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Networked Control Systems with Packet Delays and Losses

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Abstract— **We investigate the effect of packet delays and packet drops on networked control systems. First we consider the problem of where to locate a controller or state estimator in a network, and show that under a Long Packets Assumption (*LPA*) it is optimal to collocate it with the actuator. We then show that under the LPA, stabilizability is only determined by the packet drop probability and not the packet delay probabilities. We also consider a suboptimal state estimator without the LPA, based on inverting submatrices of the observability Krylov sequence.**

I. I

Contention for the medium, channel fading, and interference in networks, lead to packet delays and losses. Even though observations may be taken at regular instants, their arrivals after passage through the network may be random since collision detection and avoidance algorithms use random backoffs and delays, or they may even be dropped due to the losses in the wireless medium or collisions. Hence, we address the issue of random packet delays as well as packet drops in networked control systems.

We study an LQG system by employing a *Long Packets Assumption* (LPA) [15], which allows packets to be arbitrarily long, and in particular to contain a history of all past observations. The LPA can be realized even without long packets by having an encoder at the sensor and a decoder at the actuator, as shown in [5]. We first address the question of where the control logic should be placed within the network subject to random packet delays and losses, and show that it is optimal to collocate the controller with the actuator. This extends an earlier result for the case of packet drops only [14], [15].

Then we address the question of when such a system is stabilizable. We show that the condition

$$p_{Drop} \leq \frac{1}{\lambda_{\max}(A)^2}, \tag{1}$$

where $\lambda_{\max}(A)$ is magnitude of the eigenvalue of A with the largest magnitude is necessary, and also sufficient when the inequality is strict. This result shows that stabilizability under the LPA depends only on the loss probability and not the delay probabilities. Thus the condition for stabilizability is essentially the same as in the packet drop only case examined in [15], [18], [6], [19]. It is interesting that under LPA the delay distribution only affects system performance and *not* stabilizability. This is illustrated via simulations in Section IV. The result has implications not only for a system with long packets, but also, as mentioned above, for the encoder-decoder scheme in [5] which realizes the *LPA* but without long packets. The result on stabilizability not depending on delays but only on drop probability is analogous to that of [16], [17], where the estimation problem without the LPA is considered, and a similar independence of estimator stability on delays is shown. The stabilizability condition is different in that case since the LPA does not hold.

We next analyze "window" based schemes *without* an LPA, as has been considered in [16], [17]. We consider a family of suboptimal schemes and obtain upper bounds on packet drop probability, similar to (1), that are sufficient for their stability.

Useful references for networked control include [1], [3]. Packet delays and drops result in generally intractable non-classical information patterns [21], [12]. The effect of random sampling times on optimal controller design [2], state estimation [20], [13] and overall system performance [8] have been considered. Packet delays are considered in [11], with delay assumed to be less than one sampling period. Another approach is to focus on eliminating the effect of random delay. In [10], a buffer is maintained at the *receiving* end of the channel for randomly delayed packets, which releases them at regular intervals. A similar actuator buffer has actually been deployed in [4]. The condition (1) is studied vis-a-vis packet drops in [7], [18], [19].

II. C P

We begin by determining *where* to locate the controller; see Fig. 1. We show that placing it on a path with "best" delay characteristics is optimal.

Let $q(\cdot)$ be the probability mass function for delay on any *link* j. Packets are dropped with probability $1 - \sum_{t=0}^{\infty} q(t)$. Delays of packets on links are assumed iid. Let π_{hg} be the *path* of a packet from a node h to node g. The *path delay* is the sum of the link delays on the path, with probability distribution denoted by $F_{\pi_{hg}}$. We say that distribution F_1 *dominates* F_2 if $F_1(D) \geq$

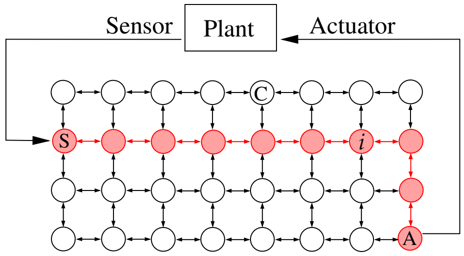


Fig. 1. The controller is at node C, and π^*_{SA} is a minimal path.

