Flatness Based Control of Some Classes of Mechanical Systems and Chemical Processes

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CAS, École des Mines de Paris

APC05
Half-day tutorial

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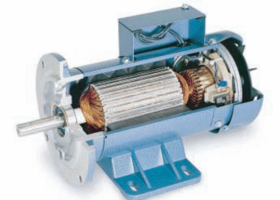
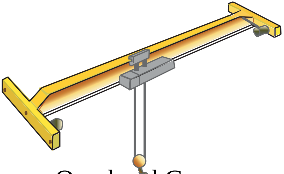
What do DC drives, cranes, motion control systems or chemical reactors hold in common ?



DC Drive



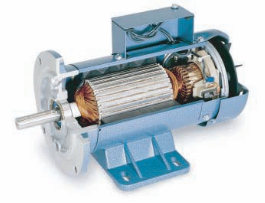
Overhead Crane

Motion Control Stage

Chemical Reactor

DC Drive

System:

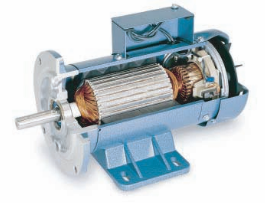


$$L\frac{dI}{dt} = U - RI - K\omega$$
$$J\frac{d\omega}{dt} = KI - K_v\omega - C_r$$

Control variable: U

DC Drive

System:



$$L\frac{dI}{dt} = U - RI - K\omega$$

$$J\frac{d\omega}{dt} = KI - K_v\omega - C_r$$

Control variable: U

Property of the output $y = \omega$ (C_r known)

$$\omega = y$$

$$I = \frac{1}{K}(J\dot{y} + K_v y + C_r)$$

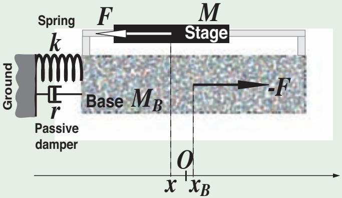
$$U = L\frac{dI}{dt} + RI + Ky$$

$$= \frac{K^2 + RK_v}{K}y + \frac{RJ + LK_v}{K}\dot{y} + \frac{JL}{K}\ddot{y}$$

$$+\frac{R}{K}C_r + \frac{L}{K}\dot{C}_r$$

Motion Control Stage

System (1 d.o.f. case):

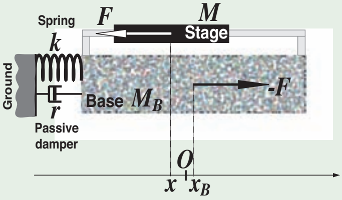


$$M\ddot{x} = F$$
$$M'_B \ddot{x}_B = -F - kx_B - r\dot{x}_B$$

Control variable: F

Motion Control Stage

System (1 d.o.f. case):



$$M\ddot{x} = F$$
$$M'_B\ddot{x}_B = -F - kx_B - r\dot{x}_B$$

Control variable: F

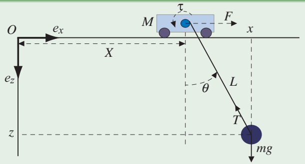
Property of the output

$$y = x - \frac{r}{k}\dot{x} + \frac{1}{M}(M'_B - \frac{r^2}{k})x_B - \frac{M'_B r}{Mk}\dot{x}_B$$

$$x = \frac{M'_B}{k}\ddot{y} + \frac{r}{k}\dot{y} + y,$$
$$x_B = -\frac{M}{k}\ddot{y}$$
$$F = M\left(\frac{M'_B}{k}y^{(4)} + \frac{r}{k}y^{(3)} + \ddot{y}\right)$$

Overhead Crane

System (2 d.o.f. case):

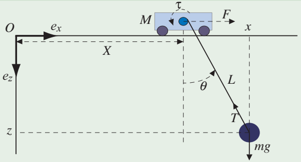


$$m\ddot{x} = -T\sin\theta$$
$$m\ddot{z} = -T\cos\theta + mg$$
$$M\ddot{X} = F + T\sin\theta$$
$$\frac{J}{\rho}\ddot{R} = -\tau + T\rho$$
$$x = X + R\sin\theta$$
$$z = R\cos\theta$$

Control variables: (F, τ)

Overhead Crane

System (2 d.o.f. case):



$$m\ddot{x} = -T\sin\theta$$
$$m\ddot{z} = -T\cos\theta + mg$$
$$M\ddot{X} = F + T\sin\theta$$
$$\frac{J}{\rho}\ddot{R} = -\tau + T\rho$$
$$x = X + R\sin\theta$$
$$z = R\cos\theta$$

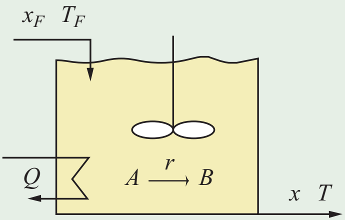
Control variables: (F, τ)

Property of the output $y = (x, z)$

$$X = x - \frac{\ddot{x}z}{\ddot{z} - g}$$
$$R = \frac{z}{\ddot{z} - g}\sqrt{\ddot{x}^2 + (\ddot{z} - g)^2}$$
$$\theta = \arctan\left(\frac{\ddot{x}}{\ddot{z} - g}\right)$$
$$T = m\sqrt{\ddot{x}^2 + (\ddot{z} - g)^2}$$
$$F = M\frac{d^2}{dt^2}\left(x - \frac{\ddot{x}z}{\ddot{z} - g}\right) + m\ddot{x}$$
$$\tau = -\frac{J}{\rho}\frac{d^2}{dt^2}\left(\frac{z}{\ddot{z} - g}\sqrt{\ddot{x}^2 + (\ddot{z} - g)^2}\right) + m\rho\sqrt{\ddot{x}^2 + (\ddot{z} - g)^2}$$

Chemical Reactor

System (Aris & Amundson, 1958):



