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Stability and Robust Stability of Integral Delay Systems

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Abstract— **In this paper we consider a special class of integral delay systems arising in several stability problems of time-delay systems. For these integral systems we derive stability and robust stability conditions in terms of Lyapunov-Krasovskii functionals. More explicitly, after providing the stability conditions we compute quadratic functionals and apply them to derive exponential estimates for solutions, and robust stability conditions for perturbed integral delay systems.**

Keywords: integral delay systems, Lyapunov-Krasovskii functionals, stability, robust stability

I. INTRODUCTION

Recently, in [3] and [5] it has been shown that some system transformations, commonly used to obtain delay-dependent stability conditions for time-delay systems, introduce additional dynamics. Additional dynamics are responsible for the lack of equivalence in the stability property of the original and the transformed systems, *i.e.,* the original system may be stable while the transformed one is not. It has been shown in [6] that the additional dynamics are described by a special integral delay system. Stability of the integral system is a necessary condition for the stability of the transformed system [6].

Similar integral systems describe the internal dynamics of the controllers used for finite spectrum assignment of time-delay systems [7]. It has been demonstrated in [2], [8] and [10] that the internal stability of such controllers is an essential condition for their successful implementation.

Integral delay systems also appear in the stability analysis of the difference operator of some neutral type functional differential equations. Stability of the difference operator is a necessary condition for stability of the neutral type functional differential equation, see [4] for details.

These three different sources of stability problems associated with time-delay systems motivate us to look for stability and robust stability conditions of a special class of integral delay systems. To the best of our knowledge, no attempt has been made to derive such conditions for integral delay systems in terms of Lyapunov-Krasovskii functionals.

We present such stability conditions, and give a statement guaranteeing the existence of Lyapunov-Krasovskii functionals for the integral delay systems, which leads to a procedure for finding the functionals. Finally, we show how the functionals can be used to derive exponential estimates for solutions, and robust stability conditions for perturbed systems.

The paper is organized as follows: In section II, we shortly describe how the additional dynamics appear in time-delay systems. We discuss the internal stability problem of controllers used for finite spectrum assignment of time-delay systems, and the stability problem of difference operators in neutral functional differential equations. We also introduce the class of integral delay systems that will be studied in the paper. Section III presents some preliminary results. The existence, uniqueness and stability of solutions are briefly discussed. The Cauchy formula for solutions of integral delay systems is also presented. In section IV, Lyapunov-Krasovskii stability conditions for integral delay systems are given. A converse result, providing a constructive procedure for computing Lyapunov-Krasovskii functionals of a given exponentially stable integral delay system, is presented in section V. Functionals are used to derive exponential estimates for the solutions in section VI. In section VII, we show how functionals can be used for the robust stability analysis of perturbed systems. Concluding remarks end the paper.

II. MOTIVATION OF THE INTEGRAL SYSTEMS

We first briefly describe how additional dynamics appear in the stability analysis of time-delay systems.

Let us consider a time-delay system of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} G(\theta)x(t+\theta)d\theta, \quad (1)$$

where A_0, A_1 are $n \times n$ real constant matrices, delay $h > 0$, and $G(\theta)$ is a continuous matrix function defined for $\theta \in [-h, 0]$.

In order to obtain delay-dependent stability conditions for (1) one usually applies a special transformation to the system, see [9]. The aim of the transformation is to present the system in a form more suitable for the stability analysis. The transformation replaces in (1) the delay terms $x(t-h)$ and $x(t+\theta)$ by the Newton-Leibnitz formula

$$\begin{aligned} x(t-h) &= x(t) - \int_{-h}^{0} \dot{x}(t+\xi)d\xi, \\ x(t+\theta) &= x(t) - \int_{\theta}^{0} \dot{x}(t+\xi)d\xi, \end{aligned}$$

and substitutes the derivative under the integral by the right-hand side of (1). As a result, the transformed system can be written in the form, see [6],

$$\begin{cases} \dot{y}(t) = A_0y(t) + A_1y(t-h) + \int_{-h}^{0} G(\theta)y(t+\theta)d\theta + z(t) \\ z(t) = \int_{-h}^{0} \left(A_1 + \int_{-h}^{\theta} G(\xi)d\xi \right) z(t+\theta)d\theta. \end{cases} \quad (2)$$

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The second equation of the above system

$$z(t) = \int_{-h}^{0} \left(A_1 + \int_{-h}^{\theta} G(\xi) d\xi \right) z(t+\theta) d\theta \qquad (3)$$

describes the additional dynamics introduced by the transformation. Stability of the dynamics is a necessary condition for stability of (2).