$F_2(D)\forall D$. A path between nodes h and g with the "best" delay characteristics, i.e., one whose delay distribution dominates those of all other paths between h and g, if one exists, is referred to as a 'shortest path' and is denoted by π^*_{hg}; see Figure 1.

Lemma 2.1: Suppose the same packet is sent along two chains of nodes C_1 and C_2, as in Fig. 2. Under the LPA, the information at each node in C_1 with more nodes is dominated by that at a corresponding node in C_2.

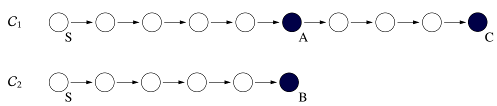


Fig. 2. Two chains of nodes with source node denoted by node S.

Proof: Node A on C_1 at the same hop distance from the source S as B (see Fig. 2, has $F_{\pi^*_{SA}} = F_{\pi^*_{SB}}$). The information arrival processes at these two nodes can be stochastically coupled. Since packets will be further delayed between nodes A and C in C_1, the information at node B stochastically dominates that at C. ∎

Corollary 2.2: Under the *LPA* there is an optimal controller placement that is on path π^*_{SA}.

Theorem 2.3: Placing the controller at the actuator is optimal.

Proof: Consider Case 1 with controller located at node $i \in \pi^*_{SA}$, and Case 2 where it is located at A. We stochastically couple the cases so that events of packets from i reaching A are identical. Hence, using LPA, node A can receive the same observation information in both cases. However, the controller located at A additionally has the history of all *implemented* controls. ∎

III. S P D M

Consider the observable and controllable system:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_j &= C_j x_j + v_j, \\ z_j &= \{y_j, y_{j-1}, \ldots y_0\}. \end{aligned} \tag{2}$$

Noises w_k and v_j are zero mean iid Gaussian processes with covariances Σ^w and Σ^v. Packet delays, x_0, $\{w_k\}$ and $\{v_k\}$ are mutually independent. y_j is the state observation made by the sensor at time j. z_j is the *transmitted observation*, and under the *LPA* is comprised of all *state observations* $\{y_k : 0 \le k \le j\}$. The sensor takes and transmits observations at each *sampling instant* k.

Control actions are implemented every s sample instants, and are held constant during the intermediate interval. These are called *actuation instants*. Transmitted observations are subject to delay. Since control actions are only implemented at actuation instants, a delayed measurement can arrive at any time, but can only be used at the next actuation instant.

We now describe the probabilistic model for the *end-to-end* packet delay; see Figure 3. A packet transmitted at time j will contain z_j under the *LPA* and is subject to delay. There is no guarantee that packets will arrive in order. Delays of individual packets are i.i.d.

p_i	:=Prob(Packet transmitted at k arrives at $k+i$).
p_{Drop}	:=Prob(Packet is dropped) $= 1 - \sum_{i=0}^{\infty} p_i$.
$\bar{p}_i$	$:= P\,[\text{Packet delay} \ge i] = p_{Drop} + \sum_{j=i}^{\infty} p_j$.
$\mathcal{I}(k)$	:=Set of observations known to *controller* at k is the *information set*.
$\alpha(k)$	:= *age* of the information set $= k-i$, if the latest packet to have arrived before actuation instant k is z_i. Note that $\mathcal{I}(k) = \{y_j : j \le k - \alpha(k)\}$.
$\Pi_\alpha(k)$	$:= P(\alpha(k) = \alpha)$. Note that $\Pi_\alpha := \lim_{k\to\infty} \Pi_\alpha(k)$ exists $\forall \alpha$. Denote $\Pi := \{\Pi_0, \Pi_1, \ldots\}$.