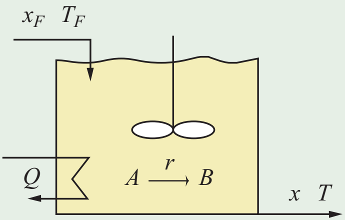
$$\dot{x} = D(x_F - x) - r(x, T)$$

$$\dot{T} = D(T_F - T) + \alpha r(x, T) + Q$$

Control variable: Q

Chemical Reactor

System (Aris & Amundson, 1958):



$$\dot{x} = D(x_F - x) - r(x, T)$$

$$\dot{T} = D(T_F - T) + \alpha r(x, T) + Q$$

Control variable: Q

Property of the output $y = x$

$T = \mathfrak{T}(x, \dot{x})$ solution to

$$r(x, T) = D(x_F - x) - \dot{x}$$

$$Q = \frac{d\mathfrak{T}}{dt}(x, \dot{x}, \ddot{x}) - D(T_F - \mathfrak{T}(x, \dot{x})) - \alpha\left(D(x_F - x) - \dot{x}\right)$$

Summary

In all these examples

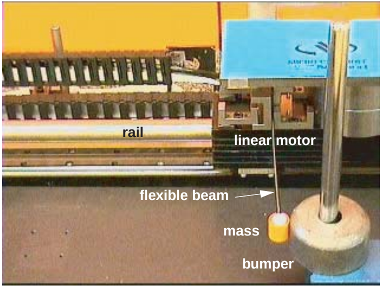
there exists an output y such that

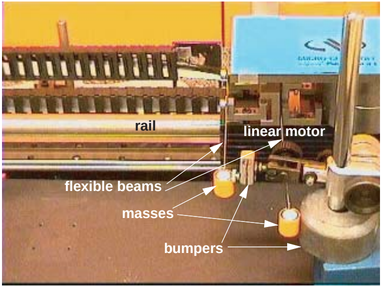
- y has the same dimension as the control vector;
- all the system variables can be expressed in function of y and a finite number of successive derivatives.

y is called a **flat output** and the corresponding system is said to be **differentially flat**.

What for?

Generate and follow fast trajectories with complex objectives, using poor actuators and sensors.





Mass=disturbance   Flatness-based

Masses=disturbance   Flatness-based

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Recalls on Differentially Flat Systems

©M. Fliess, J.L., P. Martin, P. Rouchon 1991.

Definition

The nonlinear system $\dot{x} = f(x, u)$, with $x = (x_1, \ldots, x_n)$: state and $u = (u_1, \ldots, u_m)$: control, $m \leq n$.
is (differentially) flat if and only if there exists $y = (y_1, \ldots, y_m)$ such that:

- y and its successive derivatives $\dot{y}, \ddot{y}, \ldots,$ are independent,
- $y = h(x, u, \dot{u}, \ldots, u^{(r)})$ (generalized output),
- Conversely, x and u can be expressed as:
$$x = \varphi(y, \dot{y}, \ldots, y^{(\alpha)}), \quad u = \psi(y, \dot{y}, \ldots, y^{(\alpha+1)})$$
with $\dot{\varphi} \equiv f(\varphi, \psi)$.

The vector y is called a **flat output**.

Main advantages of Flatness

1. Direct open-loop trajectory computation, without integration nor optimization.
2. Local stabilization of any reference trajectory using the equivalence between the system trajectories and those of
$$y^{(\alpha+1)} = v.$$

"Flatness-Based Control" = Trajectory Planning + Trajectory Tracking.

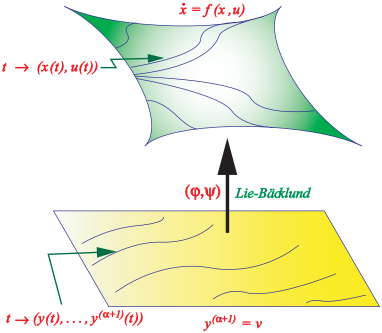
Alternative approach to Predictive Control (see e.g. Fliess, Marquez 2001).

Consequence on motion planning

To every curve $t \mapsto y(t)$ enough differentiable, there corresponds a trajectory

$$t \mapsto \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \varphi(y(t), \dot{y}(t), \ldots, y^{(\alpha)}(t)) \\ \psi(y(t), \dot{y}(t), \ldots, y^{(\alpha+1)}(t)) \end{pmatrix}$$

that identically satisfies the system equations.



1. Find the flat output initial and final conditions:

given	find
$(t_i, x(t_i), u(t_i))$	$(y(t_i), \ldots, y^{(r+1)}(t_i))$
$(t_f, x(t_f), u(t_f))$	$(y(t_f), \ldots, y^{(r+1)}(t_f))$

2. Build a curve $t \mapsto y(t)$ for $t \in [t_i, t_f]$ by **interpolation**, *possibly satisfying further constraints*.
3. Deduce the corresponding trajectory $t \mapsto (x(t), u(t))$.

Rest-to-rest trajectories:

given	thus
$\dot{x}(t_i) = 0, \dot{u}(t_i) = 0$	$\dot{y}(t_i) = \ldots = y^{(r+1)}(t_i) = 0$
$\dot{x}(t_f) = 0, \dot{u}(t_f) = 0$	$\dot{y}(t_f) = \ldots = y^{(r+1)}(t_f) = 0$

Consequence on trajectory tracking

Assume that $y, \ldots, y^{(\alpha)}$ are measured or suitably estimated.
There exists an **endogeneous dynamic feedback**

$$u = \alpha(x, z, v), \quad \dot{z} = \beta(x, z, v)$$

such that the closed-loop system is **diffeomorphic** to

$$y^{(\alpha+1)} = v.$$

Given a reference $t \mapsto (y_{ref}(t), v_{ref}(t))$ with $v_{ref}(t) = y_{ref}^{(\alpha+1)}(t)$, to **stabilize the tracking error** $\varepsilon = y - y_{ref}$ we set:

$$\varepsilon^{(\alpha+1)} = v - v_{ref} = -\sum_{i=0}^{\alpha} k_i \varepsilon^{(i)}$$

with the gains k_i, $i = 0, \ldots, \alpha$, such that all the roots of the polynomial $s^{\alpha+1} + k_\alpha s^\alpha + \ldots + k_1 s + k_0$ have negative real part.
Thus $\|\varepsilon(t)\| \leq Ce^{-a(t-t_0)}$ and, by continuity, locally,

$$\text{dist}(x(t), x_{ref}(t)) \to 0.$$

Some usual critics

- Flatness often understood as feedback linearization.
- questionable novelty
- not robust (model dependent)
- not physical (compensation of open-loop stable dynamics)
- inapplicable with constraints
- no systematic way to compute flat outputs
- and so on...

