \in C^0(\Re^n;\Re^+)$, *which satisfy the following properties:*

(P1) *For every* $(d,x_0) \in M_D \times \Re^n$, *there exists* $\hat{t}(x_0,d) \in [0,T(x_0)]$ *such that the unique solution* $x(t;x_0,d)$ *of (2.1) satisfies* $x(t;x_0,d) \in \Omega$ *for all* $t \in [\hat{t}(x_0,d), t_{\max})$ *and* $|x(t;x_0,d)| \le G(x_0)$ *for all* $t \in [0,\hat{t}(x_0,d)]$, *where* $t_{\max} = t_{\max}(x_0,d)$ *is the maximal existence time of the solution,*

(P2) $\sup_{d \in D} \left(\nabla V(x) F(d,x) \right) < 0$ *for all* $x \in \Omega$, $x \neq 0$.

Then $0 \in \Re^n$ *is URGAS for (2.1).*

Remark 1: For disturbance-free systems, a set $\Omega \subseteq \Re^n$ that satisfies condition (P1) of Theorem 1 is called an *absorbing set*.

The following lemma provides sufficient conditions for the reachability condition (P1) of Theorem 1.

Lemma 1: *Consider system (2.1) under hypotheses (H1-3) and suppose that there exist locally Lipschitz functions* $h:\Re^n \to \Re$ *with* $h(0)<0$, $a:\Re^n \to \Re$ *being bounded from above with* $a(0)=0$, $W:\Re^n \to \Re^+$ *being radially unbounded, a continuous function* $\delta:\Re^+ \to (0,+\infty)$ *and constants* $K \ge 0$, $\varepsilon > 0$, *such that* $\{x \in \Re^n : 0 < h(x) < \varepsilon\} \neq \emptyset$ *and*

$$\sup_{d \in D} \nabla h(x) F(d,x) \le 0, \text{ for almost all } x \in \Re^n \text{ with } 0 < h(x) < b \tag{2.2a}$$

$$\sup_{d \in D} \left(\nabla h(x) - \nabla a(x) \right) F(d,x) \le -\delta(h(x)), \text{ for almost all } x \in \Re^n \text{ with } h(x) > 0 \tag{2.2b}$$

$$\sup_{d \in D} \nabla W(x) F(d,x) \le K W(x), \text{ for almost all } x \in \Re^n \text{ with } h(x) > 0 \tag{2.3}$$

Then for every $\hat{\varepsilon} \in (0,b)$ *there exist functions* $T \in C^0(\Re^n;\Re^+)$, $G \in C^0(\Re^n;\Re^+)$ *such that property (P1) of Theorem 1 holds with* $\Omega := \{x \in \Re^n : h(x) \le \hat{\varepsilon}\}$.

We next consider a nonlinear dynamic system with manipulated input *u*, of the form:

$$\begin{aligned} \dot{x} &= f(d,x) + g(d,x)u \\ x &\in \Re^n, d \in D, u \in U \end{aligned} \tag{2.4}$$

where $D \subset \Re^l$ is a closed and bounded set, $U \subseteq \Re^m$ a non-empty convex set with $0 \in U$, $f: D \times \Re^n \to \Re^n$, $g: D \times \Re^n \to \Re^{n \times m}$ are continuous mappings with $f(d,0)=0$ for all $d \in D$ and the vector field $f(d,x)+g(d,x)u$ is locally Lipschitz continuous with respect to $x \in \Re^n$. The problem is to construct a continuous state feedback law $u = k(x)$ with $k:\Re^n \to U$ and $g(d,0)k(0)=0$ for all $d \in D$, which achieves robust global stabilization of $0 \in \Re^n$ for (2.4), i.e., $0 \in \Re^n$ is uniformly robustly globally asymptotically stable for the closed-loop system $\dot{x} = f(d,x) + g(d,x)k(x)$.