With (a) below representing packet $k-\alpha$ arriving by k, and (b) below packets $\{k-\alpha+1, k-\alpha+2, \ldots, k\}$ that did not arrive by time k, see Figure 3, we can write:

$$\Pi_\alpha(k) = \begin{cases} \overbrace{(1-\bar{p}_{\alpha+1})}^{(a)}\overbrace{\bar{p}_1\bar{p}_2\bar{p}_3\ldots\bar{p}_\alpha}^{(b)} & \text{for } \alpha \le k \\ 0 & \text{for } \alpha > k. \end{cases},$$

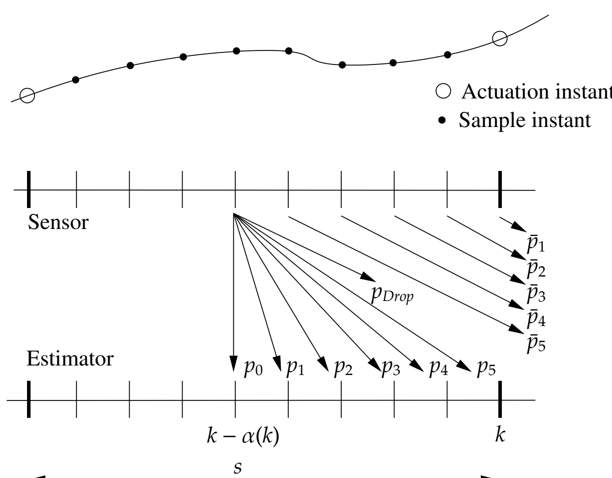


Fig. 3. State observations occur at *sample instants*. Control actions are computed and implemented at *actuation instants*. Actuation instants are separated by s sample instants. Sensor observations are sent over the network and incur a delay i with probability p_i, or are dropped with probability $p_{Drop} = 1 - \sum_{i=0}^{\infty} p_i$.

IV. N C B E E C

We address boundedness of the quadratic cost:

$$J = \limsup_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} x_k Q x_k' + u_k R u_k'\right].$$

Denoting by $\Sigma_k^{\tilde{x}}$ the state estimation error covariance at k, which is random because of the randomness in obtaining measurements, we can write

$$J = \limsup_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} \left(\hat{x}_k Q \hat{x}_k' + u_k R u_k' + Tr(\Sigma_k^{\tilde{x}} Q)\right)\right],$$

So we need only study boundedness of $E\left[\Sigma_k^{\tilde{x}}\right]$.

A. State Estimation: Kalman Filter

The "Time update" is, with the usual notation:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k, \quad (3)$$
$$\Sigma_{k+1|k}^{\tilde{x}} = A\Sigma_{k|k}^{\tilde{x}} A' + \Sigma^w, \quad (4)$$

The "Measurement update" is:

$$K_{k+1} = \Sigma_{k+1|k}^{\tilde{x}} C' \left(C\Sigma_{k+1|k}^{\tilde{x}} C' + \Sigma^v\right)^{-1}, \quad (5)$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_k - c\hat{x}_{k+1|k}\right), \quad (6)$$
$$\Sigma_{k+1|k+1}^{\tilde{x}} = \left(I - K_{k+1}C\right)\Sigma_{k+1|k}^{\tilde{x}}, \quad (7)$$

At each actuation sample instant, (3) and (4) are used to update the system state estimate. Iterating D steps,

$$\Sigma_{k+D|k}^{\tilde{x}} = A^D \Sigma_{k|k}^{\tilde{x}} A'^D + \sum_{i=0}^{D-1} A^i \Sigma^w A'^i. \quad (8)$$

If there is no packet loss or delay, then since $[A, \Sigma^{w\frac{1}{2}}]$ is controllable and $[A, C]$ detectable, the error covariance converges to a positive definite limit, which we denote by $\Sigma_{\tilde{x}}^*$. For simplicity we assume that the system is started with $\Sigma_{0|0}^{\tilde{x}} = \Sigma_{\tilde{x}}^*$. Under the *LPA*, whenever a packet z_j arrives, the estimation error covariance $\Sigma_{j|j}^{\tilde{x}}$ reverts to $\Sigma_{\tilde{x}}^*$. Hence it is the durations of the excursions from $\Sigma_{\tilde{x}}^*$ that determine boundedness.

B. Bounded Estimation Error Covariance

The expected estimation error covariance is $E\left[\Sigma^{\tilde{x}}\right] = \sum_{i=0}^{\infty} \Pi_i \Sigma_{k+i|k}^{\tilde{x}}$.

