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A Bounded Positive Non-Linear PI Control Law for Double-Pipe Heat Exchangers

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Abstract— **In this work, a PI-type control scheme for the outlet temperature regulation of double-pipe heat exchangers is proposed. Compared to previously proposed approaches, the algorithm developed here guarantees positivity and boundedness of the input flow rate. Moreover, the proposed approach takes into account and actually exploits the analytical-stability features inherent to the open-loop dynamics. As a result, outlet temperature regulation is achieved through a simple controller which does not need to feedback the whole state vector, does not depend on the exact value of the system parameters, and whose stabilization character is global in the closed-loop system state-space domain. The analytical developments are corroborated through experimental results.**

I. INTRODUCTION

Several control schemes for the outlet temperature regulation of heat exchangers have been developed in the literature through the application of various techniques. For instance, linearizing feedback algorithms have been proposed: based on a simple lumped-parameter model in [10], and considering a distributed-parameter model in [9]. Unfortunately, such a geometric control design methodology assumes the exact knowledge of the structure and parameters of the system dynamics, involves all the states of the considered model, and generally gives rise to complex expressions, neglecting the natural analytical-stability properties of the process. Other works like that in [5], which proposes an optimal control scheme, or that in [7], where a generalized predictive control algorithm is developed, focus on the optimization of a cost function, disregarding the natural qualitative properties of the system. Furthermore, works like that in [4], where conventional P, PI, and PID algorithms are tested, that in [8] where a PI fuzzy controller was proposed, or that in [14], where a multi-loop controller tuned using game theory is considered, lack of stability proofs and/or stability region estimations.

On the other hand, as far as the authors are aware, previous works on control design for double-pipe heat exchangers do not simultaneously consider the positive (unidirectional) and bounded nature of the flow rate taken as input variable. Such controllers could eventually try to force the actuators to go beyond their natural capabilities, undergoing the well-known phenomenon of saturation. In a general context, the presence of such a nonlinearity is not necessarily disadvantageous as long as it is taken into account in the control design and/or the closed loop analysis. Otherwise, it may give rise to unpredicted undesirable effects as pointed out for instance in [12, §5.2]. Thus, control design considering those input constraints turns out to be important in order to avoid such unexpected or undesirable closed-loop system behaviors.

In this work, a simple (non-linear) PI-type controller for the process (hot) fluid outlet temperature regulation of double-pipe heat exchangers, taking the cold fluid (coolant) flow rate as control input, is proposed. The algorithm takes into account the positive and bounded nature of the flow rate taken as input variable, as well as the natural analytical-stability properties of the exchanger.

The text is organized as follows. In Section II, we state the nomenclature, notation, and preliminaries that support our developments. Section III presents the system dynamics. In Section IV, the proposed controller is presented and the closed-loop stability analysis is developed. Experimental results are presented in Section V. Finally, conclusions are given in Section VI.

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II. NOMENCLATURE AND NOTATION

Throughout the paper, the system variables and parameters are denoted as follows:

F	mass flow rate [kg/sec]
C_p	specific heat [J/(°C · kg)]
M	total mass inside the tube [kg]
U	overall heat transfer coefficient [J/(°C · m^2 · sec)]
A	heat transfer surface area [m^2]
T	temperature [°C]
t	time [sec]
ΔT	temperature difference [°C]
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{R}^n$	set of n-tuples (x_j) with $x_j \in \mathbb{R}$
0_n	origin of $\mathbb{R}^n$
$\mathbb{R}^n_+$	set of n-tuples (x_j) with $x_j \in \mathbb{R}_+$

Subscripts:

u	upper bound	l	lower bound
c	cold	h	hot
i	inlet	o	outlet

Let ΔT_1 and ΔT_2 stand for the temperature difference at each terminal side of the heat exchanger, *i.e.*

$$\Delta T_1 \triangleq \begin{cases} T_{hi} - T_{co} & \text{if } \alpha = 1 \\ T_{hi} - T_{ci} & \text{if } \alpha = -1 \end{cases}$$

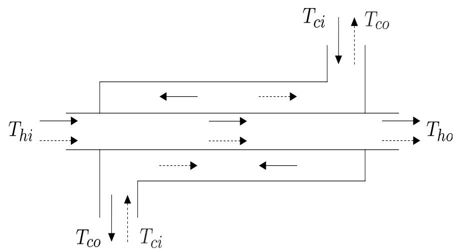

Fig. 1. Counter/parallel-flow (full/dashed arrows resp.) double-pipe heat exchanger.

and

$$\Delta T_2 \triangleq \begin{cases} T_{ho} - T_{ci} & \text{if } \alpha = 1 \\ T_{ho} - T_{co} & \text{if } \alpha = -1 \end{cases}$$

where

$$\alpha \triangleq \begin{cases} 1 & \text{if counter flow} \\ -1 & \text{if parallel flow} \end{cases}$$

(see Fig. 1). The *logarithmic mean temperature difference* (LMTD) among the fluids is typically expressed as

$$\Delta T_\ell \triangleq \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}}$$