In view of the foregoing analysis for systems of the form (2.1), we can develop "relaxed" Lyapunov-like conditions for the existence of a locally Lipschitz, globally stabilizing state feedback law $u = k(x)$ for system (2.4):
Instead of requesting the existence of a Robust Control Lyapunov Function (RCLF), i.e., the existence of a continuously differentiable, positive definite and radially unbounded function $V:\Re^n \to \Re^+$ with

$$\inf_{u \in U} \sup_{d \in D} \nabla V(x) \left(f(d,x) + g(d,x)u \right) < 0, \text{ for all } x \neq 0,\ x \in \Re^n$$

a "relaxed" condition

$$\inf_{u \in U} \sup_{d \in D} \nabla V(x) \left(f(d,x) + g(d,x)u \right) < 0, \text{ for all } x \neq 0,\ x \in \Omega$$

with $\Omega \subseteq \Re^n$, can be used to design a globally stabilizing state feedback. In particular, a continuous state feedback law $k:\Re^n \to U$ with $g(d,0)k(0)=0$ for all $d \in D$, which guarantees

$$\sup_{d\in D} \nabla V(x)\big(f(d,x)+g(d,x)k(x)\big)<0 \text{, for all } x \neq 0,\ x \in \Omega$$

with $\Omega \subseteq \Re^n$ such that every solution of (2.4) with $u=k(x)$ enters $\Omega \subseteq \Re^n$ in finite time, achieves robust global stabilization of $0 \in \Re^n$ for (2.4).

Using Theorem 1 as well as the sufficient conditions for the reachability condition (P1) from Lemma 1, we can show the following:

Theorem 2: *Consider system (2.4) and suppose that there exist continuously differentiable functions* $h:\Re^n \to \Re$ *with* $h(0)<0$, $W:\Re^n \to \Re^+$ *being radially unbounded,* $V:\Re^n \to \Re^+$ *being positive definite and radially unbounded, a continuous non-increasing function* $\delta:\Re^+ \to (0,+\infty)$ *and constants* $K \geq 0$, $\varepsilon > 0$ *such that* $\{x \in \Re^n : h(x) \geq \varepsilon\} \neq \emptyset$ *and the following properties hold:*

(R1) *For every* $x \in \Re^n$ *with* $h(x) \geq 0$ *there exists* $u \in U$ *with*

$$\sup_{d\in D} \nabla h(x)\big(f(d,x)+g(d,x)u\big) \leq -\delta(h(x)) \tag{2.5}$$

$$\sup_{d\in D} \nabla W(x)\big(f(d,x)+g(d,x)u\big) \leq KW(x) \tag{2.6}$$

(R2) *For every* $x \neq 0$ *with* $h(x) \leq \varepsilon$ *there exists* $u \in U$ *with*

$$\sup_{d\in D} \nabla V(x)\big(f(d,x)+g(d,x)u\big) < 0 \tag{2.7}$$

(R3) *For every* $x \in \Re^n$ *with* $h(x) \in [0,\varepsilon]$ *there exists* $u \in U$ *satisfying (2.5), (2.6) and (2.7).*

(R4) *There exists a neighbourhood* $\mathcal{N}$ *of* $0 \in \Re^n$ *and a locally Lipschitz mapping* $\tilde{k}:\mathcal{N} \to U$ *with* $\tilde{k}(0)=0$ *such that* $\sup_{d\in D} \nabla V(x)\big(f(d,x)+g(d,x)\tilde{k}(x)\big)<0$ *for all* $x \in \mathcal{N}$, $x \neq 0$.

Then there exists a locally Lipschitz mapping $k:\Re^n \to U$ *with* $k(0)=0$ *such that* $0 \in \Re^n$ *is URGAS for the closed-loop system (2.4) with* $u=k(x)$.

The theoretical results developed in this section can have the following types of applications:

i) in many cases, RCLFs are not available, while "relaxed" RCLFs can be found, leading to robust globally stabilizing controllers.while RCLFs are not available.
ii) even when a RCLF is known, the use of "relaxed" RCLFs can lead to alternative feedback designs that may have advantages over the ones from RCLF.

In the next section, the results will be applied to the problem of robust feedback stabilization of the chemostat.

3. Application to chemostat control

Consider again the general chemostat model (1.2) and assume that the specific growth rate function $\mu:\Re \to [0,\mu_{\max}]$ is a locally Lipschitz function with $\mu(S)=0$ for all $S \leq 0$ and $\mu(S)>0$ for all $S>0$. Additionally, we make the following assumptions:

(S1) *There exists an equilibrium point* $(X_s,S_s) \in (0,+\infty)\times(0,S_i)$ *with* $\mu(S_s)=D_s+b$ *and* $\dfrac{D_s Y(S_i-S_s)}{D_s+b+Ym}=X_s$ *for a certain value of the dilution rate* $D_s>0$.