Theorem 4.1: (1) is necessary for bounded cost.

Proof: $\Pi_\alpha = \left(\sum_{j=0}^{\alpha} p_j\right)\left(\prod_{j=1}^{\alpha} \bar{p}_j\right)$. So, using (8),

$$\begin{aligned} E\left[\Sigma^{\tilde{x}}\right] &= \sum_{i=0}^{\infty} \Pi_i \Sigma_{k+i|k}^{\tilde{x}} \\ &= \sum_{i=0}^{\infty} \Pi_i \left(A^i \Sigma_{\tilde{x}}^* A'^i + \sum_{j=0}^{i-1} A^j \Sigma^w A'^j\right) \quad (9) \\ &\geq \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \bar{p}_j\right)\left(\sum_{j=0}^{i} p_j\right) A^i \Sigma_{\tilde{x}}^* A'^i. \end{aligned}$$

If v is the eigenvector of A corresponding to the eigenvalue of A with the largest magnitude, $\lambda_{\max}(A)$, then:

$$v' E\left[\Sigma^{\tilde{x}}\right] v \geq (v'\Sigma_{\tilde{x}}^* v) \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \bar{p}_j\right)\left(\sum_{j=0}^{i} p_j\right) \lambda_{\max}(A)^{2i}.$$

We now use the ratio test, noting that $\lim_{i\to\infty} \bar{p}_i = p_{Drop}$:

$$\begin{aligned} &\limsup_{i\to\infty} \frac{\left(\prod_{j=1}^{i+1} \bar{p}_j\right)(1 - \bar{p}_{i+2})\,\lambda_{\max}(A)^{2i+2}}{\left(\prod_{j=1}^{i} \bar{p}_j\right)(1 - \bar{p}_{i+1})\,\lambda_{\max}(A)^{2i}} \\ &= \limsup_{i\to\infty} \frac{\bar{p}_{i+1}(1 - \bar{p}_{i+2})}{(1 - \bar{p}_{i+1})} \lambda_{\max}(A)^2 \\ &= p_{Drop} \lambda_{\max}(A)^2, \end{aligned}$$

establishing the necessity of (1). ■

Theorem 4.2: A sufficient condition for boundedness of $E\left[\Sigma^{\tilde{x}}\right]$ is

$$p_{Drop} < \frac{1}{\lambda_{\max}(A)^2}.$$

Proof: Because $\gamma_1 \mathbb{I} \leq \Sigma_{\tilde{x}}^* \leq \gamma_2 \mathbb{I}$, and the same is true for $A^i \Sigma_W A'^i$, we only need to consider boundedness of:

$$\begin{aligned} E\left[\Sigma^{\tilde{x}}\right] &= \sum_{i=0}^{\infty} \Pi_i \Sigma_{k+i|k}^{\tilde{x}} \\ &\leq \gamma_3 \sum_{i=0}^{\infty} \Pi_i A^i A'^i \\ &\leq \gamma_3 \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \bar{p}_j\right) \lambda_{\max}(A)^{2i}, \end{aligned}$$

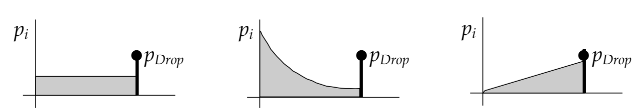
where we have upper bounded several terms by 1. By the ratio test, a sufficient condition for stability is:

$$\begin{aligned} \limsup_{i\to\infty} \frac{\left(\prod_{j=1}^{i+1} \bar{p}_j\right)\lambda_{\max}(A)^{2i+2}}{\left(\prod_{j=1}^{i} \bar{p}_j\right)\lambda_{\max}(A)^{2i}} &< 1 \\ p_{Drop}\lambda_{\max}(A)^2 &< 1. \end{aligned}$$

■

Hence stabilizability under *LPA only* depends on packet drop probability and system dynamics. Performance is adversely affected by larger delay, but not stabilizability. This has potential design implications. One can set the critical number of delivery attempts at the transport layer, similar to the MAC layer "retry limit" in IEEE 802.11, so as to meet a desired p_{Drop}.

The performance of several specific delay distributions in Figure 4 is illustrated in Figures 5, 6 and 7.