Nonetheless, this expression reduces to an indeterminate form when $\Delta T_1 = \Delta T_2$, which is specially problematic in the counterflow case. Such an indetermination is avoided if the LMTD is taken as

$$\Delta T_L \triangleq \begin{cases} \Delta T_\ell & \text{if } \Delta T_2 \neq \Delta T_1 \\ \Delta T_0 & \text{if } \Delta T_2 = \Delta T_1 = \Delta T_0 \end{cases} \quad (1)$$

This was proved in [16], together with the following analytical properties:

Lemma 1: [16, Lemma 2 & Remark 3] ΔT_L in Eq. (1) is continuously differentiable at every $(\Delta T_1, \Delta T_2) \in \mathbb{R}^2_+$. Moreover, it is positive on $\mathbb{R}^2_+$, while $\lim_{\Delta T_1 \to 0} \Delta T_L = 0$ for any $\Delta T_2 \in \mathbb{R}_+$, and $\lim_{\Delta T_2 \to 0} \Delta T_L = 0$ for any $\Delta T_1 \in \mathbb{R}_+$.

Lemma 2: [16, Lemma 3] ΔT_L in Eq. (1) is strictly increasing in its arguments, *i.e.* $\frac{\partial \Delta T_L}{\partial \Delta T_i} > 0$, $i = 1, 2$, $\forall (\Delta T_1, \Delta T_2) \in \mathbb{R}^2_+$.

Furthermore, the interior and boundary of a set, say $\mathcal{B}$, will be respectively denoted $\text{int}(\mathcal{B})$ and $\partial\mathcal{B}$. The derivative of a scalar function depending on a single scalar variable, say $\rho : \mathbb{R} \to \mathbb{R} : \varsigma \mapsto \rho(\varsigma)$, will be denoted ρ', *i.e.* $\rho'(\varsigma) = \frac{d\rho}{d\varsigma}(\varsigma)$.

III. THE SYSTEM DYNAMICS

The following assumptions are considered:

A1. Radially uniform fluid temperatures and velocities.
A2. Axially uniform and constant heat transfer coefficient.
A3. Constant fluid thermophysical properties.
A4. No heat transfer with the surroundings.
A5. Fluids are incompressible and single phase.
A6. Negligible axial heat conduction.
A7. There is no energy storage in the walls.
A8. Inlet temperatures, T_{ci} and T_{hi}, are constant.
A9. The flow rates are axially uniform and any variation is considered to take place instantaneously at every point along the whole length of the exchanger.
A10. The hot fluid flow rate, F_h, is kept constant, while the value of the cold fluid flow rate, F_c, can be arbitrarily varied within a compact interval $\mathcal{F}_c \triangleq [F_{cl}, F_{cu}]$, for some positive constants $F_{cl} < F_{cu}$.

Under these assumptions, and taking the whole exchanger as one *bi-compartmental cell*, a simplified but suitable lumped-parameter dynamical model for a double-pipe heat exchanger is (see for instance [17]):

$$\dot{T}_{co} = \frac{2}{M_c} \left[F_c \left(T_{ci} - T_{co} \right) + \frac{UA}{C_{pc}} \Delta T_L(T_{co}, T_{ho}) \right] \quad (2a)$$

$$\dot{T}_{ho} = \frac{2}{M_h} \left[F_h \left(T_{hi} - T_{ho} \right) - \frac{UA}{C_{ph}} \Delta T_L(T_{co}, T_{ho}) \right] \quad (2b)$$

where $\Delta T_L(\cdot, \cdot)$ is the LMTD (complemented) expression in Eq. (1), considered a function of (T_{co}, T_{ho}). A physically reasonable state-space domain for the system in Eqs. (2) is

$$\mathcal{D} \triangleq \begin{cases} \{(T_{o1}, T_{o2}) \in \mathbb{R}^2 \mid T_{ci} < T_{oj} < T_{hi} \, , \; j = 1, 2\} \\ \hfill \text{if } \alpha = 1 \\ \{(T_{o1}, T_{o2}) \in \mathbb{R}^2 \mid T_{ci} < T_{o1} < T_{o2} < T_{hi}\} \\ \hfill \text{if } \alpha = -1 \end{cases}$$

(see for instance [17], [15]).

The control objective consists in the regulation of the process (hot) fluid outlet temperature, T_{ho}, towards a (pre-specified) desired value, T_{hd}, through the cold fluid flow rate, F_c, as input variable, taking into account the restricted range and unidirectional nature of such an input flow rate (according to Assumption A10). The use of a simple model, like Eqs. (2), for the control design aiming at the achievement of such an objective is desirable [11], [13]. Indeed, a high order process dynamics representation would end up in a complex scheme with complicated expressions, and would involve temperature measurements of intermediate points throughout the exchanger which are not always available. In particular, the model in Eqs. (2) has been used for control design for instance in [10]. It has actually been used as a suitable dynamics representation of double-pipe heat exchangers for numerous purposes, as pointed out in [17].

