Modular Design of Discrete-Time Nonlinear Observers for State and Disturbance Estimation

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Abstract—**This work studies the problem of design of discrete-time nonlinear observers in the presence of exogenous disturbances. In particular, the present work proposes a systematic design method for discrete-time nonlinear state and disturbance observers in modular form, consisting of an observer for the disturbance-free part of the system, along with a disturbance observer and a state-estimate corrector. The modular observer is first defined and characterized in a general setting and then, a systematic design method is developed on the basis of exact linearization with eigenvalue assignment. Necessary and sufficient conditions for feasibility of the proposed nonlinear modular observer design are derived.**

I. Introduction

TECHNICAL limitations and/or high cost of sensors result in the non-availability of all state variables for direct on-line measurement, and this creates the need for on-line state estimation. Furthermore, the operation of a process or plant is subject to time-varying disturbances, associated with changes in key process parameters or improper operation of sensing instruments. For this reason, in addition to monitoring the state variables, there is a definite practical need for detection and estimation of disturbances.

The problem of combined state and disturbance estimation can be conceptually formulated as a state estimation problem for an extended system. In the case of linear systems, the well-known Luenberger observer offers a comprehensive solution. More specifically, in industrial applications of combined state and disturbance estimation, the Luenberger observer is designed and implemented in a modular configuration, consisting of an observer for the disturbance-free part of the system and, on top of it, a disturbance estimator and a state-estimate corrector ([3]).

The purpose of the present work is to develop a systematic discrete-time nonlinear observer design method for state and disturbance estimation, so that the resulting observer possesses the modular configuration that is sought for in practice. The discrete-time nonlinear observer design problem will be formulated within the general framework of exact observer linearization ([1],[2],[5],[8]–[11]), and in particular, following the invariant-manifold formulation, originally proposed in [5] and further developed in [10],[11].

The theoretical developments and results of the present paper leading to the discrete-time modular observer design are exactly parallel to the continuous-time modular observer of [7].

After a review of basic results on discrete-time observer linearization and their implication on the problem of simultaneous state and disturbance estimation, the modular observer will be defined and characterized in terms of appropriate invariance conditions. Necessary and sufficient conditions for exact linearization with eigenvalue assignment will be derived, leading to a step-by-step design procedure for the modular observer. It will be shown that the proposed discrete-time modular observer design is feasible whenever the corresponding simultaneous design is feasible. Moreover, it will be shown that, when this observer is applied to linear systems, the proposed modular design generates the standard modular Luenberger observer.

II. Problem Formulation

Consider a discrete-time nonlinear dynamic system

$$\begin{aligned} x(k+1) &= f(x(k), w(k)) \\ y(k) &= h(x(k), w(k)) \end{aligned} \tag{1}$$

that represents the dynamics of a process, where x is the process state vector, y is the vector of measurements and w is the vector of unmeasurable disturbances. The dynamics of the disturbances is generated by the exosystem

$$w(k+1) = s(w(k)) \tag{2}$$

The problem of *state and disturbance estimation* is the one of estimating both the process states x and the disturbances w given on-line measurements of y. If one considers the extended system consisting of (1) and (2), i.e.

$$\begin{aligned} x(k+1) &= f(x(k), w(k)) \\ w(k+1) &= s(w(k)) \\ y(k) &= h(x(k), w(k)) \end{aligned} \tag{3}$$

the problem of state and disturbance estimation becomes a standard state estimation problem for system (3), where the state $\begin{bmatrix} x \\ w \end{bmatrix}$ of (3) must be estimated from on-line measurements of y.

The effect of disturbances is often neglected by engineers, whenever their magnitude is viewed to be small relative to the other variables affecting the operation of the process. In this case, process dynamics is approximately represented as

$$\begin{aligned} x^{df}(k+1) &= f\left(x^{df}(k), 0\right) \\ y(k) &= h\left(x^{df}(k), 0\right) \end{aligned} \tag{4}$$

where x^{df} represents the "disturbance-free" process state, i.e. the process state when $w = 0$.
When the effect of disturbances can be neglected, (4) can be used as the basis for designing an observer. This is the most convenient approach from an engineering point of view, since it simplifies the observer equations and implementation. When disturbances are large enough so that they must be accounted for, engineers still want to monitor process state estimates obtained from (4) ("disturbance-free part of the system"), together with disturbance estimates and estimates of the process states that account for disturbances ("corrected state estimates"). In practice, a typical industrial implementation of a linear Luenberger observer for state and disturbance estimation has the structure of Figure 1 ([3]).

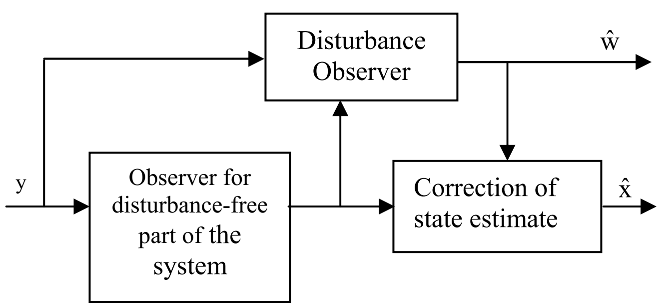


Fig. 1. Modular observer for state and disturbance estimation

The observer of Figure 1 has a modular structure: there is a basic observer for the disturbance-free part of the system and, on top of it, there is a disturbance observer. The disturbance estimate is used to correct the state estimate that was computed from the disturbance-free part of the system.

The modular configuration generates two sets of state estimates, one neglecting disturbances and another accounting for disturbances, and their comparison can be useful from a process monitoring point of view. Moreover, it gives the process engineer the flexibility of turning on or off the disturbance observer, or using alternative disturbance observers based on different assumptions on the nature of disturbances.

For continuous-time systems, the problem of simultaneous state and disturbance estimation in non-modular structure was recently studied in [6], in the context of exact observer linearization ([4]). A modular form of this continuous-time state and disturbance observer was developed in [7]. The present paper will develop a modular observer design method for discrete-time nonlinear systems, in a way that is parallel to the continuous-time modular observer in [7].