Some usual critics

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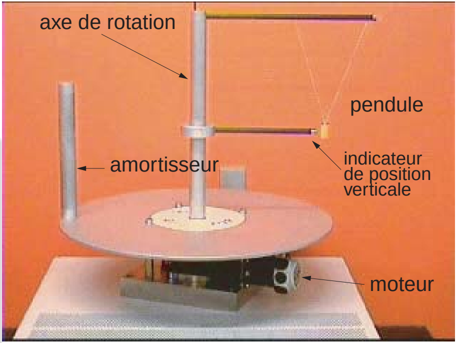
- inapplicable with constraints
- no systematic way to compute flat outputs
- and so on...

Are they right?

No!!! See next slide...

Benchmark :

- Displacements of an undamped pendulum
- at high speed
- without oscillation at the end
- without overshoot
- without position sensor



PID on the motor position | Input filtering

Flatness-based

Thanks to the help of Micro-Contrôle (subsidiary of Newport Corporation.).

Conclusions :

- no feedback linearization (only the motor position is measured)
- novelty: improvement w.r.t. input filtering
- robustness: depends only on the pendulum period
- physical aspects: no dynamical compensation (open loop)
- constraints: no problem.
- flat output computation:
  - *in the linear case*: see J.L. and D.V. Nguyen, Systems & Control Letters, 2003,
  - *in the general case*: J.L., Proc. NOLCOS 2004, Stuttgart.

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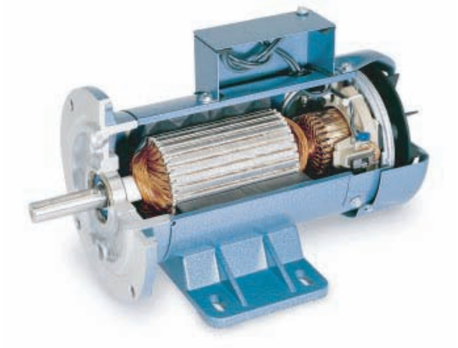
DC Motor Start-Up



From rest at time t_i.
To stabilized speed ω_f at time t_f.

System (recall):

$$L\frac{dI}{dt} = U - RI - K\omega$$

$$J\frac{d\omega}{dt} = KI - K_v\omega - C_r$$

System (recall):

$$L\frac{dI}{dt} = U - RI - K\omega$$
$$J\frac{d\omega}{dt} = KI - K_v\omega - C_r$$

Flat output: $y = \omega$ (C_r known)

$$\omega = y$$
$$I = \frac{1}{K}(J\dot{y} + K_v y + C_r)$$
$$U = L\frac{dI}{dt} + RI + Ky$$
$$= \frac{K^2 + RK_v}{K}y + \frac{RJ + LK_v}{K}\dot{y} + \frac{JL}{K}\ddot{y} + \frac{R}{K}C_r + \frac{L}{K}\dot{C}_r$$

DC Motor Start-Up

Step Reference Tracking

Speed step reference:

$$\omega^*(t) = \omega_f H_{t_i+\varepsilon}(t), \quad U^*(t) = \frac{K^2 + RK_v}{K} \omega_f H_{t_i+\varepsilon}(t)$$

$$\text{with } H_{t_i+\varepsilon}(t) = \begin{cases} 0 & \text{if } t \in [t_i, t_i + \varepsilon[\\ 1 & \text{if } t \in [t_i + \varepsilon, t_f[\end{cases}$$

PID :

$$U = U^* - K_P(\omega - \omega^*) - K_D\dot{\omega} - K_I \int_{t_i}^{t} (\omega(\tau) - \omega^*(\tau))d\tau$$

Constraints:

$$|U| \leq U_{max}, \quad \left|\frac{dI}{dt}\right| \leq \delta, \quad |I| \leq I_{max}$$

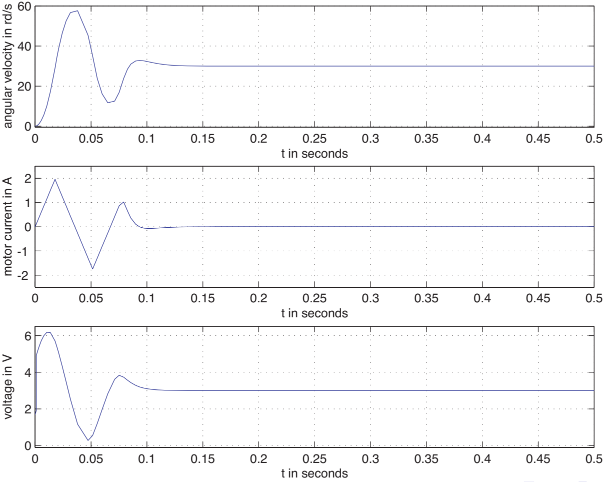
In every simulation:

- $\omega_f = 30$ rd/s, duration $T = t_f - t_i = 0.1$s.
- Initial error $\omega(t_i) = 0.087$ rd/s ($\approx 5°$/s), $C_r = 0.5$ Nm.
- $U_{max} = 25$ V and $I_{max} = 10$ A.
- $K_P = 0.056$, $K_I = 7.45$ et $K_D = 10^{-5}$ (time constants: 10^{-2}, $6.6.10^{-3}$ and 4.10^{-3}s).

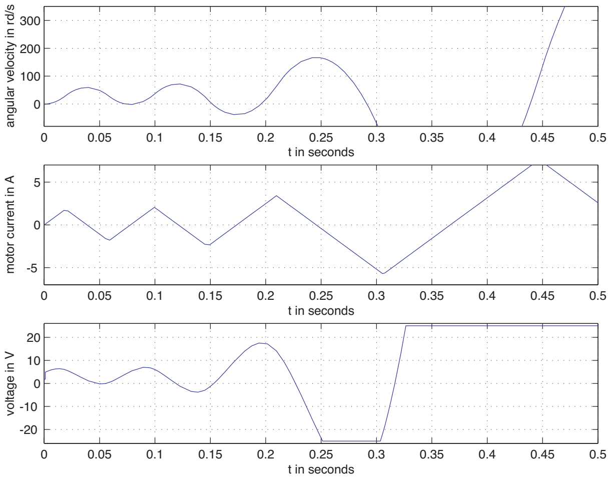
Variation of δ (current rate limitation):

1. 100 A/s
2. 95 A/s
3. 500 A/s.