Remark 1: Notice that by considering ΔT_L a function of (T_{co}, T_{ho}) on $\mathcal{D}$, continuous differentiability and positivity hold for all (T_{co}, T_{ho}) on $\mathcal{D}$, and 0 (zero) may be considered the value that ΔT_L takes at any (T_{co}, T_{ho}) on $\partial\mathcal{D}$ such that $\Delta T_1(T_{co}, T_{ho}) \cdot \Delta T_2(T_{co}, T_{ho}) = 0$ (see Lemma 1 in Section II). Furthermore, strict monotonicity in its arguments holds as (applying the chain rule): $\frac{\partial \Delta T_L}{\partial T_{ho}} > 0$ and $\frac{\partial \Delta T_L}{\partial T_{co}} < 0$, $\forall (T_{co}, T_{ho}) \in \mathcal{D}$ (see Lemma 2 in Section II).

Remark 2: Let y denote the open-loop state vector, *i.e.* $y \triangleq (T_{co}, T_{ho})^T$, and let $\dot{y} = \bar{f}(y; \theta)$ represent the open-loop system dynamics in Eqs. (2) assuming constant flow rates, where $\theta \in \mathbb{R}^p$ (for some positive integer p) is the system parameter vector. Considering Lemma 1, it can be seen (from Eqs. (2)) that $\bar{f}$ is continuously differentiable in

$(y;\theta)$ on $\mathcal{D}\times\mathbb{R}^p_+$. Then, the system solutions, $y(t;y_0,\theta)$ with $y_0 \triangleq y(0) \in \mathcal{D}$, do not only exist and are unique, but are also continuously differentiable with respect to initial conditions and parameters, for all $y_0 \in \mathcal{D}$ and all θ sufficiently close to any nominal parameter vector $\theta_0 \in \mathbb{R}^p_+$ [6, §2.4].

In [17], it was shown that, considering constant flow rates, the system dynamics (2) possesses a unique equilibrium point $(T^*_{co}, T^*_{ho}) \in \mathcal{D}$, where

$$\begin{pmatrix} T^*_{co} \\ T^*_{ho} \end{pmatrix} = \begin{pmatrix} 1-P & P \\ RP & 1-RP \end{pmatrix} \begin{pmatrix} T_{ci} \\ T_{hi} \end{pmatrix} \triangleq \begin{pmatrix} g_c(F_c) \\ g_h(F_c) \end{pmatrix} \quad (3)$$

with $R = \frac{F_c C_{pc}}{F_h C_{ph}}$,

$$P = \begin{cases} \frac{1-S}{1+(-S)^\beta R} & \text{if } R-\alpha \neq 0 \\ \frac{UA}{UA+F_c C_{pc}} & \text{if } R=\alpha=1 \end{cases}$$

$S = e^{UA\left(\frac{\alpha}{F_h C_{ph}} - \frac{1}{F_c C_{pc}}\right)}$, and $\beta \triangleq \frac{\alpha+1}{2}$.

Claim 1: g_h in Eq. (3) is a one-to-one strictly decreasing continuously differentiable function of F_c.

Proof: Continuous differentiability of g_h with respect to F_c follows from the arguments given in Remark 2. Hence, from Eq. (3), we have

$$g'_h(F_c) = \begin{cases} \frac{RS[1+\gamma-e^\gamma](T_{hi}-T_{ci})}{F_c(1+(-S)^\beta R)^2} & \text{if } R-\alpha \neq 0 \\ -\frac{C_{pc}U^2A^2(T_{hi}-T_{ci})}{2C_{ph}F_h(UA+C_{ph}F_h)^2} & \text{if } R=\alpha=1 \end{cases}$$

where $\gamma \triangleq \frac{UA}{C_{pc}F_c} - \frac{\alpha UA}{C_{ph}F_h}$. Thus, from Formula 4.2.30 in[1] [1], we see that $g'_h(F_c) < 0, \forall F_c > 0$, showing that $g_h(F_c)$ is strictly decreasing on its domain. This, in turn, corroborates its one-to-one character. ■

Remark 3: Observe that through Claim 1, two important facts are concluded: 1) T^*_{ho} is restricted to a *reachable steady-state space* defined by $\mathcal{R}_h \triangleq [g_h(F_{cu}), g_h(F_{cl})]$. 2) Any value of $T^*_{ho} \in \mathcal{R}_h$ is uniquely defined by a specific flow rate value $F^*_c \in \mathcal{F}_c$ (see Assumption A10), which in turn defines a unique value of T^*_{co} according to Eq. (3).