In the proposed discrete-time modular observer design method, the observer for the disturbance-free part of the system will be designed first and then the disturbance observer plus state-estimate corrector will be designed. The observer error linearization approach ([5]) will be utilized for both steps of the design, for eigenvalue assignment.

III. BACKGROUND

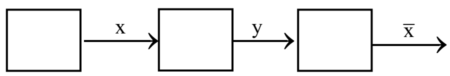
Definition 1: *Given a dynamic system*

$$\begin{aligned} x(k+1) &= f\left(x(k)\right) \\ y(k) &= h\left(x(k)\right) \end{aligned} \tag{5}$$

where $f:\mathbb{R}^n \to \mathbb{R}^n$, $h:\mathbb{R}^n \to \mathbb{R}^p$*, the system*

$$\begin{aligned} \overline{x}(k+1) &= \Gamma\left(\overline{x}(k), y(k)\right) \\ \hat{x}(k) &= \Xi\left(\overline{x}(k)\right) \end{aligned} \tag{6}$$

where $\Gamma:\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, $\Xi:\mathbb{R}^n \to \mathbb{R}^n$*, is called an* <u>*observer*</u> *for (5) if in the series connection*



the overall dynamics

$$\begin{aligned} x(k+1) &= f\left(x(k)\right) \\ \overline{x}(k) &= \Gamma\left(\overline{x}(k), h\left(x(k)\right)\right) \end{aligned} \tag{7}$$

has the property that $x = \Xi\left(\overline{x}\right)$ *is an invariant manifold. In particular, when* Ξ *is the identity map, (6) is called an* <u>*identity observer*</u>.

In the above definition, the requirement that $x = \Xi\left(\overline{x}\right)$ is an invariant manifold of (7), i.e. that $x(0) = \Xi\left(\overline{x}(0)\right) \Rightarrow x(k) = \Xi\left(\overline{x}(k)\right) \quad \forall k > 0$, translates to the following condition:

$$\Xi\left(\Gamma\left(\overline{x}, h\left(\Xi\left(\overline{x}\right)\right)\right)\right) = f\left(\Xi\left(\overline{x}\right)\right) \tag{8}$$

In the special case of identity observer, the above condition collapses to

$$\Gamma\left(\overline{x}, h\left(\overline{x}\right)\right) = f\left(\overline{x}\right) \tag{9}$$

In the observer linearization design method, the observer is chosen so that it can be transformed to a linear system

$$z(k+1) = Az(k) + By(k) \tag{10}$$

with A, B being $n \times n$ and $n \times p$ matrices respectively and A having pre-assigned eigenvalues, under an appropriate coordinate transformation. Considering the case of identity observer for reasons of simplicity, and denoting by

$$z = \theta\left(\overline{x}\right) \tag{11}$$

a coordinate transformation that maps

$$\overline{x}(k+1) = \Gamma\left(\overline{x}(k), y(k)\right) \tag{12}$$

to (10), one immediately concludes that the transformation of (12) to (10) will be feasible if and only if

$$\theta\left(\Gamma\left(\overline{x}, y\right)\right) = A\theta\left(\overline{x}\right) + By \tag{13}$$

for every $\overline{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

Applying condition (13) for $y = h\left(\overline{x}\right)$ and using (9), it follows that $\theta\left(\overline{x}\right)$ must satisfy

$$\theta\left(f\left(\overline{x}\right)\right) = A\theta\left(\overline{x}\right) + Bh\left(\overline{x}\right) \tag{14}$$

Combining (13) with (14), it follows that

$$\theta\left(\Gamma\left(\overline{x}, y\right)\right) = \theta\left(f\left(\overline{x}\right)\right) + B\left(y - h\left(\overline{x}\right)\right) \tag{15}$$

from which

$$\Gamma\left(\overline{x}, y\right) = \theta^{-1}\left(\theta\left(f\left(\overline{x}\right)\right) + B\left(y - h\left(\overline{x}\right)\right)\right) \tag{16}$$

The following Proposition summarizes the result of the above derivation:

<u>Proposition 1</u>: *Given* $f:\mathbb{R}^n \to \mathbb{R}^n$, $h:\mathbb{R}^n \to \mathbb{R}^p$*, it is possible to find* $\Gamma:\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ *that satisfies condition (9) and a coordinate transformation that maps (12) to (10) if and only if there exists an invertible function* $\theta:\mathbb{R}^n \to \mathbb{R}^n$ *that satisfies the linear functional equation (LFE)*

$$\theta\left(f(x)\right) = A\theta(x) + Bh(x) \tag{17}$$

Assuming that such a function θ *exists, the choice*

$$\Gamma\left(x, y\right) = \theta^{-1}\left(\theta\left(f(x)\right) + B\left(y - h(x)\right)\right) \tag{18}$$

satisfies condition (9) and makes (12) transformable to (10) via the coordinate transformation $z = \theta(\bar{x})$.

The result of Proposition 1 reduces the problem of observer linearization to the one of solving the LFE (16). Alternative local solvability conditions for (17) are available in the literature ([5], [10]), depending on the smoothness assumptions on $f(x)$ and $h(x)$ and also, depending on the nature of the spectrum of $\frac{\partial f}{\partial x}(x)$ evaluated at the equilibrium point. The following Proposition provides local solvability conditions for the real-analytic Poincaré-domain case.