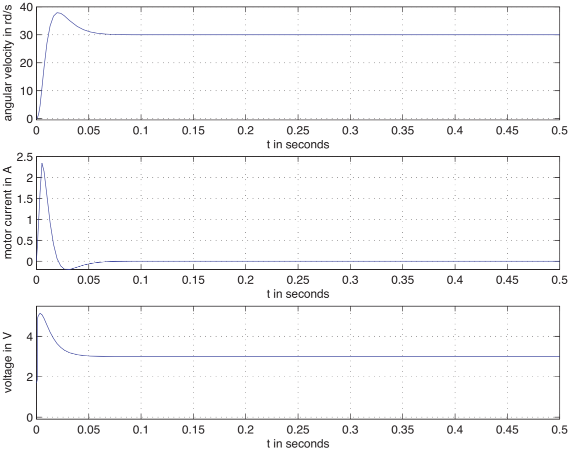
$\delta = 100$ A/s.


angular velocity in rd/s
60
40
20
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t in seconds
motor current in A
2
1
0
-1
-2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t in seconds
voltage in V
6
4
2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t in seconds

$\delta = 95$ A/s.


angular velocity in rd/s
300
200
100
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t in seconds
motor current in A
5
0
-5
t in seconds
voltage in V
20
10
0
-10
-20
t in seconds

$\delta = 500$ A/s.


angular velocity in rd/s
40
30
20
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0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
t in seconds
motor current in A
2.5
2
1.5
1
0.5
0
t in seconds
voltage in V
4
2
0
t in seconds

DC Motor Start-Up
Flatness-Based Reference Tracking

Speed and voltage reference trajectories:

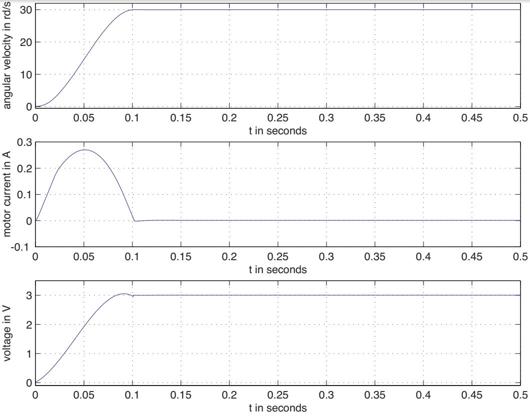
$$\omega^{**} = \omega_f \left(\frac{t-t_i}{T}\right)^2 \left(3 - 2\left(\frac{t-t_i}{T}\right)\right)$$

$$U^{**} = \frac{K^2 + RK_v}{K}\omega^{**} + \frac{RJ + LK_v}{K}\dot{\omega}^{**} + \frac{JL}{K}\ddot{\omega}^{**}$$

PID :

$$U = U^{**} - K_P(\omega - \omega^{**}) - K_D\dot{\omega} - K_I \int_{t_i}^{t}(\omega(\tau) - \omega^{**}(\tau))d\tau$$

Same gains K_P, K_I and K_D, same constraints, same initial errors.



Final angular speed reached in 0.1 s without overshoot, with a precision of $\approx$ 0.1 mm/s for a speed current rate limitation 50 times smaller (10A/s).

Consequences: energy savings, increased life duration.

Linear Motor with Auxiliary Masses

Single Mass Case

Model:

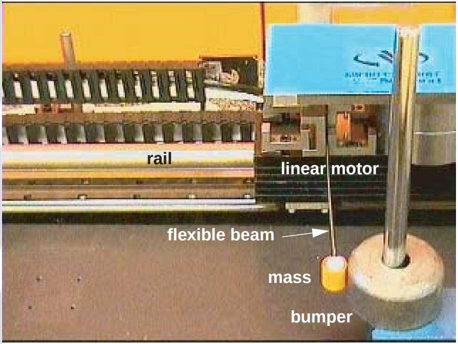
$$M\ddot{x} = F - k(x - z) - r(\dot{x} - \dot{z})$$
$$m\ddot{z} = k(x - z) + r(\dot{x} - \dot{z})$$

Aims:

Rest-to-rest fast and high precision displacements.

Measurements:

- x and $\dot{x}$ measured,
- z not measured.



In collaboration with Micro-Contrôle.

Diplacement without taking into account the auxiliary mass:

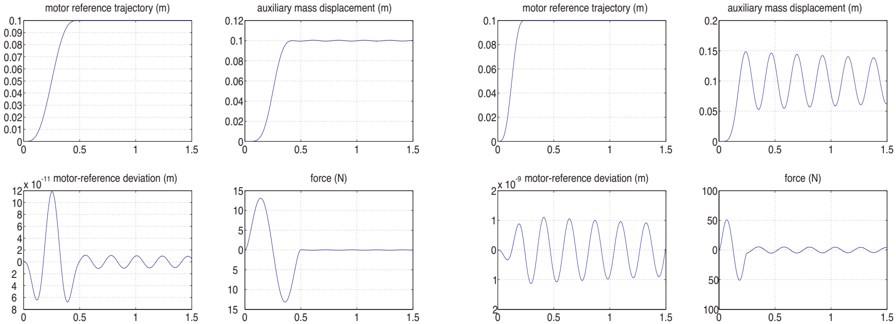
$$M\ddot{x} = F + p \qquad (p : \text{unknown disturbance})$$

Rest-to-rest trajectory ($p = 0$):

$$\begin{aligned} x_{ref}(t) &= x_0 + (x_1 - x_0)\left(\frac{t}{T}\right)^4 \\ &\times \left(35 - 84\left(\frac{t}{T}\right) + 70\left(\frac{t}{T}\right)^2 - 20\left(\frac{t}{T}\right)^3\right) \\ F_{ref}(t) &= M\ddot{x}_{ref}(t) = 420\, M\left(\frac{x_1 - x_0}{T^2}\right)\left(\frac{t}{T}\right) \\ &\times \left(1 - 4\left(\frac{t}{T}\right) + 5\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right). \end{aligned}$$

PID controller:

$$F = F_{ref} - k_P \left(x - x_{ref}\right) - k_D \left(\dot{x} - \dot{x}_{ref}\right)$$
$$-k_I \int_0^t \left(x(\tau) - x_{ref}(\tau)\right) d\tau$$



Duration $T = 0,5$s (left) and $T = 0,25$s (right), error of 20% on k and r.