IV. THE PROPOSED CONTROLLER

Proposition 1: Consider the dynamical system in (2) with $F_c \in \mathcal{F}_c$. Let the value of F_c be continuously computed as follows

$$F_c(\phi, T_{ho}) = k_p\eta(\phi)\,(T_{ho} - T_{hd}) + \phi \quad (4)$$

for any constant $T_{hd} \in \text{int}(\mathcal{R}_h)$, where

$$\eta(\phi) \triangleq (\phi - F_{cl})(F_{cu} - \phi)$$

ϕ is an auxiliary state whose instantaneous value is dynamically calculated according to the following auxiliary dynamics

$$\dot{\phi} = k_i\eta(\phi)\,(T_{ho} - T_{hd}) \quad (5)$$

k_p is a nonnegative scalar satisfying

$$k_p < \frac{1}{(F_{cu}-F_{cl})(T_{hi}-T_{ci})} \quad (6)$$

and k_i is a positive constant. Then, provided that k_p and k_i are sufficiently small, for any initial closed-loop (extended) state vector $(T_{co}, T_{ho}, \phi)(0) \in \mathcal{D}\times\text{int}(\mathcal{F}_c)$: $T_{ho}(t) \to T_{hd}$ as $t \to \infty$, with $F_c(t) \in \text{int}(\mathcal{F}_c)$, $\forall t \geq 0$, and $\big(T_{co}, T_{ho}\big)(t) \in \mathcal{D}$, $\forall t \geq 0$.

Proof: From the satisfaction of inequality (6):

$$\begin{aligned} 0 &< -k_p(F_{cu}-F_{cl})(T_{hi}-T_{ci}) + 1 \\ &\leq k_p(F_{cu}+F_{cl}-2\phi)(T_{ho}-T_{hd}) + 1 \\ &= k_p\eta'(\phi)(T_{ho}-T_{hd}) + 1 = \frac{\partial F_c}{\partial \phi} \end{aligned}$$

$\forall(\phi, T_{ho}) \in \mathcal{F}_c \times \mathcal{T}$, where

$$\mathcal{T} \triangleq \{T_o \in \mathbb{R} \mid T_{ci} \leq T_o \leq T_{hi}\}$$

i.e.

$$\frac{\partial F_c}{\partial \phi} > 0 \quad \forall(\phi, T_{ho}) \in \mathcal{F}_c \times \mathcal{T} \quad (7)$$

Moreover, $\frac{\partial F_c}{\partial T_{ho}} = k_p\eta(\phi) \geq 0$, $\forall(\phi, T_{ho}) \in \mathcal{F}_c \times \mathcal{T}$ (with strict inequality on $\text{int}(\mathcal{F}_c) \times \text{int}(\mathcal{T})$ if $k_p > 0$). Then $F_c(F_{cl}, T_{ci}) = F_{cl} < F_c(\phi, T_{ho}) < F_{cu} = F_c(F_{cu}, T_{hi})$, $\forall(\phi, T_{ho}) \in \text{int}(\mathcal{F}_c) \times \text{int}(\mathcal{T})$, or equivalently

$$F_c(\phi, T_{ho}) \in \text{int}(\mathcal{F}_c) \quad \forall(\phi, T_{ho}) \in \text{int}(\mathcal{F}_c) \times \text{int}(\mathcal{T}) \quad (8)$$

Now, let x denote the closed-loop (extended) state vector, *i.e.* $x \triangleq (T_{co}, T_{ho}, \phi)^T$, and $\dot{x} = f(x)$ represent the closed-loop system dynamics. Based on Lemma 1 (see also Remark 1), and in view of expression (8), it can be verified that, with $\alpha = 1$:

$$f_1(T_{hi}, T_{ho}, \phi) = \frac{2F_c(\phi, T_{ho})}{M_c}(T_{ci} - T_{hi}) < 0 \quad \forall(T_{ho}, \phi) \in \text{int}(\mathcal{T}) \times \text{int}(\mathcal{F}_c)$$

$$f_2(T_{co}, T_{ci}, \phi) = \frac{2F_h}{M_h}(T_{hi} - T_{ci}) > 0 \quad \forall(T_{co}, \phi) \in \text{int}(\mathcal{T}) \times \text{int}(\mathcal{F}_c)$$

with $\alpha = -1$:

$$f_1(T_{co}, T_{co}, \phi) = \frac{2F_c(\phi, T_{co})}{M_c}(T_{ci} - T_{co}) < 0 \quad \forall(T_{co}, \phi) \in \text{int}(\mathcal{T}) \times \text{int}(\mathcal{F}_c)$$

$$f_2(T_{ho}, T_{ho}, \phi) = \frac{2F_h}{M_h}(T_{hi} - T_{ho}) > 0 \quad \forall(T_{ho}, \phi) \in \text{int}(\mathcal{T}) \times \text{int}(\mathcal{F}_c)$$

and for any $\alpha \in \{-1, 1\}$:

$$f_1(T_{ci}, T_{ho}, \phi) = \frac{2UA}{M_c C_{pc}}\Delta T_L(T_{ci}, T_{ho}) > 0 \quad \forall(T_{ho}, \phi) \in \text{int}(\mathcal{T}) \times \text{int}(\mathcal{F}_c)$$

[1] Formula 4.2.30 in [1] states the following well-known inequality: $e^x > 1 + x$, $\forall x \neq 0$.