(a) Uniform (b) Exponential (c) Positve

Fig. 4. The drop probability, p_{Drop}, is chosen to be the same. The density function, is chosen so to satisfy $\sum_{i=0}^{\infty} p_i + p_{Drop} = 1$.

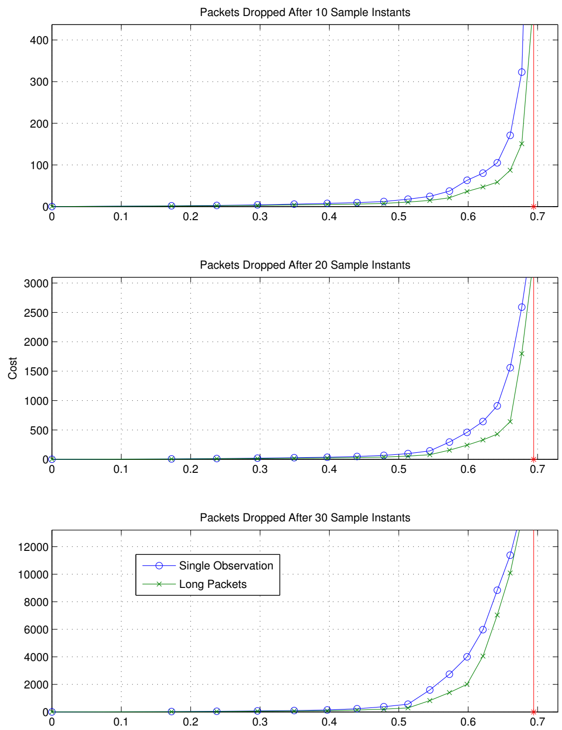


Fig. 5. Packets are delivered with uniform probability of delay as shown in Fig. 4(a). In the top figure, packets delayed by more than 10 sample instants are dropped. In the second and third figures the delay threshold is 20 and 30, respectively. The x-axis is the packet drop probability. The lower curve in each is the cost under LPA, and the upper curve that for system which only transmits a single observation. Notice that the cost diverges at the same packet drop probability in each figure.

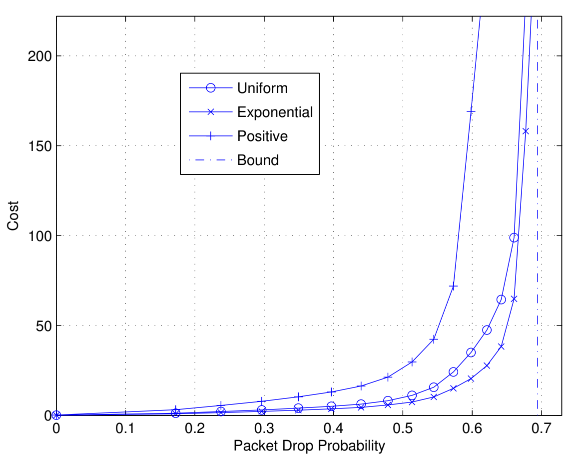


Fig. 6. Packets delayed 30 time steps are dropped. The lowest curve is the cost for exponential packet delay distribution (Figure 4(b)), the middle curve for uniform distribution (Figure 4(a)), and upper curve for linear distribution (Figure 4(c)).

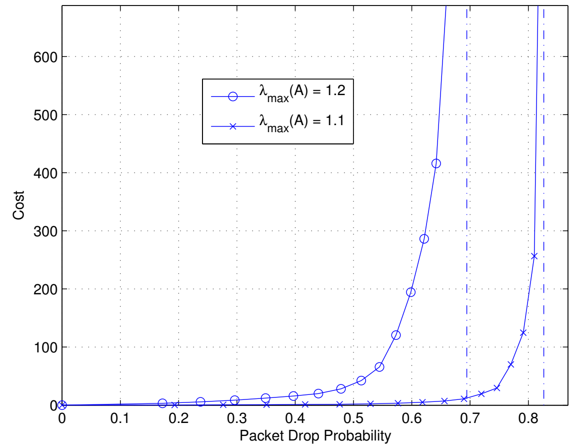


Fig. 7. Each curve represents a different system. Packets are delayed with uniform probability (Figure 4(a)), and dropped when delay exceeds 30. The stability upper bound on p_{Drop} is indicated by the vertical asymptotes. The upper curve represents a system with a largest eigenvalue of 1.2 and hence a stability bound of $p_{Drop} = 0.69$. The lower curve is a system with largest eigenvalue as 1.1 and stability bound of $p_{Drop} = 0.83$.