Taking into account the auxiliary mass

$$M\ddot{x} = F - k(x-z) - r(\dot{x}-\dot{z})$$
$$m\ddot{z} = k(x-z) + r(\dot{x}-\dot{z})$$

Flat output (J.L. and D.V. Nguyen, S&CL, 2003) :

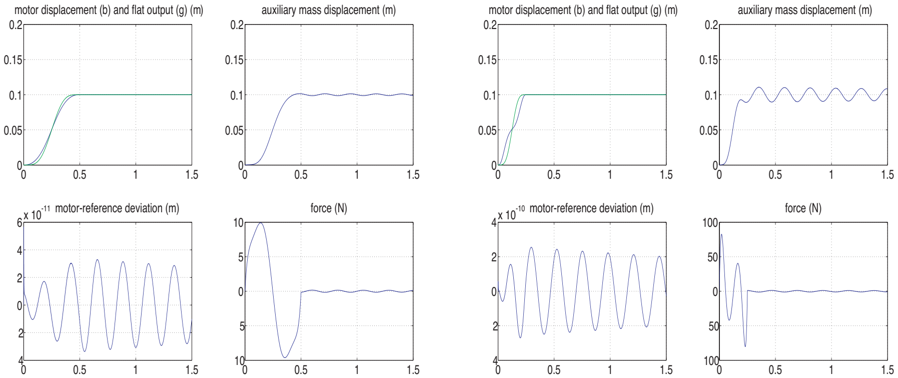
$$y = \frac{r^2}{mk}x + \left(1 - \frac{r^2}{mk}\right)z - \frac{r}{k}\dot{z}$$
$$x = y + \frac{r}{k}\dot{y} + \frac{m}{k}\ddot{y}, \quad z = y + \frac{r}{k}\dot{y}$$
$$F = (M+m)\left(\ddot{y} + \frac{r}{k}y^{(3)} + \frac{Mm}{(M+m)k}y^{(4)}\right)$$

Trajectory planning:

$$
\begin{aligned}
y_{ref}(t) = x_0 + (x_1 - x_0)\left(\frac{t}{T}\right)^5 \\
\times \Bigg(126 - 420\left(\frac{t}{T}\right) + 540\left(\frac{t}{T}\right)^2 \\
-315\left(\frac{t}{T}\right)^3 + 70\left(\frac{t}{T}\right)^4\Bigg).
\end{aligned}
$$

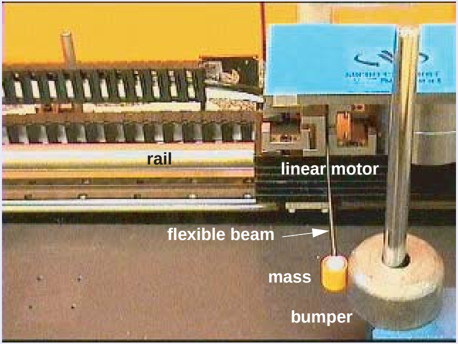
PID controller:

$$
\begin{aligned}
F = F_{ref} - k_P\left(x - x_{ref}\right) - k_D\left(\dot{x} - \dot{x}_{ref}\right) \\
-k_I \int_0^t \left(x(\tau) - x_{ref}(\tau)\right) d\tau
\end{aligned}
$$



Duration $T = 0,5$s (left) and $T = 0,25$s (right), error of 20% on k and r.

Videos



Mass=disturbance

Input filtering

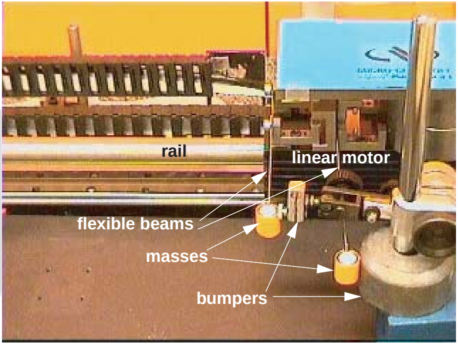
Flatness-based

Linear Motor with Auxiliary Masses

The Case of Two Masses

Model:

$$
\begin{aligned}
M\ddot{x} &= F - k(x-z) - r(\dot{x}-\dot{z}) \\
&\quad -k'(x-z') - r'(\dot{x}-\dot{z}') \\
m\ddot{z} &= k(x-z) + r(\dot{x}-\dot{z}) \\
m'\ddot{z}' &= k'(x-z') + r'(\dot{x}-\dot{z}')
\end{aligned}
$$



In collaboration with Micro-Contrôle.

Flatness:

$$x = y + \left(\frac{r}{k} + \frac{r'}{k'}\right)\dot{y} + \left(\frac{m}{k} + \frac{m'}{k'} + \frac{rr'}{kk'}\right)\ddot{y} + \left(\frac{mr' + rm'}{kk'}\right)y^{(3)} + \frac{mm'}{kk'}y^{(4)}$$

$$z = y + \left(\frac{r}{k} + \frac{r'}{k'}\right)\dot{y} + \left(\frac{m'}{k'} + \frac{rr'}{kk'}\right)\ddot{y} + \frac{rm'}{kk'}y^{(3)}$$

$$z' = y + \left(\frac{r}{k} + \frac{r'}{k'}\right)\dot{y} + \left(\frac{m}{k} + \frac{rr'}{kk'}\right)\ddot{y} + \frac{mr'}{kk'}y^{(3)}$$