$$f_2(T_{co},T_{hi},\phi) = -\frac{2UA}{M_hC_{ph}}\Delta T_L(T_{co},T_{hi}) < 0 \quad \forall (T_{co},\phi)\in \mathrm{int}(\mathcal{T})\times\mathrm{int}(\mathcal{F}_c)$$

$$f_3(T_{co},T_{ho},F_{cl}) = f_3(T_{co},T_{ho},F_{cu}) = 0 \quad \forall (T_{co},T_{ho})\in\mathcal{D}$$

This shows that there is no point on the boundary of $\mathcal{D}\times\mathcal{F}_c$ where the vector field f have a normal component pointing outwards. Consequently, for any initial extended state vector in $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$, the closed-loop system solution cannot leave the system state-space domain $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$. Moreover, it is clear that the points on $\partial\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ cannot even be approached. On the other hand, from expression (7), it can be seen that, for any given cold fluid flow rate steady-state value $F_c^*\in\mathcal{F}_c$, corresponding to a specific hot fluid outlet steady-state temperature $T_{ho}^*\in\mathcal{T}$ (according to Remark 3), there corresponds a unique auxiliary state equilibrium value $\phi^*\in\mathcal{F}_c$. With this and Remark 3 in mind, it can be seen from Eq. (5) that the closed-loop system has a unique equilibrium point $x^*=(T_{co}^*,T_{ho}^*,\phi^*)$ in $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$, where $T_{ho}^*=T_{hd}$, and ϕ^* takes the unique value on $\mathcal{F}_c$ through which T_{ho}^* can adopt the desired value T_{hd}. Besides, letting $x_l^*\triangleq\big(g_c(F_{cl}),g_h(F_{cl}),F_{cl}\big)$ and $x_u^*\triangleq\big(g_c(F_{cu}),g_h(F_{cu}),F_{cu}\big)$ (see Eq. (3); observe that $F_c(F_{cl},g_h(F_{cl}))=F_{cl}$ and $F_c(F_{cu},g_h(F_{cu}))=F_{cu}$), with $g_h(F_{cl})=\max\{T_{ho}^*\in\mathcal{R}_h\}$ and $g_h(F_{cu})=\min\{T_{ho}^*\in\mathcal{R}_h\}$ (see Remark 3), we have that $f(x_l^*)=f(x_u^*)=0_3$. Actually, x_l^* and x_u^* are the only equilibrium points on the boundary of $\mathcal{D}\times\mathcal{F}_c$. The Jacobian matrix of f, *i.e.*

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{2\left[k_p\eta'(\phi)(T_{ho}-T_{hd})+1\right](T_{ci}-T_{co})}{M_c} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 0 \\ 0 & k_i\eta(\phi) & k_i\eta'(\phi)(T_{ho}-T_{hd}) \end{pmatrix}$$

where

$$\frac{\partial f_1}{\partial x_1} = \frac{2}{M_c}\left[-F_c(\phi,T_{ho})+\frac{UA}{C_{pc}}\frac{\partial\Delta T_L}{\partial T_{co}}\right]$$

$$\frac{\partial f_1}{\partial x_2} = \frac{2}{M_c}\left[k_p\eta(\phi)(T_{ci}-T_{co})+\frac{UA}{C_{pc}}\frac{\partial\Delta T_L}{\partial T_{ho}}\right]$$

$$\frac{\partial f_2}{\partial x_1} = -\frac{2UA}{M_hC_{ph}}\frac{\partial\Delta T_L}{\partial T_{co}}$$

$$\frac{\partial f_2}{\partial x_2} = -\frac{2}{M_h}\left[F_h+\frac{UA}{C_{ph}}\frac{\partial\Delta T_L}{\partial T_{ho}}\right] \tag{9}$$

and $\eta'(\phi)=F_{cu}+F_{cl}-2\phi$, evaluated at x_l^* and x_u^*, *i.e.* $\frac{\partial f}{\partial x}\Big|_{x=x_l^*}$ and $\frac{\partial f}{\partial x}\Big|_{x=x_u^*}$, have eigenvalues $k_i(F_{cu}-F_{cl})(g_h(F_{cl})-T_{hd})>0$ and $k_i(F_{cl}-F_{cu})(g_h(F_{cu})-T_{hd})>0$, respectively. Then x_l^* and x_u^* are repulsive (unstable) and consequently the points on $\mathcal{D}\times\partial\mathcal{F}_c$ cannot be asymptotically approached from the interior of the system state-space domain either. Consequently, for any $x_0\in\mathcal{D}\times\mathrm{int}(F_c)$, $x(t;x_0)\in\mathcal{D}\times\mathrm{int}(F_c)$, $\forall t\geq 0$, or equivalently $(T_{co},T_{ho},\phi)(t)\in\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$, $\forall t\geq 0$. This and expression (8) prove that $F_c(\phi,T_{ho})(t)=F_c(\phi(t),T_{ho}(t))\in\mathrm{int}(\mathcal{F}_c)$, $\forall t\geq 0$. Now, let us consider the Jacobian matrix of f at x^*, *i.e.* $\frac{\partial f}{\partial x}\Big|_{x^*}$. Its characteristic polynomial is given by $P(\lambda)=\lambda^3+a_2\lambda^2+a_1\lambda+a_0$, where