V. S S C LPA

We consider a suboptimal scheme without LPA.

A. A Sub-Optimal Estimation Scheme

For an n-dimensional observable system, n consecutive observations yield a state estimate with bounded error covariance [9], denoted $\bar{\Sigma}^*_{\tilde{x}}$. We consider a sub-optimal filter that uses only the most recent set of n *consecutive* measurements that have arrived. Open loop prediction is done in between such batches of n or more consecutive observation arrivals.

Denote by n_k the elapsed time since the most recent time at which n consecutive packets were delivered and the current time k. Denote this set of n observations as:

$$Y(k) \quad = \quad \left\{ y_j : k - n_k - n + 1 \le j \le k - n_k \right\}.$$

The resulting estimation error covariance at k, $\hat{\Sigma}^{\tilde{x}}_k$ is:

$$\hat{\Sigma}^{\tilde{x}}_{k|n_k} = A^{n_k\prime} \bar{\Sigma}^*_{\tilde{x}} A^{n_k} + \sum_{j=0}^{n_k-1} A^{j\prime} \Sigma^w A^j.$$

n_k being random, we compute the expectation $E[E[\hat{\Sigma}^{\tilde{x}}_{k|n_k} | n_k]]$.

Theorem 5.1: The expected estimation error covariance is bounded if

$$\liminf_{k\to\infty} \frac{p_k}{\bar{p}_{k+1}} > 1 - \frac{1}{\lambda_{\max}(A)^2}. \tag{10}$$

Proof:

If packets $(k-n_k-n+1, k-n_k-1-n+2, \ldots, k-n_k)$ are the *latest* n consecutive packets to arrive before k, then packet $k-n_k+1$ should have *not* arrived by k; see Fig. 8.

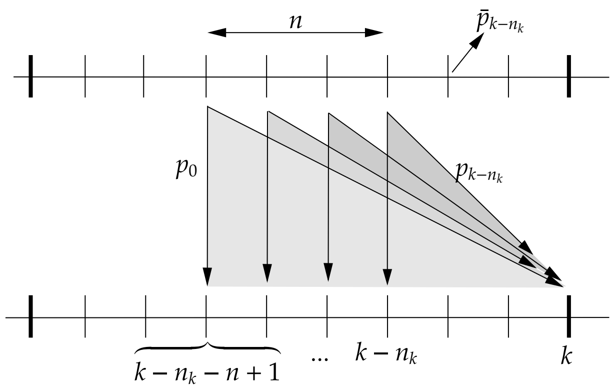


Fig. 8. The Figure illustrates an event where n consecutive packets arrive before time k. The packets which arrive may do so at anytime between their transmission time and time k. The potential arrival times are represented by the shaded area. The sequence of consecutive packets is broken by a packet being delayed with probability $\bar{p}_{n_k}$.

This non-arrival occurs with probability $\bar{p}_{n_k}$. Hence,

$$P\left[\begin{array}{c}\text{Packets } (k-n_k-n+1, k-n_k-n+2, \ldots, k-n_k)\\ \text{are the last } n \text{ consecutive ones to arrive before } k\end{array}\right]$$
$$\leq \left(\sum_{i=0}^{n_k+n-1} p_i\right)\left(\sum_{i=0}^{n_k+n-2} p_i\right)\ldots\left(\sum_{i=0}^{n_k} p_i\right)\bar{p}_{n_k}. \qquad (11)$$