We now address the problem of finite spectrum assignment for time-delay systems. Consider the linear system with delayed input

$$\dot{x}(t) = Ax(t) + Bu(t-h), \qquad (4)$$

where $h > 0$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent the state and control vectors, and A, B are real constant matrices of appropriate dimensions. The control law

$$u(t) = C \left[e^{Ah} x(t) + \int_{-h}^{0} e^{-A\theta} Bu(t+\theta) d\theta \right] \qquad (5)$$

assigns a finite spectrum to the closed-loop system (4)-(5) which coincides with the spectrum of the matrix $(A+BC)$, see [7]. Recently, in [2], [8] and [10] it has been shown that if the integral at the right-hand side of (5) is approximated by a finite sum, then the closed-loop system may become unstable if the controller (5) is not internally stable. So, the practical implementation of the control law (5) demands its internal stability. The internal dynamics of (5) are described by the integral system

$$z(t) = \int_{-h}^{0} Ce^{-A\theta} Bz(t+\theta) d\theta. \qquad (6)$$

Now consider a neutral functional differential equation of the form

$$\frac{d}{dt}[Dx_t] = Lx_t, \qquad (7)$$

where

$$\begin{aligned} Dx_t &= x(t) - \int_{-h}^{0} M(\theta) x(t+\theta) d\theta, \text{and} \\ Lx_t &= A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} N(\theta) x(t+\theta) d\theta. \end{aligned}$$

Here $M(\theta)$ and $N(\theta)$ are continuous real matrix functions defined for $\theta \in [-h, 0]$. Stability of the difference operator Dx_t is a necessary condition for stability of (7), see [4].

Operator Dx_t is stable if and only if the integral system

$$x(t) = \int_{-h}^{0} M(\theta) x(t+\theta) d\theta \qquad (8)$$

is stable. Comparing systems (3), (6) and (8) one can conclude that all of them are of the form

$$z(t) = \int_{-h}^{0} F(\theta) z(t+\theta) d\theta, \quad t \geq 0, \qquad (9)$$

where $F(\theta) = A_1 + \int_{-h}^{\theta} G(\xi) d\xi$ for the case of (3), $F(\theta) = Ce^{-A\theta}B$ for the case of (6), while for (8) $F(\theta) = M(\theta)$.

Therefore, the stability and robust stability of (9) turns out to be an interesting issue arising in several problems associated with time-delay systems.

We will assume that in (9), h is a positive constant and $F(\theta)$ is a continuously differentiable real matrix function on $[-h, 0]$, where a right-hand side continuous derivative at $-h$ and a left-hand side continuous derivative at 0 are assumed to exist. This differentiability assumption holds for the cases of additional dynamics and internal dynamics of control laws with distributed delay, while for the case of difference operator Dx_t, it imposes certain restrictions on matrix $M(\theta)$.

Throughout this paper we will use the Euclidean norm for vectors and the induced norm for matrices, both denoted by $\|\cdot\|$. We denote by A^T the transpose of A, I stands for the identity matrix, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of a symmetric matrix A, respectively.

III. Preliminaries

A. Solutions

In order to define a particular solution of (9) an initial vector function $\varphi(\theta)$, $\theta \in [-h, 0)$ should be given. We assume that φ belongs to the space of piecewise continuous vector functions $\mathbb{C}^0([-h, 0), \mathbb{R}^n)$, equipped with the uniform norm $\|\varphi\|_h = \sup_{\theta \in [-h,0)} \|\varphi(\theta)\|$.