$$a_2 \triangleq \left[\frac{2F_c(\phi,T_{ho})}{M_c}+\frac{2F_h}{M_h}-\frac{2UA}{M_cC_{pc}}\frac{\partial\Delta T_L}{\partial T_{co}}+\frac{2UA}{M_hC_{ph}}\frac{\partial\Delta T_L}{\partial T_{ho}}\right]_{x=x^*}$$

$$a_1 \triangleq \left[\frac{4F_hF_c(\phi,T_{ho})}{M_cM_h}+\frac{4UAF_c(\phi,T_{ho})}{M_cM_hC_{ph}}\frac{\partial\Delta T_L}{\partial T_{ho}}-\frac{4UAF_h}{M_hM_cC_{pc}}\frac{\partial\Delta T_L}{\partial T_{co}} + \frac{4UAk_p\eta(\phi)(T_{ci}-T_{co})}{M_cM_hC_{ph}}\frac{\partial\Delta T_L}{\partial T_{co}}\right]_{x=x^*}$$

and $a_0\triangleq k_i\bar{a}_0$ with

$$\bar{a}_0 \triangleq \left[\frac{4UA\eta(\phi)(T_{ci}-T_{co})}{M_cM_hC_{ph}}\frac{\partial\Delta T_L}{\partial T_{co}}\right]_{x=x^*}$$

From these expressions and Lemma 2 (see also Remark 1), it can be seen that

$$a_2 > b_2 \triangleq \frac{2F_{cl}}{M_c}+\frac{2F_h}{M_h} > 0$$

$$a_1 > b_1 \triangleq -\frac{4F_hUA}{M_hM_cC_{pc}}\left[\frac{\partial\Delta T_L}{\partial T_{co}}\right]_{x=x^*} > 0$$

$$0 < \bar{a}_0 < \bar{b}_0 \triangleq \frac{4UA\eta\left(\frac{F_{cl}+F_{cu}}{2}\right)(T_{ci}-T_{hi})}{M_cM_hC_{ph}}\left[\frac{\partial\Delta T_L}{\partial T_{co}}\right]_{x=x^*}$$

(where the fact that $\eta(\phi)\leq\eta\left(\frac{F_{cl}+F_{cu}}{2}\right)$, $\forall\phi\in\mathcal{F}_c$, has been taken into account). Furthermore, let us consider that k_i satisfies

$$k_i \leq \frac{b_1b_2}{\bar{b}_0} = \frac{8F_hC_{ph}(F_{cl}M_h+F_hM_c)}{M_hM_cC_{pc}(F_{cu}-F_{cl})^2(T_{hi}-T_{ci})} \tag{10}$$

Under the satisfaction of this inequality, it turns out that $a_0=k_i\bar{a}_0<k_i\bar{b}_0\leq b_1b_2<a_1a_2$, *i.e.* $a_0<a_1a_2$ which is a necessary and sufficient condition for the three roots of $P(\lambda)$ to have negative real part (see for instance Example 6.2 in [3]). Thus, x^* is asymptotically stable. Its attractivity is global on $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ if $\{x^*\}$ is the only invariant in $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$, which is the case for small enough values of k_p and k_i. Indeed, from boundedness of $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ and its positive invariance with respect to the closed-loop system dynamics, every solution $x(t;x_0\in\mathcal{D}\times\mathrm{int}(\mathcal{F}_c))$ has a nonempty, compact, and invariant positive limit set $\mathcal{L}^+$, and $x(t;x_0)\to\mathcal{L}^+$ as $t\to\infty$, $\forall x_0\in\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ (see [6, Lemma 3.1]). Then, the global attractivity of x^* on $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ is subject to the absence of periodic orbits on $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$ (implying $\mathcal{L}^+=\{x^*\}$). Sufficiently small values of k_p and k_i render the closed loop a *slowly varying system* (see [6, §5.7]). Then, the 3rd-order closed-loop dynamics can be approximated by the 2nd-order system in (2) with (quasi) constant F_c. Since under such representation no closed orbits can take place[2], we deduce the absence of periodic solutions of the closed-loop (3rd-order) system on $\mathcal{D}\times\mathrm{int}(\mathcal{F}_c)$. Thus, in conclusion: $T_{ho}(t)\to T_{hd}$ as $t\to\infty$. ■

[2]This is verified through Bendixon's Criterion [6, Thrm. 7.2], since $\frac{\partial\bar{f}_1}{\partial y_1}+\frac{\partial\bar{f}_2}{\partial y_2}=-a_2<0$, $\forall y\in\mathcal{D}$, as was stated and shown in [17].

Remark 4: Notice, from the proof of Proposition 1, that inequality (10) may be taken as an *a priori* control gain tuning criterion for k_i. However, it is worth to note that such a condition is not necessary and that it might be conservative.