Proposition 2 [5]: *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$, $h: \mathbb{R}^n \to \mathbb{R}^p$ *be real analytic functions with* $f(0) = 0$, $h(0) = 0$ *and denote* $F = \frac{\partial f}{\partial x}(0)$, $H = \frac{\partial h}{\partial x}(0)$. *Also, denote by* $\sigma(F)$ *the set of eigenvalues of* F.
Suppose:

1. *There exists an invertible matrix* T *such that* $TF = AT + BH$
2. *All the eigenvalues of* A *are non-resonant with* $\sigma(F)$*, i.e. no eigenvalue* λ_j *of* A *is of the form* $\lambda_j = \prod_{i=1}^{n} \kappa_i^{m_i}$ *with* $\kappa_i \in \sigma(F)$ *and* m_i *nonnegative integers, not all zero.*
3. $\sigma(F)$ *is either a subset of the unit disc or disjoint with the unit disc.*

Then there exists a unique analytic solution of the LFE (17) locally around $x = 0$. *The solution has the property that* $\frac{\partial \theta}{\partial x}(0) = T$ *and so,* θ *is a local diffeomorphism.*

Note that assumptions 1 and 2 of the above Proposition imply that (H, F) is an observable pair. On the other hand, if (H, F) is an observable pair, it is always possible to find matrices A, B, T which satisfy the matrix equation of assumption 1, with T invertible and A having prescribed eigenvalues.

The results of Propositions 1 and 2 can now be applied to the problem of state and disturbance estimation, considering the extended system (3), where $x \in \mathbb{R}^m$ is the vector process states, $w \in \mathbb{R}^\ell$ is the vector of disturbances and $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{m+\ell}$ is the state of the entire system (3) that must be estimated. The corresponding observer LFE (17) for system (3) is

$$\theta(f(x,w), s(w)) = A\theta(x,w) + Bh(x,w) \tag{19}$$

and, assuming that it admits an invertible solution, the observer has the form

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{w}(k+1) \end{bmatrix} = \theta^{-1}\left(\theta(f(\hat{x}(k), \hat{w}(k)), s(\hat{w}(k))) + B(y(k) - h(\hat{x}(k), \hat{w}(k)))\right) \tag{20}$$

or equivalently

$$\theta(\hat{x}(k+1), \hat{w}(k+1)) = \theta(f(\hat{x}(k), \hat{w}(k)), s(\hat{w}(k))) + B(y(k) - h(\hat{x}(k), \hat{w}(k))) \tag{21}$$

Note that the computational effort in the observer design can be somewhat reduced if A is taken to be block-diagonal, with diagonal blocks of sizes $m \times m$ and $\ell \times \ell$. In particular, setting $A = \begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix}$, $B = \begin{bmatrix} B_0 \\ B' \end{bmatrix}$, where A_0, A', B_0, B' are $m \times m$, $\ell \times \ell$, $m \times p$, $\ell \times p$ matrices respectively and partitioning $\theta(x,w)$ accordingly, $\theta(x,w) = \begin{bmatrix} \pi(x,w) \\ \eta(x,w) \end{bmatrix}$, with $\pi: \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$, $\eta: \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^\ell$, the observer LFE (19) breaks up into two uncoupled LFE's :

$$\pi(f(x,w), s(w)) = A_0 \pi(x,w) + B_0 h(x,w) \tag{22}$$

$$\eta(f(x,w), s(w)) = A' \eta(x,w) + B' h(x,w) \tag{23}$$

As an immediate consequence of Proposition 2, we have

Proposition 3: *Let* $f: \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$, $s: \mathbb{R}^\ell \to \mathbb{R}^\ell$ *and* $h: \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^p$ *be real analytic functions with*

$f(0,0)=0,\ s(0)=0,\ h(0,0)=0$ *and denote* $F=\frac{\partial f}{\partial x}(0,0),\ P=\frac{\partial f}{\partial w}(0,0),\ S=\frac{\partial s}{\partial w}(0),\ H=\frac{\partial h}{\partial x}(0,0),\ Q=\frac{\partial h}{\partial w}(0,0)$.

Also, denote by $\sigma(F)$ *and* $\sigma(S)$ *the sets of eigenvalues of* F *and* S *respectively.*

Suppose:

1. *There exists an invertible matrix* T *such that* $T\begin{bmatrix} F & P \\ S & 0 \end{bmatrix}=\begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix}T+\begin{bmatrix} B \\ B' \end{bmatrix}\begin{bmatrix} H & Q \end{bmatrix}$
2. *All the eigenvalues of* A_0 *and* A' *are non-resonant with* $\sigma(F)\cup\sigma(S)$.
3. $\sigma(F)\cup\sigma(S)$ *is either a subset of the unit disc or disjoint with the unit disc.*

Then the system of LFE's (22) and (23) admits a unique solution locally around $(x,w)=(0,0)$. *The solution has the property that* $\begin{bmatrix} \frac{\partial \pi}{\partial x}(0,0) & \frac{\partial \pi}{\partial w}(0,0) \\ \frac{\partial \eta}{\partial x}(0,0) & \frac{\partial \eta}{\partial w}(0,0) \end{bmatrix}=T$

IV. Modular Observer

In the present section, a modular design of an observer for system (3) will be developed, as an alternative to the "simultaneous" design outlined in the previous section.

The proposed modular observer will consist of two distinct parts. An identity observer for the disturbance-free part of the system (system (4)), followed by a disturbance observer together with a corrector for the state estimate. The identity observer for the disturbance-free part of the system will have the form

$$\tilde{x}(k+1)=\Gamma_0(\tilde{x}(k),y(k)) \tag{24}$$

whereas the disturbance observer and corrector will have the form

$$\hat{w}(k+1)=\Gamma_w(\tilde{x}(k),\hat{w}(k),y(k)) \tag{25}$$

$$\hat{x}(k+1)=\Upsilon(\tilde{x}(k),\hat{w}(k)) \tag{26}$$

Figure 2 depicts the structure of the proposed modular observer:

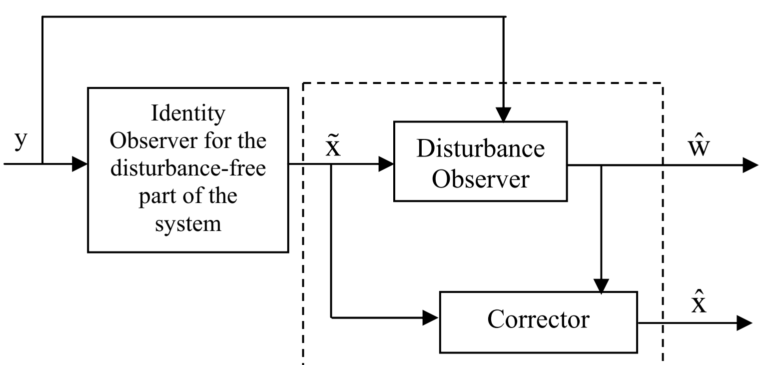


Figure 2: Structure of the modular observer (24) – (26).