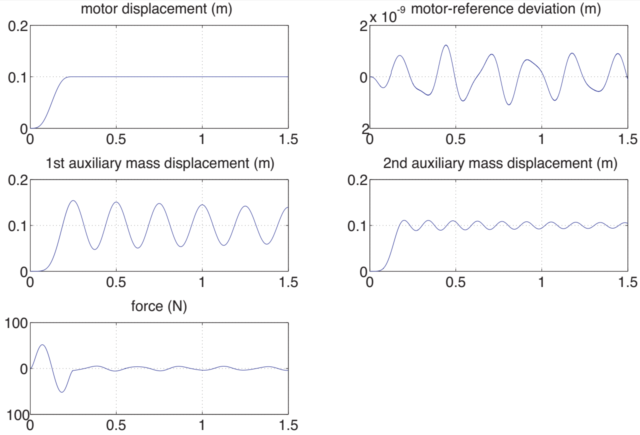
$$F = \hat{M}\ddot{y} + \hat{M}\left(\frac{r}{k} + \frac{r'}{k'}\right)y^{(3)} + \left(\frac{m}{k}\bar{M}' + \bar{M}\frac{m'}{k'} + \hat{M}\frac{rr'}{kk'}\right)y^{(4)} + \left(\frac{mr'}{kk'}\bar{M}' + \bar{M}\frac{rm'}{kk'}\right)y^{(5)} + \frac{Mmm'}{kk'}y^{(6)}$$

with $\hat{M} = (M + m + m')$, $\bar{M} = M + m$ and $\bar{M}' = M + m'$.

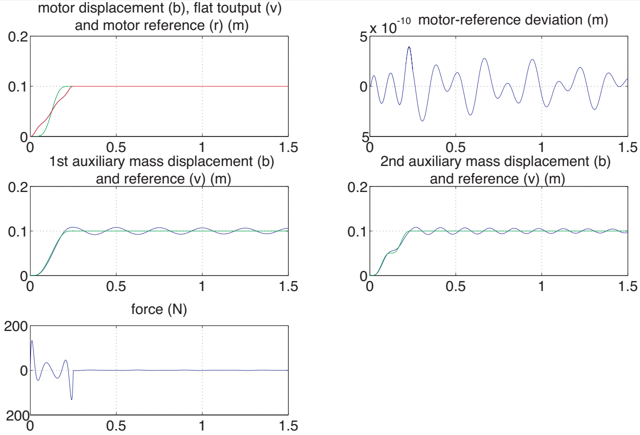
Rest-to-rest trajectory:

$$y(t) = x_0 + (x_1 - x_0)\left(\frac{t}{T}\right)^7$$
$$\times \left(1716 - 9009\left(\frac{t}{T}\right) + 20020\left(\frac{t}{T}\right)^2\right.$$
$$\left. -24024\left(\frac{t}{T}\right)^3 16380\left(\frac{t}{T}\right)^4 - 6006\left(\frac{t}{T}\right)^5 + 924\left(\frac{t}{T}\right)^6\right).$$

x and F deduced from y by the previous formulas.

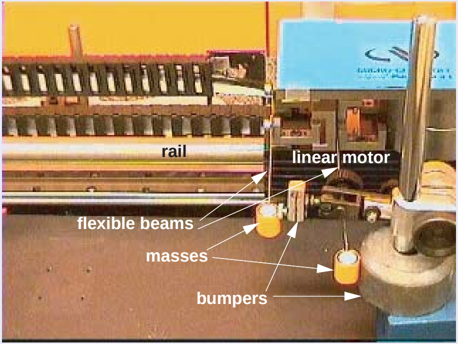


Without taking into account the auxiliary masses (same PID controller as in the single mass case) duration $T = 0, 25$s, error of 20% on k and r



Taking into account the auxiliary masses
(same PID controller with rest-to-rest reference trajectory)
duration $T = 0, 25$s, error of 20% on k and r

Videos



Masses=disturbance

Input filtering

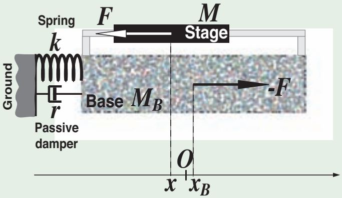
Flatness-based

Contents

Flatness-based control of oscillating systems

High-precision positionning system

System (recall):



$$M\ddot{x} = F$$

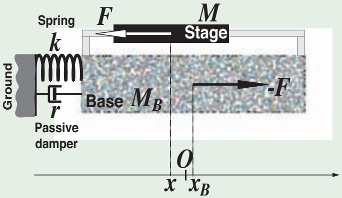
$$M'_B\ddot{x}_B = -F - kx_B - r\dot{x}_B$$

Control variable: F

Flatness-based control of oscillating systems

High-precision positionning system

System (recall):



$$M\ddot{x} = F$$

$$M'_B \ddot{x}_B = -F - kx_B - r\dot{x}_B$$

Control variable: F

Flat output

$$y = x - \frac{r}{k}\dot{x} + \frac{1}{M}(M'_B - \frac{r^2}{k})x_B - \frac{M'_B r}{Mk}\dot{x}_B$$

$$x = \frac{M'_B}{k}\ddot{y} + \frac{r}{k}\dot{y} + y,$$

$$x_B = -\frac{M}{k}\ddot{y}$$

$$F = M\left(\frac{M'_B}{k}y^{(4)} + \frac{r}{k}y^{(3)} + \ddot{y}\right)$$

Rest-to-rest trajectories

Initial and final conditions:

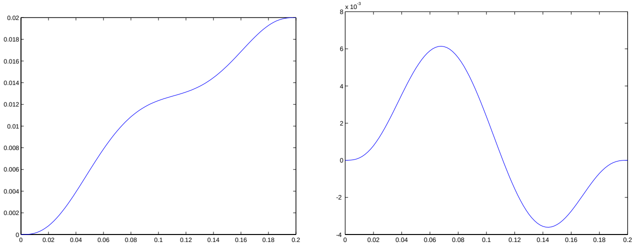
	initial time t_0	final time t_1
stage	$x(t_0) = x_0$ $\dot{x}(t_0) = 0,$ $F(t_0) = 0$	$x(t_1) = x_1$ $\dot{x}(t_1) = 0,$ $F(t_1) = 0$
base	$x_B(t_0) = \dot{x}_B(t_0) = 0$	$x_B(t_1) = \dot{x}_B(t_1) = 0$
flat output	$y(t_0) = x_0$ $\dot{y}(t_0) = \ddot{y}(t_0) = 0$ $y^{(3)}(t_0) = y^{(4)}(t_0) = 0$	$y(t_1) = x_1$ $\dot{y}(t_1) = \ddot{y}(t_1) = 0$ $y^{(3)}(t_1) = y^{(4)}(t_1) = 0$

10 conditions $\Longrightarrow$ 9th degree polynomial:

$$y(t) = y_0 + (y_1 - y_0)\,\tau^5 \left(126 - 420\tau + 540\tau^2 - 315\tau^3 + 70\tau^4\right)$$

with

$$\tau = \frac{t - t_0}{t_1 - t_0}.$$



Stage (left) and base (right) rest-to-rest displacements.