Remark 5: Observe that the proposed approach does not need to feedback the whole extended state vector. No measurements of T_{co} are required for its implementation. Furthermore, the exact knowledge of the accurate values of the system parameters is not needed. Such features characterize the proposed algorithm as a simple controller that gives rise to a control signal evolving within its physical limits. This way, undesirable phenomena, such as *wind-up*, are avoided.

Remark 6: Observe that the proposed controller may be equivalently expressed in the following PI form

$$F_c(t) = [\kappa(\phi; k_p)e](t) + \int_0^t [\kappa(\phi; k_i)e](\tau)d\tau + \phi_0$$

where e is the error variable defined as $e \triangleq T_{ho} - T_{hd}$; κ is a variable gain defined as $\kappa(\phi; k) \triangleq k\eta(\phi)$, with $k = k_p$ for the *proportional* (P) term and $k = k_i$ for the *integral* (I) action; $[\kappa(\phi; k)e](t) = \kappa(\phi(t); k)e(t)$; ϕ is the dynamical auxiliary variable calculated through the auxiliary dynamics in Eq. (5); and ϕ_0 is the value of ϕ at the initial time $t = 0$, *i.e.* $\phi_0 = \phi(0) \in \text{int}(\mathcal{F}_c)$. Note that the non-linear term $\eta(\phi)$, involved in the variable gains κ, states the difference with respect to a conventional PI control law. It is actually thanks to such a non-linear term, $\eta(\phi)$, that the input flow rate, F_c, is kept within its physical limits. Indeed, observe that, in view of expression (8) (guaranteed through the satisfaction of inequality (6)), for any $\phi(0) \in \text{int}(\mathcal{F}_c)$, F_c is not able to go beyond the lower and upper bounds of $\mathcal{F}_c$ since, at F_{cl} or F_{cu}, ϕ stops evolving. Moreover, due to the repulsive (unstable) nature of the consequent equilibrium points, x_l^* and x_u^*, appearing on the boundary of $\mathcal{D} \times \mathcal{F}_c$, such limit values of F_c, *i.e.* F_{cl} and F_{cu}, cannot even be asymptotically approached.

V. Experimental results

In order to verify the effectiveness of the proposed controller, experiments were carried out on a bench-scale pilot plant consisting of a completely instrumented double-pipe heat exchanger. The plant operates as a water-cooling process, with the hot water flowing through the internal tube and the cooling water flowing through the external pipe, and may be configured in either counter or parallel flow configuration. More details (such as the parameter calculated values) of such an experimental setup can be found in [2]. The inlet temperatures were kept constant at $T_{ci} = 29\,°\text{C}$ (measured by means of a RIY-Moore temperature transmitter) and $T_{hi} = 65\,°\text{C}$ (measured *via* an Engelhard Pyro-Controle Pt-100 temperature transmitter). The flow rates were measured *via* Platon flowmeters. The hot fluid flow rate was fixed at $F_h = 16.7 \times 10^{-3}$ kg/sec. The cold fluid flow rate, F_c, was made vary between $F_{cl} = 0.8 \times 10^{-3}$ kg/sec and $F_{cu} = 10.8 \times 10^{-3}$ kg/sec by means of a pneumatic valve. The cold fluid outlet temperature, T_{co}, was measured using

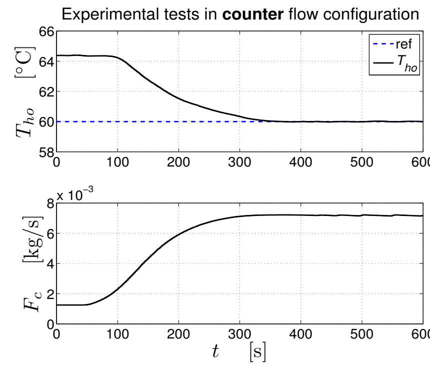

Fig. 2. Closed-loop response and control signal with the proposed approach (Prop. 1) in counter flow configuration

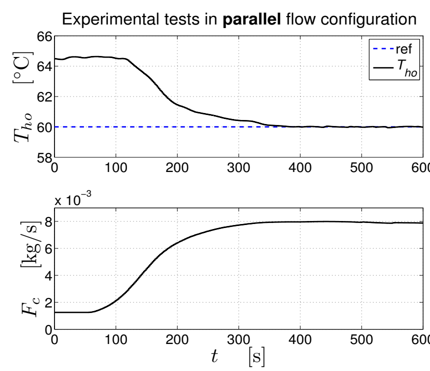

Fig. 3. Closed-loop response and control signal with the proposed approach (Prop. 1) in parallel flow configuration

an Engelhard Pyro-Controle Pt-100 temperature transmitter. The controller gains and initial condition were fixed at $k_p = 2.5\,\text{sec}/(°\text{C}\cdot\text{kg})$, $k_i = 0.67\,[1/(°\text{C}\cdot\text{kg})]$ and $F_c(0) = 1.25 \times 10^{-3}$ kg/sec. The desired outlet temperature was defined as $T_{hd} = 60\,°\text{C}$. Keeping these values and reproducing the same initial (steady-state) conditions at every test, experiments were run in counter flow configuration.