The following Proposition is an immediate consequence of Definition 1, applied to the modular structure considered here.

Proposition 4: a) *System (24) will be an identity observer for (4) if*

$$\Gamma_0(\tilde{x},h(\tilde{x},0)) = f(\tilde{x},0) \tag{27}$$

b) *The overall system (24) – (26) will be an observer for (3) if the following conditions are met*

$$\Upsilon\big(\Gamma_0(\tilde{x},h(\Upsilon(\tilde{x},\hat{w}),\hat{w})),s(\hat{w})\big)=f(\Upsilon(\tilde{x},\hat{w}),\hat{w}) \tag{28}$$

and

$$\Gamma_w\left(\tilde{x},\hat{w},h\left(\Upsilon(\tilde{x},\hat{w}),\hat{w}\right)\right)= s(\hat{w}) \tag{29}$$

We can now proceed with the design of the modular observer via linearization and eigenvalue assignment. The rationale of the previous section will be followed, while adhering to the modular structure of (24)–(26).

Proposition 5: *It is possible to find* $\Gamma_0:\mathbb{R}^m\times\mathbb{R}^p\to\mathbb{R}^m$ *that satisfies (27) and a coordinate transformation that maps (24) to*

$$z_0(k+1)= A_0z_0(k) + B_0y(k) \tag{30}$$

where A_0 *and* B_0 *are* $m\times m$ *and* $m\times p$ *matrices respectively, if and only if there exists an invertible function* $\theta_0:\mathbb{R}^m\to\mathbb{R}^m$ *that satisfies the LFE*

$$\theta_0\left(f(x,0)\right)= A_0\theta_0(x) + B_0h(x,0) \tag{31}$$

Assuming that such a function θ_0 *exists, the choice*

$$\Gamma_0(x,y)=\theta_0^{-1}\left(\theta_0(f(x,0))+B_0(y-h(x,0))\right) \tag{32}$$

satisfies condition (27) and makes (24) transformable to (30) via the coordinate transformation $z_0=\theta_0(\tilde{x})$.

Proof: In order for $z_0 = \theta_0(\tilde{x})$ to map (24) to (30), the condition

$$\theta_0\left(\Gamma_0(\tilde{x},y)\right) = A_0\theta_0(\tilde{x}) + B_0y \tag{33}$$

is necessary and sufficient.

Applying (33) for $y = h(\tilde{x},0)$ and taking into account (27), it follows that $\theta_0(x)$ must satisfy (31).

Conversely, suppose that θ_0 satisfies (31). Then, the function $\Gamma_0(x,y)$ given by (32) automatically satisfies (27) and, moreover,

$$\theta_0\left(\Gamma_0(x,y)\right)=\theta_0(f(x,0))+B_0(y-h(x,0))$$

which, combined with (31), gives (33).

Proposition 6: *Assume that* Γ_0 *is given by (32), with* θ_0 *being an invertible function that satisfies (31). It is possible to find* $\Gamma_w:\mathbb{R}^m\times\mathbb{R}^\ell\times\mathbb{R}^p\to\mathbb{R}^m$ *that satisfies (29) and a coordinate transformation that maps the dynamics of (24) and (25) to*

$$\begin{bmatrix} z_0(k+1) \\ z'(k+1) \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix}\begin{bmatrix} z_0(k) \\ z'(k) \end{bmatrix} + \begin{bmatrix} B_0 \\ B' \end{bmatrix} y(k) \tag{34}$$

if and only if there exists a function $\Phi:\mathbb{R}^m\times\mathbb{R}^\ell\to\mathbb{R}^\ell$ *such that the equation* $z'=\Phi(x,w)$ *is uniquely solvable with respect to* w *and*

$$\Phi\left(\Gamma_0\left(x,h\left(\Upsilon(x,w),w\right)\right),s(w)\right)= A'\Phi(x,w) + B'h(\Upsilon(x,w),w) \tag{35}$$

Assuming that such a function Φ exists, the implicit function $\omega=\Gamma_w(x,w,y)$ *defined as the solution of the algebraic equation*

$$\Phi\left(\Gamma_0(x,y),\omega\right)-\Phi\left(\Gamma_0\left(x,h\left(\Upsilon(x,w),w\right)\right),s(w)\right)= B'\left(y-h\left(\Upsilon(x,w),w\right)\right) \tag{36}$$

satisfies condition (29) and makes (24)–(25) transformable to (34) via the coordinate transformation $\begin{bmatrix} z_0 \\ z' \end{bmatrix} = \begin{bmatrix} \theta_0(\tilde{x}) \\ \Phi(\tilde{x},\hat{w}) \end{bmatrix}$.

Proof: Since θ_0 invertible, the transformation $\begin{bmatrix} z_0 \\ z' \end{bmatrix} = \begin{bmatrix} \theta_0(\tilde{x}) \\ \Phi(\tilde{x},\hat{w}) \end{bmatrix}$ will be invertible if and only if the equation $z' = \Phi(\tilde{x},\hat{w})$ is uniquely solvable with respect to $\hat{w}$.

Since Γ_0 is given by (32) with θ_0 satisfying (31), $z_0(k+1)= A_0z_0(k) + B_0y(k)$ holds, as a result of Proposition 5.

Therefore, the question is the transformation of (25) to $z'(k+1)=A'z'(k)+B'y(k)$. In order for this to happen, it is necessary and sufficient to have

$$\Phi\left(\Gamma_0(x,y),\Gamma_w(x,w,y)\right)= A'\Phi(x,w) + B'y \tag{37}$$

Applying (37) for $y = h(\Upsilon(x,w),w)$ and taking into account (29), it follows that Φ must satisfy LFE (35).