Assumption is satisfied for Monod, Haldane and generalized Haldane kinetics, as long as the value of the dilution rate D_s is not too high (smaller than the washout dilution rate).

(S2) *There exists* $S^+ \in (0,S_s)$ *and* $p>0$ *such that* $\dfrac{1}{Y}\mu(S)+m \geq p$ *and* $\mu(S)-b \geq 2p$ *for all* $S \in [S^+,S_i]$.

Assumption is satisfied for Monod, Haldane and generalized Haldane kinetics, as long as $\min\left(\mu(S_s),\mu(S_i)\right) > \max\left(b,-Ym\right)$, in which case, any $S^+ > \max\left(b,-Ym\right)$ satisfies it.

The goal is the robust global stabilization of the non-trivial equilibrium point $(X_s,S_s)\in(0,+\infty)\times(0,S_i)$ with $\mu(S_s)=D_s+b$ and $\dfrac{D_sY(S_i-S_s)}{D_s+b+Ym}=X_s$ involved in hypotheses (S1-2) for system (1.2).

As a first step, we apply the change of coordinates:

$$S=\frac{S_i\exp(x_1)}{c+\exp(x_1)}\;;\;\frac{X}{S_i-S}=G\exp(x_2) \tag{3.1}$$

and the input transformation:

$$D=D_s+u \tag{3.2}$$

where $c:=\dfrac{S_i}{S_s}-1$ and $G:=\dfrac{D_sY}{D_s+b+Ym}$.

The above coordinate change maps the strip $\{(X,S)\in\Re^2: X>0,\ 0<S<S_i\}$ onto $\Re^2$.

Under the above transformations, the uncertain system (1.2) is transformed to the following:

$$\begin{aligned}
\dot{x}_1 &=\left(c\exp(-x_1)+1\right)\left(D_s+u-\left(\frac{1}{Y}\tilde{\mu}(x_1)+m\right)G\exp(x_2)\right)\\
\dot{x}_2 &=\left(\tilde{\mu}(x_1)+d_1\frac{cS_s}{c+\exp(x_1)}\left|\exp(x_1)-1\right|-d_2\frac{cS_s}{c+\exp(x_1)}\max\left(0,1-\exp(x_1)\right)-b\right)-\left(\frac{1}{Y}\tilde{\mu}(x_1)+m\right)G\exp(x_2)\\
x &=(x_1,x_2)\in\Re^2,u\in U:=[-D_s,+\infty),d=(d_1,d_2)\in[0,a]^2
\end{aligned} \tag{3.3}$$

where $\tilde{\mu}(x_1):=\mu\left(\dfrac{S_i\exp(x_1)}{c+\exp(x_1)}\right)$. Notice that $x=0$ is an equilibrium point for the above system for $u\equiv 0$. Therefore, we seek for a locally Lipschitz feedback law $k:\Re^2\to\Re$ with $k(0)=0$ so that the origin is uniformly robustly globally asymptotically stable for the closed-loop system under $u=k(x)$ in the sense described in the previous section.

Insights for the solution of the feedback stabilization problem for (3.3) may be obtained by considering the transformation of the nominal system (1.1) (i.e. for $a=b=m=0$):

$$\begin{aligned}
\dot{x}_1 &=\left(c\exp(-x_1)+1\right)\left(D_s+u-\tilde{\mu}(x_1)\exp(x_2)\right)\\
\dot{x}_2 &=\tilde{\mu}(x_1)-\tilde{\mu}(x_1)\exp(x_2)\\
x &=(x_1,x_2)\in\Re^2,u\in U:=[-D_s,+\infty)
\end{aligned} \tag{3.4}$$

For the control system (3.4), families of Control Lyapunov Functions (CLF) are known (see [8]). Let $\gamma:\Re\to\Re^+$, $\beta:\Re\to\Re^+$ be non-negative, continuously differentiable functions with $\gamma(0)=\beta(0)=0$ and such that

$$x\gamma'(x)>0,\ x\beta'(x)>0,\ \text{for all } x\neq 0 \tag{3.5a}$$

$$\text{if } x\to\pm\infty \text{ then } \gamma(x)\to+\infty \text{ and } \beta(x)\to+\infty \tag{3.5b}$$