Fig. 2 shows the evolution of the controlled outlet temperature, T_{ho}, and the control variable, F_c, for the tests developed in counter flow configuration. The evolution of the same variables resulting from the implementation of the proposed controller in parallel flow configuration are shown in Fig. 3. Observe that, in both configuration cases, the control objective is achieved in less than 400 s, with a control signal varying within its physical limits.

For comparison purposes, the linearizing feedback approach developed in [10] was implemented in counter flow configuration. The complete control law, considering the complemented LMTD expression in Eq. (1), is shown in the appendix. The controller parameter values were tuned as suggested in [10] (see the appendix). Fig. 4 shows the closed-loop outlet temperature response and the input flow rate arisen with such a linearizing feedback scheme. Note that the stabilization time is considerably longer: more than 1000 s were needed to achieve the control objective. Moreover, notice that the closed-loop response is characterized by an

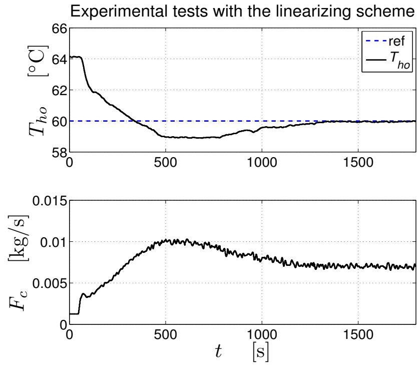

Fig. 4. Closed-loop response and control signal with the linearizing feedback scheme (appendix) in counter flow configuration

important overshoot. Furthermore, observe that the resulting control signal is noisy. This may be a consequence of the high dependence of this controller on the system states, on the one hand, and on the exact system dynamics, on the other (in view of the modelling errors). Finally, compare the complexity of the control expression in the appendix with the simplicity of the algorithm in Proposition 1.

VI. Conclusions

In this work, a bounded positive PI-type non-linear control scheme for the outlet temperature global regulation of double-pipe heat exchangers was proposed. The algorithm guarantees a control signal varying within its physical positive limits, which agrees with the bounded and unidirectional nature of the corresponding flow rate. Moreover, the proposed scheme turns out to be a simple algorithm that does not need to feedback the whole closed-loop state vector and does not depend on the exact knowledge of the system parameters. Experimental results corroborated the theoretical developments.

Appendix

Considering the LMTD complemented expression in Eq. (1), the linearizing feedback control scheme developed in [10] for countercurrent heat exchangers is given by

$$F_c = \frac{v - \ddot{T}_{hm} + \frac{\partial f_2}{\partial x_2}\dot{T}_{ho} - \frac{(2UA)^2}{M_c M_h C_{pc} C_{ph}} \frac{\partial \Delta T_L}{\partial T_{co}} \Delta T_L}{\frac{4UA}{M_c M_h C_{ph}} \frac{\partial \Delta T_L}{\partial T_{co}} (T_{ci} - T_{co})}$$

with $\frac{\partial f_2}{\partial x_2}$ as defined in (9),

$$v = -k_p e - k_i \int_0^t e(\tau) d\tau - k_d \dot{e}$$

$$e = T_{hm} - T_{ho}$$

T_{hm} is the state of a first order reference model defined as

$$\dot{T}_{hm} = -\lambda_m T_{hm} + \lambda_m T_{hd}$$

for some positive scalar λ_m, and

$$\Delta T_L = \begin{cases} \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}} & \text{if } \Delta T_2 \neq \Delta T_1 \\ \Delta T_0 & \text{if } \Delta T_2 = \Delta T_1 = \Delta T_0 \end{cases}$$

$$\frac{\partial \Delta T_L}{\partial T_{co}} = \begin{cases} \frac{\left[\ln \frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_1}\right]}{\left[\ln \frac{\Delta T_2}{\Delta T_1}\right]^2} & \text{if } \Delta T_2 \neq \Delta T_1 \\ -\frac{1}{2} & \text{if } \Delta T_2 = \Delta T_1 \end{cases}$$

$$\frac{\partial \Delta T_L}{\partial T_{ho}} = \begin{cases} \frac{\left[\ln \frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_2}\right]}{\left[\ln \frac{\Delta T_2}{\Delta T_1}\right]^2} & \text{if } \Delta T_2 \neq \Delta T_1 \\ \frac{1}{2} & \text{if } \Delta T_2 = \Delta T_1 \end{cases}$$

with $\Delta T_1 = T_{hi} - T_{co}$ and $\Delta T_2 = T_{ho} - T_{ci}$. The following tuning criterion is proposed in [10]: $k_p = \frac{25}{t_s^2 \xi^2}$, $k_i = \frac{125}{3 t_s^3 \xi^2}$, and $k_d = \frac{10}{t_s}$, for some positive constants t_s and ξ. Finally, the following values are suggested in [10] for a good regulatory response: $\lambda_m = 0.05\,[1/\text{sec}]$, $t_s = 60\,\text{sec}$, and $\xi = 0.75$ (these were, consequently, the values taken for the experimental tests in Section V).

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