Using (11) and (8) we can compute an upper bound

$$E[\bar{\Sigma}^{\tilde{x}}_{k|k-n_k}] \leq \sum_{n_k=0}^{\infty}\left\{\left(\sum_{i=0}^{n_k+n-1} p_i\right)\left(\sum_{i=0}^{n_k+n-2} p_i\right)\ldots\left(\sum_{i=0}^{n_k} p_i\right)\bar{p}_{n_k} \cdot \left(A^{n_k}\hat{\Sigma}^*_{\tilde{x}}A'^{n_k} + \sum_{j=0}^{n_k-1} A^j\Sigma^w A'^j\right)\right\}.$$

It is sufficient to consider the boundedness of

$$\sum_{n_k=0}^{\infty}\left\{\left(\sum_{i=0}^{n_k+n-1} p_i\right)\left(\sum_{i=0}^{n_k+n-2} p_i\right)\ldots\left(\sum_{i=0}^{n_k} p_i\right)\bar{p}_{n_k}\left(A^{n_k}A'^{n_k}\right)\right\}$$
$$\leq \sum_{n_k=0}^{\infty}\left\{\left(\sum_{i=0}^{n_k+n-1} p_i\right)\left(\sum_{i=0}^{n_k+n-1} p_i\right)\ldots\left(\sum_{i=0}^{n_k+n-1} p_i\right)\bar{p}_{n_k}\left(A^{n_k}A'^{n_k}\right)\right\}$$
$$= \sum_{n_k=0}^{\infty}\underbrace{\left(\sum_{i=0}^{n_k+n-1} p_i\right)^n}_{\leq 1}\bar{p}_{n_k}\left(A^{n_k}A'^{n_k}\right)$$
$$\leq \sum_{n_k=0}^{\infty}\bar{p}_{n_k}\left(A^{n_k}A'^{n_k}\right) \leq \sum_{n_k=0}^{\infty}\bar{p}_{n_k}\lambda_{\max}(A)^{2n_k}.$$

For boundedness, by the ratio test it is sufficient if

$$\limsup_{n_k\to\infty}\frac{\bar{p}_{n_k+1}\lambda_{\max}(A)^{2n_k+2}}{\bar{p}_{n_k}\lambda_{\max}(A)^{2n_k}} < 1.$$

This is assured if

$$\limsup_{n_k\to\infty}\left(1-\frac{p_{n_k}}{\bar{p}_{n_k}}\right) < \frac{1}{\lambda_{\max}(A)^2}. \qquad (12)$$

■

Interestingly, the condition is independent of n. For a geometric delay $p_k = (1-p)^k p$ and $\bar{p}_k = (1-p)^k$ the sufficient condition is $p > 1 - \frac{1}{\lambda_{\max}(A)^2}$. This is illustrated in Figure 9 as the $\gamma = 0$ curve.

B. An improved suboptimal scheme

An improved estimator can use *any* n observations from $n+\gamma$ consecutive observations. Denote by γ the largest integer such that *any* r observations drawn from $\{y_{\bar{k}-r-\gamma+1}, y_{\bar{k}-r-\gamma+2}, \ldots, y_{\bar{k}}\}$ yield an estimate of $x_{\bar{k}}$ with bounded error covariance, and call $\bar{k}$ an *estimation epoch*, γ the *number of allowable misses*, and the estimator an *allowable misses estimator*.

Theorem 5.2: With γ allowable misses, a state estimate with bounded error covariance can be formed if

$$\liminf_{k\to\infty}\frac{p_k}{\bar{p}_k} > 1 - \left(\frac{1}{\lambda_{\max}(A)^2}\right)^{\frac{1}{\gamma+1}}. \qquad (13)$$

Proof: The set of observations used at $\bar{k}$ is

$$Y(\bar{k}) := \{y_j : \bar{k}-r-\gamma+1 \leq j \leq \bar{k} \text{ and } y_j \text{ delivered}\}. \qquad (14)$$

Define the oldest possible observation in $Y(\bar{k})$ as $\underline{k} := \bar{k}-r-\gamma+1$. Say that $Y(\bar{k})$ is *full* if it is missing exactly γ observations in the interval $\bar{k}-n-\gamma+1 \leq j \leq \bar{k}$.