Conversely, suppose that Φ satisfies (35) and that the equation $z' = \Phi(x,w)$ is uniquely solvable with respect to w. Then, the function $\Gamma_w(x,w,y)$ defined as the solution of (36) automatically satisfies (29) and, moreover, $\Phi(\Gamma_0(x,y),\Gamma_w(x,w,y)) = \Phi(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)) + B'(y-h(\Upsilon(x,w),w))$
which, combined with (35), gives (37).

Combining the results of Propositions 4, 5 and 6, leads the following design procedure for the modular observer:

1. Design of the identity observer for the disturbance-free part of the system:

$$\Gamma_0(x,y) = \theta_0^{-1}(\theta_0(f(x,0)) + B_0(y-h(x,0))) \quad (32)$$

where $\theta_0(x)$ *satisfies the LFE*

$$\theta_0(f(x,0)) = A_0\theta_0(x) + B_0 h(x,0) \quad (31)$$

2. Design of the corrector:

$\Upsilon(x,w)$ *satisfies the functional equation (FE)*

$$\Upsilon(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)) = f(\Upsilon(x,w),w) \quad (28)$$

3. Design of the disturbance observer:

$\omega = \Gamma_w(x,w,y)$ *is the implicit function defined as the solution of the algebraic equation*

$$\Phi(\Gamma_0(x,y),\omega) - \Phi(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)) = B'(y-h(\Upsilon(x,w),w)) \quad (36)$$

where $\Phi(x,w)$ *satisfies the LFE*

$$\Phi(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)) = A'\Phi(x,w) + B'h(\Upsilon(x,w),w) \quad (35)$$

The conclusion is that the modular design of the observer reduces to solvability of three FE's, (31) for θ_0, (28) for Υ and (35) for Φ.

Local solvability conditions for the LFE (31) immediately arise from Proposition 2. However, FE's (28) and (35) need further attention.

In the next section, it will be shown that if the LFE's (22) and (23) for simultaneous design are locally solvable, then FE's (28) and (35) for modular design are also solvable.

V. Modular Versus Simultaneous Design

The following proposition provides the solution of the modular observer design problem if the solution of the simultaneous design problem is available.

Proposition 7: *Suppose that the LFE's (22) and (23) are locally solvable in a neighborhood of* $(x = 0, w = 0)$ *and that the solution is differentiable, with* $\left[\frac{\partial \pi}{\partial x}(x,0)\right]$ *and* $\begin{bmatrix} \frac{\partial \pi}{\partial x}(x,w) & \frac{\partial \pi}{\partial w}(x,w) \\ \frac{\partial \eta}{\partial x}(x,w) & \frac{\partial \eta}{\partial w}(x,w) \end{bmatrix}$ *invertible matrices, in a neighborhood of* $(x = 0, w = 0)$. *Then*

i. The function $\theta_0(x) = \pi(x,0)$ *is an invertible function in a neighborhood of* $x = 0$ *and satisfies the LFE (31).*

ii. The algebraic equation $\pi(\hat{x},w) = \pi(x,0)$ *is locally solvable with respect to* $\hat{x}$ *and its solution* $\hat{x} = \Upsilon(x,w)$ *satisfies the FE (28).*

iii. The function $\Phi(x,w) = \eta(\Upsilon(x,w),w)$ *satisfies the LFE (35). Also, the algebraic equation* $z' = \Phi(x,w)$ *is locally solvable with respect to* w.

Proof: i) Applying (22) for $w=0$ leads to $\pi(f(x,0),0)=A_0\pi(x,0)+B_0h(x,0)$, i.e. the function $\theta_0(x)=\pi(x,0)$ satisfies (31). Moreover, the function $\theta_0(x)=\pi(x,0)$ is locally invertible, since $\frac{\partial\pi}{\partial x}(x,0)$ is an invertible matrix in a neighborhood of the origin, as a consequence of the inverse function theorem.

ii) The equation $\pi(\hat{x},w)=\pi(x,0)$ is locally solvable with respect to $\hat{x}$ since $\frac{\partial\pi}{\partial x}(0,0)$ is an invertible matrix, as a consequence of the implicit function theorem. The implicit function $\hat{x}=\Upsilon(x,w)$ will satisfy $\pi(\Upsilon(x,w),w)=\pi(x,0)$, i.e.

$$\pi(\Upsilon(x,w),w)=\theta_0(x) \tag{38}$$

Notice that the function $\Gamma_0(x,y)$ defined by (32) satisfies $\theta_0(\Gamma_0(x,y))=\theta_0(f(x,0))+B_0(y-h(x,0))$, which, substituting $y=h(\Upsilon(x,w),w)$, gives

$$\begin{aligned}\theta_0(\Gamma_0(x,h(\Upsilon(x,w),w)))&=\theta_0(f(x,0))+B_0(h(\Upsilon(x,w),w)-h(x,0))\\ &=A_0\theta_0(x)+B_0h(\Upsilon(x,w),w) && \text{[because of (31)]}\\ &=A_0\pi(\Upsilon(x,w),w)+B_0h(\Upsilon(x,w),w) && \text{[because of (38)]}\\ &=\pi(f(\Upsilon(x,w),w),s(w)) && \text{[because of (22)]}\end{aligned}$$

Hence,

$$\theta_0(\Gamma_0(x,h(\Upsilon(x,w),w)))=\pi(f(\Upsilon(x,w),w),s(w)) \tag{39}$$

On the other hand, equation (38), under the substitutions $x\to\Gamma_0(x,h(\Upsilon(x,w),w))$ and $w\to s(w)$, gives

$$\pi(\Upsilon(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)),s(w))=\theta_0(\Gamma_0(x,h(\Upsilon(x,w),w)))$$

which, combined with (39), leads to

$$\pi(\Upsilon(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)),s(w))=\pi(f(\Upsilon(x,w),w),s(w))$$