Then the family of functions:

$$V(x)=\gamma(x_1)+\beta(x_2) \tag{3.6}$$

are radially unbounded, positive definite and continuously differentiable, and constitute CLFs for system (3.4). The knowledge of the above family of CLFs allows us to obtain a family of stabilizing feedback laws for (3.4). The reader may verify that the following family of feedback laws:

$$k(x) := -D_s + \tilde{\mu}(x_1)\exp(x_2)\varphi(x_1) + q(x_1, x_2) \tag{3.7}$$

where $\varphi : \Re \to \Re^+$ is a locally Lipschitz, non-negative function with $\varphi(0) = 1$, $\varphi(x) < 1$ for $x > 0$ and $\varphi(x) > 1$ for $x < 0$, $q : \Re^2 \to \Re^+$ is a locally Lipschitz, non-negative function with $q(x_1, x_2) = 0$ for $x_1 \geq 0$, is a family of globally stabilizing feedback laws for (3.4).

Back-transforming (3.7) to the original variables, gives the following family of globally stabilizing control laws for the nominal system (1.1):

$$D = \frac{\mu(S)X}{Y(S_i - S)}\varphi(S) + q(S, X) \tag{3.8}$$

where

a) $\varphi : \Re^+ \to \Re$ is a locally Lipschitz function with $\varphi(S) \geq 0$ for all $0 < S < S_i$,

$\varphi(S) > 1$ for $0 < S < S_s$, $\varphi(S_s) = 1$, $\varphi(S) < 1$ for $S > S_s$.

b) $q : \Re^+ \times \Re^+ \to \Re^+$ is a locally Lipschitz, non-negative function with $q(S, X) \equiv 0$ for $S > S_s$,

Notice hat the special choices $\varphi(S) = \frac{S_i - S}{S_i - S_s}$ and $q(S, X) = 0$ lead to $D = \frac{\mu(S)X}{Y(S_i - S_s)}$, which is the feedback law in [9] (see also [8] and references therein).

The method of CLF can be extended to the uncertain system (1.2) or (3.4), leading to globally stabilizing controllers. However, the results are extremely complicated and are omitted here for brevity.

The method of "relaxed" Control Lyapunov Functions proposed in the previous section can now be applied to the problem in hand and give rise to simple feedback controllers. Details are omitted, but the main point is that the Lyapunov differential inequality is imposed over the set $\Omega := \{(x_1, x_2) \in \Re^2 : x_1 \geq x_1^*\}$, where $x_1^* = \ln\left(\frac{\frac{S_i}{S_s} - 1}{\frac{S_i}{S^+} - 1}\right)$, instead of the entire $\Re^2$.

Theorem 3: *Let $\psi : \Re \to \Re^+$ be a locally Lipschitz non-increasing function with $\psi(x) = 0$ for all $x \geq 0$ and $\psi(x) > 0$ for all $x < 0$ and let $L : \Re^2 \to (0, +\infty)$ be a locally Lipschitz function with $\inf\{L(x) : x \in \Re^2\} > 0$. Under hypotheses (S1-2), for every $a \geq 0$, $0 \in \Re^2$ is URGAS for the closed-loop system (3.3) with*

$$u = -D_s + \max(0, \frac{1}{Y}\tilde{\mu}(x_1) + m)\, G\exp(x_2 - x_1) + L(x_1, x_2)\psi(x_1) \tag{3.9}$$

Transforming the above result in terms of the original variables, we obtain:

Theorem 3′: *Let $\psi : \Re^+ \to \Re^+$ be a locally Lipschitz non-increasing function with $\psi(S) = 0$ for all $S \geq S_s$ and $\psi(S) > 0$ for all for $0 < S < S_s$ and let $L : \Re^+ \times \Re^+ \to (0, +\infty)$ be a locally Lipschitz function with $\inf\left\{L(X, S) : X \in \Re^+,\ 0 < S < S_i\right\} > 0$. For every $a \geq 0$, the origin is URGAS for the closed-loop system (1.2) under the feedback law*

$$D = \frac{S_s}{S_i - S_s} \cdot \frac{X}{S} \cdot \max\left(0, \frac{1}{Y}\mu(S) + m\right) + L(S, X) \cdot \psi(S) \tag{3.10}$$