For a given initial function $\varphi \in \mathbb{C}^0([-h, 0), \mathbb{R}^n)$, let $z(t, \varphi), t \geq 0$, be a solution of (9) satisfying $z(t, \varphi) = \varphi(t), t \in [-h, 0)$. This solution is continuously differentiable for $t \in [0, \infty)$, at $t = 0$ the right-hand side derivative is assumed, and it suffers a jump discontinuity

$$\Delta z(0, \varphi) = z(0, \varphi) - z(-0, \varphi) = \int_{-h}^{0} F(\theta) \varphi(\theta) d\theta - \varphi(-0).$$

Let us consider the time-delay system

$$\dot{y}(t) = F(0)y(t) - F(-h)y(t-h) - \int_{-h}^{0} \dot{F}(\theta) y(t+\theta) d\theta, t \geq 0. \qquad (10)$$

For a given initial function $\varphi \in \mathbb{C}^0([-h, 0), \mathbb{R}^n)$, define the function

$$\tilde{\varphi}(\theta) = \begin{cases} \varphi(\theta), & \theta \in [-h, 0) \\ \int_{-h}^{0} F(\xi) \varphi(\xi) d\xi, & \theta = 0 \end{cases}.$$

Denote by $y(t, \tilde{\varphi}), t \geq 0$, the solution of (10) with initial function $\tilde{\varphi}$. Existence and uniqueness of $z(t, \varphi)$, as well as some other properties of the solution, can be easily derived from the following statement.

Lemma 1: $y(t, \tilde{\varphi}) = z(t, \varphi)$.

Proof: Function $y(t, \tilde{\varphi})$ satisfies

$$\begin{aligned} \dot{y}(t, \tilde{\varphi}) &= F(0)y(t, \tilde{\varphi}) - F(-h)y(t-h, \tilde{\varphi}) \\ &\quad - \int_{-h}^{0} \dot{F}(\theta) y(t+\theta, \tilde{\varphi}) d\theta, \; t \geq 0. \end{aligned}$$

Then, integrating the equality from 0 to t we obtain

$$\begin{aligned} & y(t, \tilde{\varphi}) - \tilde{\varphi}(0) \\ &= F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi - F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi \\ &\quad - \int_{-h}^{0} \dot{F}(\theta) \left(\int_{\theta}^{t+\theta} y(\xi, \tilde{\varphi}) d\xi \right) d\theta \\ &= F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi - F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi \\ &\quad - F(0) \int_0^t y(\xi, \tilde{\varphi}) d\xi + F(-h) \int_{-h}^{t-h} y(\xi, \tilde{\varphi}) d\xi \\ &\quad + \int_{-h}^{0} F(\theta) \left[y(t+\theta, \tilde{\varphi}) - y(\theta, \tilde{\varphi}) \right] d\theta \\ &= \int_{-h}^{0} F(\theta) y(t+\theta, \tilde{\varphi}) d\theta - \int_{-h}^{0} F(\theta) \tilde{\varphi}(\theta) d\theta, \end{aligned}$$

which means that $y(t,\tilde{\varphi})$ satisfies (9). Assume now that $z(t,\varphi)$, $t\geq 0$, satisfies (9). Observe first that for $t\geq 0$

$$z(t,\varphi)=\int_{-h}^{0}F(\theta)z(t+\theta,\varphi)d\theta=\int_{t-h}^{t}F(\xi-t)z(\xi,\varphi)d\xi.$$

It follows that

$$\begin{aligned}\dot{z}(t,\varphi) &= F(0)z(t,\varphi)-F(-h)z(t-h,\varphi)\\ &\quad -\int_{t-h}^{t}\dot{F}(\xi-t)z(\xi,\varphi)d\xi\\ &= F(0)z(t,\varphi)-F(-h)z(t-h,\varphi)\\ &\quad -\int_{-h}^{0}\dot{F}(\theta)z(t+\theta,\varphi)d\theta,\end{aligned}$$

which implies that $z(t,\varphi)$ satisfies (10). By definition function $\tilde{\varphi}(\theta)$ coincides with $\varphi(\theta)$ for $\theta\in[-h,0)$, and

$$z(0,\varphi)=y(0,\tilde{\varphi})=\tilde{\varphi}(0)=\int_{-h}^{0}F(\xi)\varphi(\xi)d\xi.$$

■

Definition 1: System (9) is said to be exponentially stable if there exist $\alpha>0$ and $\mu>0$ such that every solution of (9) satisfies the inequality

$$\|z(t,\varphi)\|\leq\mu\|\varphi\|_{h}e^{-\alpha t},\ t\geq 0. \qquad (11)$$

Remark 1: System (10) is not exponentially stable. Indeed, any constant vector is a solution of (10).