Taking into account the unique local solvability of the algebraic equation $\pi(x,w)=\zeta$ with respect to x, it follows that $\Upsilon(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w))=f(\Upsilon(x,w),w)$, which is exactly equation (28).

iii) The composite function $\Phi(x,w)=\eta(\Upsilon(x,w),w)$ satisfies

$$\begin{aligned}\Phi(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w))&=\eta(\Upsilon(\Gamma_0(x,h(\Upsilon(x,w),w)),s(w)),s(w))\\ &=\eta(f(\Upsilon(x,w),w),s(w)) && \text{[because of (28)]}\\ &=A'\eta(\Upsilon(x,w),w)+B'h(\Upsilon(x,w),w) && \text{[because of (23)]}\\ &=A'\Phi(x,w)+B'h(\Upsilon(x,w),w)\end{aligned}$$

This proves (35). Moreover, since

$$\begin{aligned}\frac{\partial\Phi}{\partial w}(x,w)&=\frac{\partial\eta}{\partial x}(\Upsilon(x,w),w)\frac{\partial\Upsilon}{\partial w}(x,w)+\frac{\partial\eta}{\partial w}(\Upsilon(x,w),w)\\ &=-\left[\frac{\partial\eta}{\partial x}(\Upsilon(x,w),w)\right]\left[\frac{\partial\pi}{\partial x}(\Upsilon(x,w),w)\right]^{-1}\left[\frac{\partial\pi}{\partial w}(\Upsilon(x,w),w)\right]+\left[\frac{\partial\eta}{\partial w}(\Upsilon(x,w),w)\right]\end{aligned}$$

it follows that $\det\frac{\partial\Phi}{\partial w}(x,w)=\dfrac{\det\begin{bmatrix}\frac{\partial\pi}{\partial x}(\Upsilon(x,w),w) & \frac{\partial\pi}{\partial w}(\Upsilon(x,w),w)\\ \frac{\partial\eta}{\partial x}(\Upsilon(x,w),w) & \frac{\partial\eta}{\partial w}(\Upsilon(x,w),w)\end{bmatrix}}{\det\left[\frac{\partial\pi}{\partial x}(\Upsilon(x,w),w)\right]}$

Given the assumptions made, $\det\frac{\partial\Phi}{\partial w}(x,w)\neq 0$ in a neighborhood of $(x=0,w=0)$ and therefore the algebraic equation $z'=\Phi(x,w)$ is locally solvable with respect to w, as a result of the implicit function theorem.

VI. EXAMPLE

In the present section, the proposed observer design methodology will be illustrated through a simple example. Consider

$$x(k+1) = \frac{x(k) - w(k)}{2 + x(k) + \left(2x(k) + 3\right) w(k)}$$
$$w(k+1) = w(k) \tag{40}$$
$$y(k) = \frac{2x(k)}{1+x(k)} + \frac{w(k)}{1+w(k)}$$

where x is the process state and w is a constant (step) disturbance, the objective being to design a nonlinear observer with linearizable dynamics with prescribed eigenvalues α_0 and α' ($|\alpha_0| < 1$, $|\alpha'| < 1$, $\alpha_0 \neq \alpha'$).

Applying the modular observer design procedure of Section IV, the following equations must be solved in a sequential manner:

$$\theta_0\left(\frac{x}{2+x}\right) = \alpha_0\theta_0(x) + b_0\frac{2x}{1+x} \quad ,$$
$$\theta_0\left(\Gamma_0(x,y)\right) = \theta_0\left(\frac{x}{2+x}\right) + b_0\left(y - \frac{2x}{1+x}\right),$$
$$\Upsilon\left(\Gamma_0\left(x\,,\frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)} + \frac{w}{1+w}\right), w\right) = \frac{\Upsilon(x,w) - w}{2 + \Upsilon(x,w) + \left(2\Upsilon(x,w) + 3\right) w} \;,$$
$$\Phi\left(\Gamma_0\left(x\,,\frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)} + \frac{w}{1+w}\right), w\right) = \alpha'\Phi(x,w) + b'\left(\frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)} + \frac{w}{1+w}\right),$$
$$\Phi\left(\Gamma_0(x,y), \Gamma_w(x,w,y)\right) = \Phi\left(\Gamma_0\left(x\,,\frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)} + \frac{w}{1+w}\right), w\right) + b'\left(y - \frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)} - \frac{w}{1+w}\right).$$

It turns out that the results can be obtained in closed form:

$$\theta_0(x) = \frac{2b_0}{\frac{1}{2} - \alpha_0}\cdot\frac{x}{1+x}, \quad \Gamma_0(x,y) = \frac{x + \left(\frac{1}{2} - \alpha_0\right)(1+x)\left(y - \frac{2x}{1+x}\right)}{2 + x - \left(\frac{1}{2} - \alpha_0\right)(1+x)\left(y - \frac{2x}{1+x}\right)}, \quad \Upsilon(x,w) = \frac{x - \frac{1}{2}\cdot\frac{\frac{3}{2} - \alpha_0}{1-\alpha_0}\cdot(1+x)\left(\frac{w}{1+w}\right)}{1 + \frac{1}{2}\cdot\frac{\frac{3}{2} - \alpha_0}{1-\alpha_0}\cdot(1+x)\left(\frac{w}{1+w}\right)},$$

$$\Phi(x,w) = \frac{b'}{\frac{1}{2} - \alpha'}\left(\frac{2x}{1+x} + \frac{1}{2}\left(\frac{1}{1-\alpha'} - \frac{1}{1-\alpha_0}\right)\frac{w}{1+w}\right), \quad \Gamma_w(x,w,y) = \frac{\frac{w}{1+w} - 2(1-\alpha')(1-\alpha_0)\left(y - \frac{2x}{1+x} + \frac{1}{2}\cdot\frac{1}{1-\alpha_0}\cdot\frac{w}{1+w}\right)}{\frac{1}{1+w} + 2(1-\alpha')(1-\alpha_0)\left(y - \frac{2x}{1+x} + \frac{1}{2}\cdot\frac{1}{1-\alpha_0}\cdot\frac{w}{1+w}\right)}$$