For $\bar{k}$ to be the most recent estimation epoch, clearly packet $y_{\bar{k}+1}$ should have been dropped, and in $[\bar{k}-r-\gamma+1, \bar{k}]$ there need to be γ drops. Hence,

$$P\left[\begin{array}{c}\bar{k}\text{ is the most}\\ \text{recent epoch before } k\end{array}\right] \leq \underbrace{\binom{r+\gamma}{\gamma}\bar{p}_{k-\bar{k}}\bar{p}_{k-\bar{k}}\ldots\bar{p}_{k-\bar{k}}}_{\gamma\text{ missing from } r+\gamma}\cdot\underbrace{\bar{p}_{k-\bar{k}}}_{\substack{y_{\bar{k}+1}\\ \text{missing}}} \leq K\bar{p}^{\gamma+1}_{k-\bar{k}},$$

for a sufficiently large constant K. Substituting $j = k-\bar{k}$:

$$E[\Sigma^{\tilde{x}}_{k|k}] \leq \sum_{j=0}^{\infty}K\bar{p}_j^{\gamma+1}\left(A^j\Sigma^{\tilde{x}}_{k|Y(k)}A'^j + \sum_{i=0}^{j-1}A^i\Sigma^w A'^i\right),$$

It is enough to consider conditions for boundedness of

$$\sum_{j=0}^{\infty}\bar{p}_j^{\gamma+1}\left(A^jA'^j\right) \leq \sum_{j=0}^{\infty}\bar{p}_j^{\gamma+1}\lambda_{\max}(A)^{2j}. \qquad (15)$$

By the ratio test, this is bounded if

$$\limsup_{j\to\infty}\frac{\bar{p}_{j+1}^{\gamma+1}\lambda_{\max}(A)^{2(j+1)}}{\bar{p}_j^{\gamma+1}\lambda_{\max}(A)^{2j}} < 1, \quad \text{i.e., if}$$

$$\limsup_{j\to\infty}\left(1-\frac{p_j}{\bar{p}_j}\right) < \left(\frac{1}{\lambda_{\max}(A)^2}\right)^{\frac{1}{\gamma+1}}.$$

■

For a geometric delay distribution, the sufficient condition is $p > 1 - \left(\frac{1}{\lambda_{\max}(A)^2}\right)^{\frac{1}{\gamma+1}}$. This condition is illustrated in Figure 9 for different values of the term $(\gamma + 1)$.

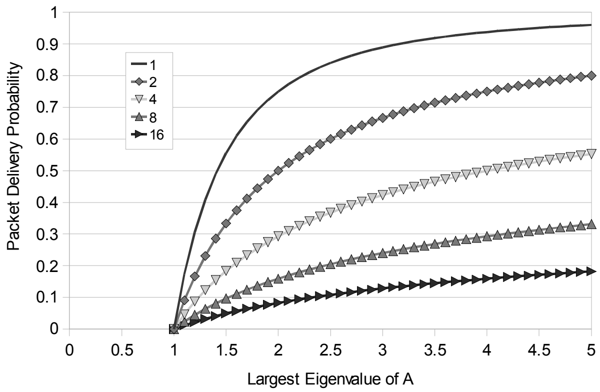


Fig. 9. Illustration of Thm. 5.2 for a geometric delay distribution for different values of the term $(\gamma + 1)$. The upper curve $(\gamma = 0)$ corresponds to Thm. 5.1.

C. Existence of γ

For random A, usually $\gamma \geq 1$. For example, $\gamma = 6$ for

$$A = \begin{bmatrix} 0.65510 & 0.58530 & 0.89090 & 0.84070 \\ 0.16260 & 0.22380 & 0.95930 & 0.25430 \\ 0.11900 & 0.75130 & 0.54720 & 0.81430 \\ 0.49840 & 0.25510 & 0.13860 & 0.24350 \end{bmatrix}, \qquad (16)$$

$(C = [1\ 0\ 0\ \ldots]$ throughout). Systems can have large γ. For example, A_1 has $\gamma > 70$, but A_2 yields $\gamma = 0$:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

VI. C

We have considered the effect of packet losses and delay on networked control system stabilizability under a Long Packets Assumption. We have established an optimal controller location and obtained a necessary condition for stability which is sufficient if the inequality is strict. The latter depends only on the drop probability and not the delay probability. We have also considered a sub-optimal scheme that may possibly be of interest.

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