Notice that the special choices $L(x_1,x_2)\equiv L>0$ and $\psi(x_1)=\max\left(0,1-\exp(x_1)\right)$, or, in terms of the original variables, $L(S,X)\equiv L>0$ and $\psi(S)=\frac{S_i}{S_i-S}\max\left(0,1-\frac{S}{S_s}\right)$, lead to the following more concrete form of feedback control law

$$D=\frac{S_s}{S_i-S_s}\cdot\frac{X}{S}\cdot\max\left(0,\frac{1}{Y}\mu(S)+m\right)+L\frac{S_i}{S_i-S}\max\left(0,1-\frac{S}{S_s}\right) \tag{3.11}$$

Finally, it is important to point out certain possible simplifications of the derived controller:

- for $m=0$, control law (3.10) simplifies to $D=\frac{\mu(S)X}{Y(S_i-S_s)}\cdot\frac{S_s}{S}+L(S,X)\cdot\psi(S)$, which is of the form (3.8).
- for $m>0$, control law (3.10) simplifies to $D=\frac{\left(\mu(S)+Ym\right)X}{Y(S_i-S_s)}\cdot\frac{S_s}{S}+L(S,X)\cdot\psi(S)$,

  or equivalently $D=\frac{\mu(S)X}{\breve{Y}(S)\cdot(S_i-S_s)}\cdot\frac{S_s}{S}+L(S,X)\cdot\psi(S)$, where

  $\breve{Y}(S)$ is an apparent variable yield factor defined by $\frac{1}{\breve{Y}(S)}=\frac{1}{Y}+\frac{m}{\mu(S)}$.

The performance of the derived controller is illustrated through simulations in Figures 4-6. Figures 4 and 5 depict the response of system (1.2) under the feedback law (3.11) and for the following parameter values and design steady state conditions:

$$S_i=10000\ mg/l,\ \frac{1}{Y}=20\ mg/mg,\quad \mu_{\max}=0.5\ d^{-1},\ K_S=100\ mg/l, K_I=4000\ mg/l$$

$$a=0,\ b=0,\ m=0.1\ d^{-1}\cdot mg/mg,\quad S_s=506.72\ mg/l,\ X_s=468.46\ mg/l$$

In Figure 4, the system's initial conditions were $X(0)=10\,mg/l$, $S(0)=10\,mg/l$, whereas in Figure 5, $X(0)=1000\,mg/l$, $S(0)=1000\,mg/l$. Note that the initial conditions were chosen to be very far from the design steady state conditions in order to test the capabilities of the proposed controller. It is observed that, despite the adverse conditions, the controller is very effective in bringing the system to its design steady state.

In Figure 6, parameter values, design steady state and initial conditions are as in Figure 5, except that an oscillating perturbation in the biomass growth rate is assumed to be present in system (1.2). Despite this persistent perturbation, the controller is very effective in bringing the system to its design steady state, even though the time response is not as fast.

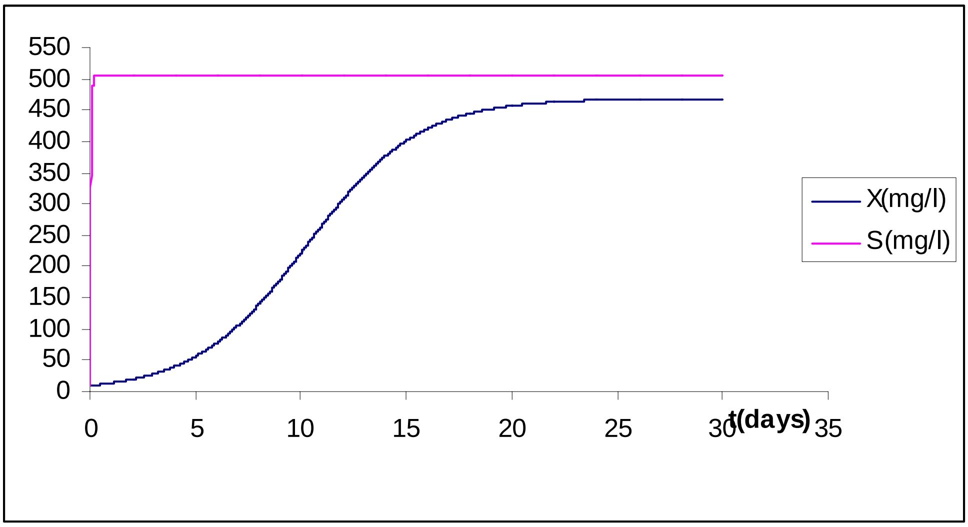

Figure 4: Evolution of states for system (1.2) with $a=b=0$, $m=0.1\ d^{-1}\cdot mg/mg$ under the control law (3.11) with $L=1$ and initial conditions $X(0)=10\,mg/l$, $S(0)=10\,mg/l$