Closing the loop. . .

Recall: base position and velocity not measured.

$$F = F_{ref} - k_P\left(x - x_{ref}\right) - k_D\left(\dot{x} - \dot{x}_{ref}\right)$$
$$-k_I \int_{t_0}^{t} \left(x(\tau) - x_{ref}(\tau)\right) d\tau$$

. . . in the classical way

x_B: perturbation.

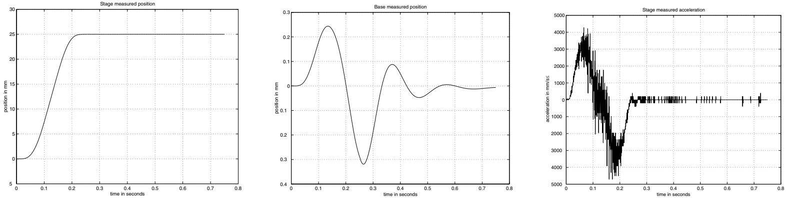
Model in this case: $M\ddot{x} = F$

and reference trajectories:

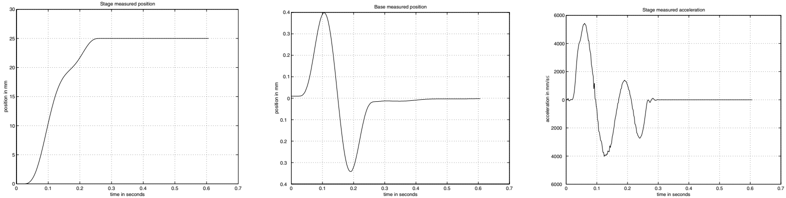
$x_{ref} = y$, $F_{ref} = M\ddot{y}$.

. . . in the flatness-based way

Reference trajectories x_{ref} and F_{ref} defined as before.



Classical controller: 25 mm displacement of the stage in 0.25s (left), base displacement (center) and stage acceleration (right).



Flatness-based controller: 25 mm displacement of the stage in 0.25s (left), base displacement (center) and stage acceleration (right).

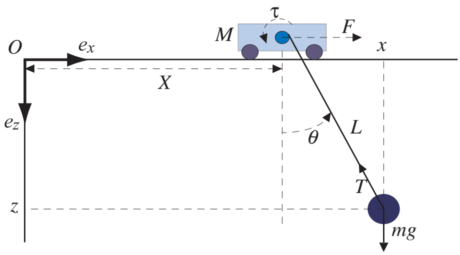
Crane Control

Model (recall)

$$m\ddot{x} = -T\sin\theta$$
$$m\ddot{z} = -T\cos\theta + mg$$

$$M\ddot{X} = F - \lambda(\dot{X}) + T\sin\theta$$
$$\frac{J}{\rho}\ddot{L} = -\tau - \mu\left(\frac{\dot{L}}{\rho}\right) + T\rho$$

$$x = L\sin\theta + X$$
$$z = L\cos\theta$$



Flatness (recall) :

$$\left\{\begin{array}{l} x = L\sin\theta + X \\ z = L\cos\theta \end{array}\right. \iff \left\{\begin{array}{l} (x-X)^2 + z^2 = L^2 \\ \tan\theta = \dfrac{x-X}{z} \end{array}\right.$$

$$\left\{\begin{array}{l} m\ddot{x} = -T\sin\theta \\ m\ddot{z} = -T\cos\theta + mg \end{array}\right. \iff \left\{\begin{array}{l} \tan\theta = \dfrac{\ddot{x}}{\ddot{z}-g} \\ T^2 = m^2\left(\ddot{x}^2 + (\ddot{z}-g)^2\right) \end{array}\right.$$

$$\tan\theta = \frac{x-X}{z} = \frac{\ddot{x}}{\ddot{z}-g} \implies X = x - \frac{\ddot{x}z}{\ddot{z}-g}$$

Summary: (x, z) flat output

$$X = x - \frac{\ddot{x}z}{\ddot{z}-g},\ L = \left((x-X)^2 + z^2\right)^{\frac{1}{2}},\ \tan\theta = \frac{\ddot{x}}{\ddot{z}-g}$$

$$T = m\left(\ddot{x}^2 + (\ddot{z}-g)^2\right)^{\frac{1}{2}}$$

Rest-to-rest trajectory of the load

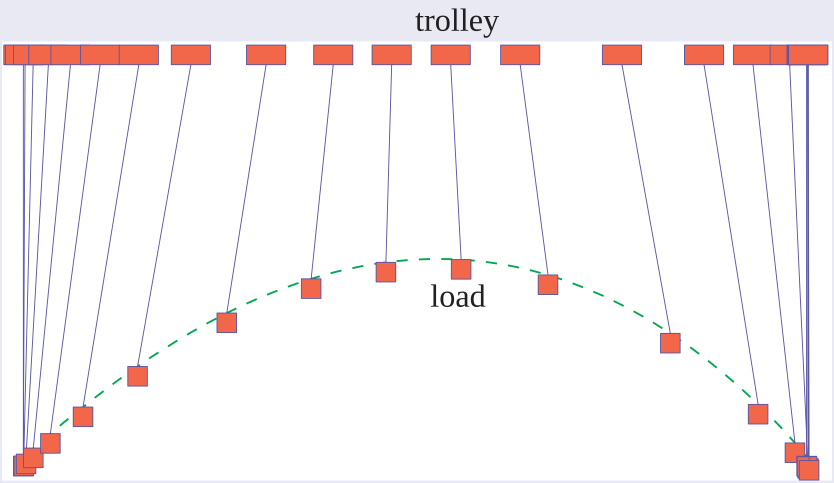
$$x_{ref}(t) = x_i + (x_f - x_i)\left(\frac{t-t_i}{t_f-t_i}\right)^5 \times \left(126 - 420\left(\frac{t-t_i}{t_f-t_i}\right) + 540\left(\frac{t-t_i}{t_f-t_i}\right)^2 - 315\left(\frac{t-t_i}{t_f-t_i}\right)^3 + 70\left(\frac{t-t_i}{t_f-t_i}\right)^4\right).$$

straight line : $z_{ref}(x) = z_i + \frac{z_f - z_i}{x_f - x_i}(x - x_i)$

parabola : $z_{ref}(t) =$

$$z_i + (z_f - z_i)\left(\frac{x_{ref}(t) + x_i - 2\bar{x}}{x_f + x_i - 2\bar{x}}\right)\left(\frac{x_{ref}(t) - x_i}{x_f - x_i}\right)$$