The resulting modular observer is

$$\tilde{x}(k+1)=\frac{\tilde{x}(k)+\left(\frac{1}{2}-\alpha_0\right)(1+\tilde{x}(k))\left(y(k)-\frac{2\tilde{x}(k)}{1+\tilde{x}(k)}\right)}{2+\tilde{x}(k)-\left(\frac{1}{2}-\alpha_0\right)(1+\tilde{x}(k))\left(y(k)-\frac{2\tilde{x}(k)}{1+\tilde{x}(k)}\right)}$$

$$\hat{w}(k+1)=\frac{\frac{\hat{w}(k)}{1+\hat{w}(k)}-2(1-\alpha')(1-\alpha_0)\left(y(k)-\frac{2\tilde{x}(k)}{1+\tilde{x}(k)}+\frac{1}{2}\cdot\frac{1}{1-\alpha_0}\cdot\frac{\hat{w}(k)}{1+\hat{w}(k)}\right)}{\frac{1}{1+\hat{w}(k)}+2(1-\alpha')(1-\alpha_0)\left(y(k)-\frac{2\tilde{x}(k)}{1+\tilde{x}(k)}+\frac{1}{2}\cdot\frac{1}{1-\alpha_0}\cdot\frac{\hat{w}(k)}{1+\hat{w}(k)}\right)} \tag{41}$$

$$\hat{x}(k)=\frac{\tilde{x}(k)-\frac{1}{2}\cdot\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot(1+\tilde{x}(k))\left(\frac{\hat{w}(k)}{1+\hat{w}(k)}\right)}{1+\frac{1}{2}\cdot\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot(1+\tilde{x}(k))\left(\frac{\hat{w}(k)}{1+\hat{w}(k)}\right)}$$

The first state equation of the modular observer (41) is an identity observer based on the assumption of zero disturbance, the second equation is the disturbance observer, and the third equation is the corrector of the state estimates.
Alternatively, the modular observer can be designed by applying the result of Proposition 7. One must then solve the LFE's (22) and (23) which, for system (40), become

$$\pi\left(\frac{x-w}{2+x+(2x+3)w}\ ,\ w\right)=\alpha_0\pi(x,w)+b_0\left(\frac{2x}{1+x}+\frac{w}{1+w}\right) \tag{42}$$

$$\eta\left(\frac{x-w}{2+x+(2x+3)w}\ ,\ w\right)=\alpha'\eta(x,w)+b'\left(\frac{2x}{1+x}+\frac{w}{1+w}\right) \tag{43}$$

Again, the solutions of the LFE's can be obtained in closed form:

$$\pi(x,w)=\frac{b_0}{\frac{1}{2}-\alpha_0}\left(\frac{2x}{1+x}+\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot\frac{w}{1+w}\right)$$
$$\eta(x,w)=\frac{b'}{\frac{1}{2}-\alpha'}\left(\frac{2x}{1+x}+\frac{\frac{3}{2}-\alpha'}{1-\alpha'}\cdot\frac{w}{1+w}\right) \tag{44}$$

from which, applying Proposition 7,

$$\theta_0(x)=\pi(x,0)=\frac{2b_0}{\frac{1}{2}-\alpha_0}\cdot\frac{x}{1+x},$$

$$\pi(\hat{x},w)=\pi(x,0)\ \Rightarrow\ \frac{b_0}{\frac{1}{2}-\alpha_0}\left(\frac{2\hat{x}}{1+\hat{x}}+\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot\frac{w}{1+w}\right)=\frac{2b_0}{\frac{1}{2}-\alpha_0}\cdot\frac{x}{1+x}\ \Rightarrow\ \hat{x}=\Upsilon(x,w)=\frac{x-\frac{1}{2}\cdot\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot(1+x)\left(\frac{w}{1+w}\right)}{1+\frac{1}{2}\cdot\frac{\frac{3}{2}-\alpha_0}{1-\alpha_0}\cdot(1+x)\left(\frac{w}{1+w}\right)},$$

$$\Phi(x,w)=\eta(\Upsilon(x,w),w)=\frac{b'}{\frac{1}{2}-\alpha'}\left(\frac{2\Upsilon(x,w)}{1+\Upsilon(x,w)}+\frac{\frac{3}{2}-\alpha'}{1-\alpha'}\cdot\frac{w}{1+w}\right)=\frac{b'}{\frac{1}{2}-\alpha'}\left(\frac{2x}{1+x}+\frac{1}{2}\left(\frac{1}{1-\alpha'}-\frac{1}{1-\alpha_0}\right)\frac{w}{1+w}\right)$$

i.e. the same results as before, leading to the same modular observer (41).

The system (40) and the observer (41) were simulated under the following initial conditions:

$x(0)=2, \qquad \tilde{x}(0)=1$

$w(0)=1, \qquad \hat{w}(0)=0$

These account for a step change in the disturbance w from 0 to 1, while the process state is initially away from equilibrium. The design parameters of the observer were selected as