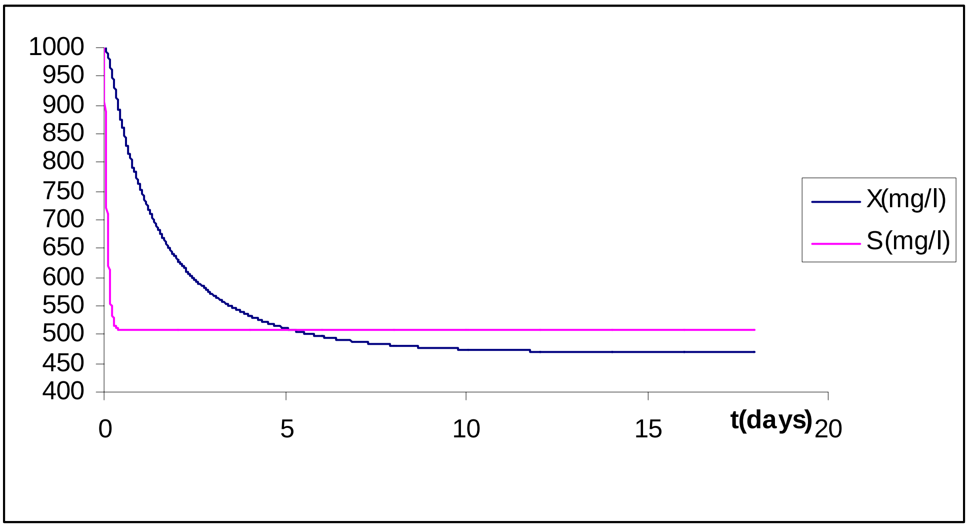

Figure 5: Evolution of states for system (1.2) with $a = b = 0$, $m = 0.1\ d^{-1} \cdot mg/mg$ under the control law (3.11) with $L = 1$ and initial conditions $X(0) = 1000\, mg/l$, $S(0) = 1000\, mg/l$

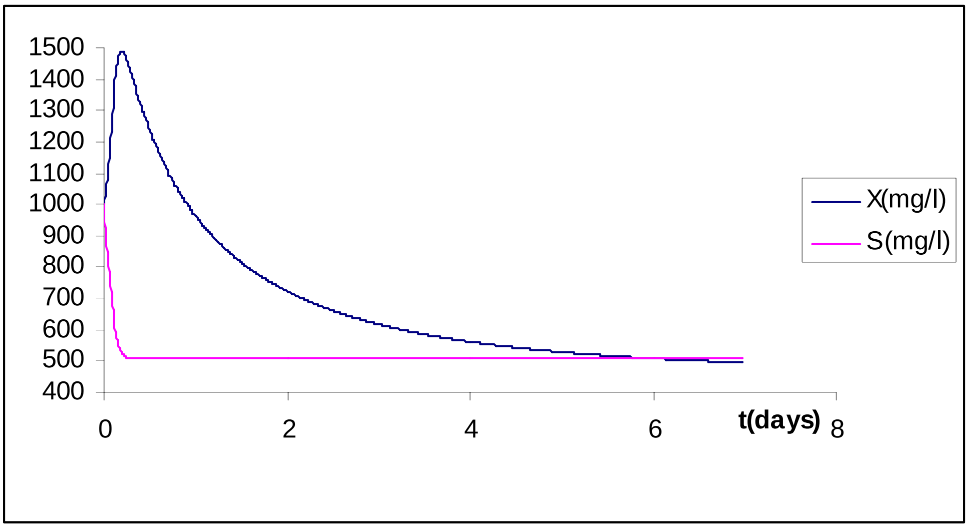

Figure 6: Evolution of states for system (1.2) with $a = 0.19$, $d_1(t) = |\sin(t)|$, $d_2(t) \equiv 0$, $b = 0$, $m = 0.1\ d^{-1} \cdot mg/mg$ under the control law (3.11) with $L = 1$ and initial conditions $X(0) = 1000\, mg/l$, $S(0) = 1000\, mg/l$

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