Parabola:



Output Feedback Trajectory Tracking

Only X, $\dot{X}$, L and $\dot{L}$ measured ((x, z, θ) not measured).

PID controller:

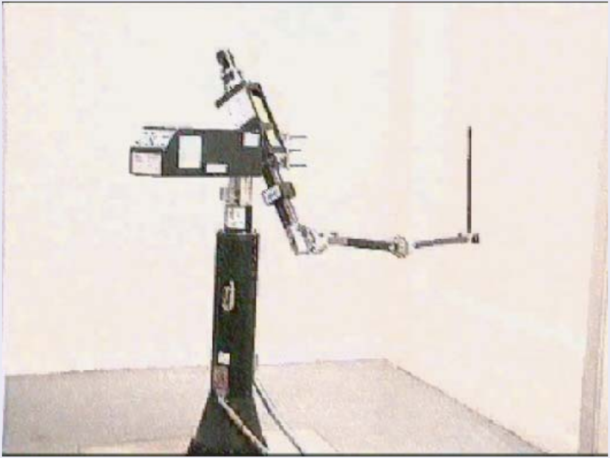
$$F = F_{ref} - k_{F,P}(X - X_{ref}) - k_{F,D}(\dot{X} - \dot{X}_{ref})$$
$$\tau = \tau_{ref} + k_{\tau,P}(L - L_{ref}) + k_{\tau,D}(\dot{L} - \dot{L}_{ref})$$

Theorem (consequence of the LaSalle Theorem):

To every rest-to-rest trajectory and every $k_{F,P}, k_{F,D}, k_{\tau,P}, k_{\tau,D} > 0$, there correspond a tubular neighborhood of the trajectory such that for every initial condition in this neighborhood and every perturbation such that the trajectory remains in it, the closed-loop system with the above PID controller is asymptotically stable.

Reduced size US Navy crane (B. Kiss, J.L., P. Müllhaupt 2000)

More and more difficult: the juggling robot



Contents

Example of Extension to Infinite Dimensional Systems

Polymerization Reactor (N. Petit et al., J. Process Control, 2002)

Model (delayed system)

$$\dot{Q}_a = u(t-\delta) - \frac{1}{\tau} Q_a$$

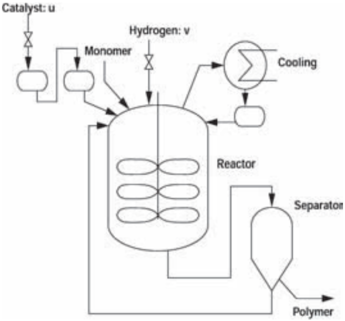
$$\dot{X} = Q_a(\alpha X + \beta) - \gamma X + \xi \frac{X}{1-X}$$

$$\dot{C}_{H_2} = \nu - g(C_{H_2}, Q_a)$$

$$\dot{\lambda} = \frac{1}{\tau}\left(a \log C_{H_2} + b - \lambda\right)$$

$$y_1 = \varphi \frac{X}{1-X}$$

$$y_2 = exp(\lambda)$$



Q_a amount of catalyst, X rate of solid, C_{H_2} concentration of hydrogen, λ log of melt-index of polymer, u amount of input catalyst per unit of time, ν amount of input hydrogen per unit of time per unit of volume.

Flat output: (X, λ)

$$
\begin{aligned}
Q_a &= \frac{\dot{X} + \gamma X - \xi \frac{X}{1-X}}{\alpha X + \beta} \\
u(t-\delta) &= \dot{Q}_a + \frac{1}{\tau} Q_a \triangleq U(X, \dot{X}, \ddot{X}) \\
C_{H_2} &= \exp\left(\frac{1}{a}\left(\tau\dot{\lambda} + \lambda - b\right)\right) \\
v &= \dot{C}_{H_2} + g(C_{H_2}, Q_a) \triangleq N(X, \dot{X}, \lambda, \dot{\lambda}, \ddot{\lambda})
\end{aligned}
$$

Consequences:

Motion planning as in the finite dimensional case.
Trajectory tracking controller using the deviations with respect to the predicted references.

Contents

Conclusions et perspectives

Numerous industrial applications:

- asynchronous motors (Schneider-Electric)
- mecatronics (Bosch)
- automotive equipements (Valeo, Bosch, PSA, Siemens Automotive (IFAC Congress Applications Paper Prize 2002 to Horn, Bamberger, Michau, Pindl))
- underwater applications (IFP)
- high-precision positionning (Micro-Contrôle)
- magnetic bearings (Alcatel, Axomat GmbH)
- chemical reactors (Total-Fina-Elf)
- biotechnological processes (Ifremer)
- . . .

Flat output computation: linear motor with a single auxiliary mass

Model (recall)

$$M\ddot{x} = F - k(x - z) - r(\dot{x} - \dot{z})$$
$$m\ddot{z} = k(x - z) + r(\dot{x} - \dot{z})$$

Setting $s = \frac{d}{dt}$:

$$(Ms^2 + rs + k)x = (rs + k)z + F$$
$$(ms^2 + rs + k)z = (rs + k)x$$

Definition polynomials:

$$x = P_x(s)y, \quad z = P_z(s)y, \quad F = Q(s)y$$
$$(ms^2 + rs + k)P_z(s) = (rs + k)P_x(s)$$

thus $P_x(s) = (ms^2 + rs + k)P_0, \quad P_z(s) = (rs + k)P_0.$