$\alpha_0 = 0.4, \ \ \alpha' = 0.6$

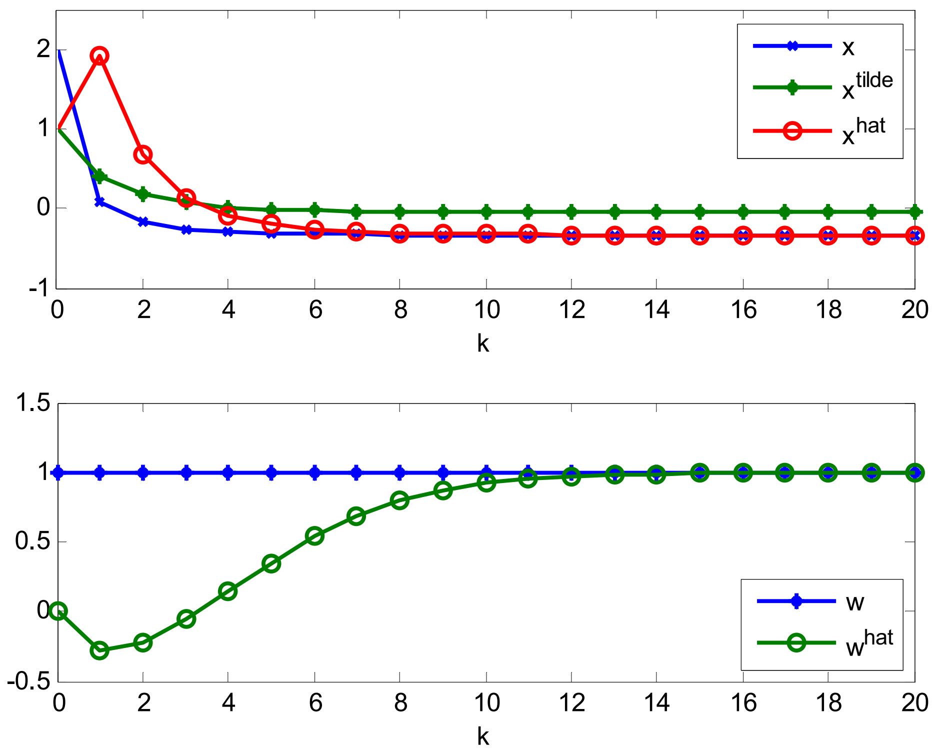


Figure 3: Representative responses for the example system (40) and the observer (41) with $\alpha_0 = 0.4$ and $\alpha' = 0.6$.

In the first plot in Figure 3, the response of the process state is compared with two alternative state estimates – one neglecting the disturbance ($\tilde{x}$), and one accounting for it ($\hat{x}$). Neglecting the disturbance gives rise to a permanent error (offset). The corrector part of the modular observer provides the necessary correction to completely eliminate the offset.

The second plot compares the applied step disturbance with the disturbance estimate. The disturbance observer part of the modular observer quickly detects the presence of the step disturbance and accurately estimates it without permanent error.

VII. Computational Issues

The example of the previous section is a rare case where the design equations can be solved in closed form. In general, the calculations have to be carried out following a numerical approximation scheme. The functional equations (31), (28) and (35) for the modular design or (22) and (23) for the simultaneous design, can be solved via a power series approach (postulating the solution in the form of power series with unknown coefficients, substituting it into the FE, equating coefficients of like terms and solving the resulting the algebraic equations for the unknown coefficients). Of course, the computations for the power series approach will have to be carried out symbolically and will have to terminate at a certain truncation order, large enough to provide an adequate approximation of the unknown solution. This approach has been successfully applied in [5] for the solution of the observer LFE in the absence of disturbances (equation (31)), which is of the same form as the observer LFE's for the simultaneous design in the presence of disturbances (equations (22) and (23)).

Questions of relative computational advantages of solving modular versus simultaneous design equations will arise in applications. At this point, there seems to be no clear general answer of whether direct solution of FE's (31), (28) and (35) is computationally advantageous over solution of LFE's (22) and (23) and subsequent application the of Proposition 7.

VIII. Modular Luenberger Observer for State and Disturbance Estimation in Linear Systems

In the linear case, where the process follows linear dynamics

$$\begin{aligned} x(k+1) &= Fx(k) + Pw(k) \\ y(k) &= Hx(k) + Qw(k) \end{aligned} \tag{45}$$

and the disturbances are generated by a linear exosystem

$$w(k+1) = Sw(k) \tag{46}$$

where F, P, H, Q, S are $m \times m$, $m \times \ell$, $p \times m$, $p \times \ell$, $\ell \times \ell$ matrices respectively, the modular form of the Luenberger observer is:

$$\begin{aligned} \tilde{x}(k+1) &= F\tilde{x}(k) + L_0\left(y(k) - H\tilde{x}(k)\right) \\ \hat{w}(k+1) &= S\hat{w}(k) + L'\left(y(k) - H\left(\tilde{x}(k) + V\hat{w}(k)\right) - Q\hat{w}(k)\right) \\ \hat{x}(k) &= \tilde{x}(k) + V\hat{w}(k) \end{aligned} \tag{47}$$

where V is the solution of the matrix equation

$$VS - (F - L_0 H)V = P - L_0 Q \tag{48}$$

and the gains L_0 and L' can be selected by eigenvalue assignment.

In particular, the choices

$$L_0 = T_0^{-1} B_0 \tag{49}$$

where T_0 is the solution of

$$T_0 F - A_0 T_0 = B_0 H \tag{50}$$

and

$$L' = R^{-1}(B' - T' L_0) \tag{51}$$

where T', R are the solutions of

$$\begin{aligned} T'F - A'T' &= B'H \\ RS - A'R &= (B' - T'L_0)(HV + Q) \end{aligned} \tag{52}$$

assign the observer eigenvalues to the ones of $\begin{bmatrix} A_0 & 0 \\ 0 & A' \end{bmatrix}$.

It is important to point out that the above standard modular observer design for linear systems emerges directly from the general design procedure developed in section IV for nonlinear systems. In fact, when the proposed design approach is applied to the system of (45) and (46), the result is

$$\theta_0(x) = T_0 x\,,\quad \Gamma_0(x,y) = Fx + L_0(y - Hx) \quad \text{where} \quad L_0 = T_0^{-1} B_0\,,$$

$$\Upsilon(x,w) = x + Vw$$

$$\Phi(x,w) = T'x + Rw\,,\quad \Gamma_w(x,w,y) = Sw + R^{-1}(B' - T'L_0)\left(y - H\left(x + Vw\right) - Qw\right)$$

where T_0, V, T', R are the solutions of (50), (48) and (52), i.e. exactly the same as the standard linear result.

IX. Conclusions

This work developed a systematic method for the design of discrete-time nonlinear observers for state and disturbance estimation, in the context of the observer linearization approach. The proposed design method leads to a nonlinear observer with modular structure. The results provide a direct generalization of the modular form of the Luenberger observer for linear systems.

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