B. Cauchy formula

In this subsection, we present the Cauchy formula for solutions of (9). This formula will play an important role in the construction of quadratic Lyapunov-Krasovskii functionals for (9). Let $K_1(t)$ be the fundamental matrix of (10). Matrix $K_1(t)$ is the solution of the matrix equation

$$\begin{aligned}\dot{K}_1(t)=K_1(t)F(0)-K_1(t-h)F(-h)\\ -\int_{-h}^{0}K_1(t+\theta)\dot{F}(\theta)d\theta,t\geq 0,\end{aligned}$$

with initial condition $K_1(t)=0$, $t\in[-h,0)$ and $K_1(0)=I$, see [1]. Then, the solution $y(t,\tilde{\varphi})$ of (10) can be written as

$$\begin{aligned}y(t,\tilde{\varphi})=K_1(t)\tilde{\varphi}(0)-\int_{-h}^{0}K_1(t-h-\theta)F(-h)\tilde{\varphi}(\theta)d\theta\\ -\int_{-h}^{0}\left(\int_{-h}^{\theta}K_1(t+\xi-\theta)\dot{F}(\xi)d\xi\right)\tilde{\varphi}(\theta)d\theta,t\geq 0.\end{aligned} \qquad (12)$$

The spectrum of (9) consists of all roots of the characteristic equation $\det\left(I-\int_{-h}^{0}F(\theta)e^{s\theta}d\theta\right)=0$.

Lemma 2: Suppose that the spectrum of (9) does not contain the point $s=0$, then matrix $K_1(t)$ can be written as

$$K_1(t)=K(t)+K_0, \qquad (13)$$

where $K_0=\left(I-\int_{-h}^{0}F(\theta)d\theta\right)^{-1}$, and $K(t)$ is the solution of the matrix equation

$$K(t)=\int_{-h}^{0}K(t+\theta)F(\theta)d\theta,\ t\geq 0, \qquad (14)$$

with initial condition $K(t)=-K_0$, $t\in[-h,0)$.

Proof: Note first that if $s=0$ does not belong to the spectrum of (9) then matrix $I-\int_{-h}^{0}F(\theta)d\theta$ is non-singular, so matrix K_0 is well defined. Observe now that matrix $K_1(t)$ satisfies the equality

$$K_1(t)=\int_{-h}^{0}K_1(t+\theta)F(\theta)d\theta+I,\ t\geq 0.$$

Introducing (13) into the above we obtain (14). The initial condition for matrix $K(t)$ follows directly from (13). ■

From (14) it follows that in spite of the fact that matrix $K_1(t)$ does not admit a strictly decreasing exponential upper bound, matrix $K(t)$ may do it.

Lemma 3: If $s=0$ does not belong to the spectrum of (9), then $z(t,\varphi),t\geq 0$, can be written as follows

$$\begin{aligned}z(t,\varphi) &= K(t)\int_{-h}^{0}F(\theta)\varphi(\theta)d\theta\\ &\quad -\int_{-h}^{0}K(t-h-\theta)F(-h)\varphi(\theta)d\theta\\ &\quad -\int_{-h}^{0}\left(\int_{-h}^{\theta}K(t+\xi-\theta)\dot{F}(\xi)d\xi\right)\varphi(\theta)d\theta.\end{aligned} \qquad (15)$$

Proof: From (12) and (13) we get, for $t\geq 0$,

$$\begin{aligned}&z(t,\varphi)=K(t)\int_{-h}^{0}F(\theta)\varphi(\theta)d\theta\\ &-\int_{-h}^{0}K(t-h-\theta)F(-h)\varphi(\theta)d\theta\\ &-\int_{-h}^{0}\left(\int_{-h}^{\theta}K(t+\xi-\theta)\dot{F}(\xi)d\xi\right)\varphi(\theta)d\theta\\ &+K_0\int_{-h}^{0}\left[F(\theta)-F(-h)-\int_{-h}^{\theta}\dot{F}(\xi)d\xi\right]\varphi(\theta)d\theta.\end{aligned}$$

Since the expression in the square brackets above is equal to zero, it follows that the Cauchy formula (15) holds. ■

Matrix $K(t)$ is known as the fundamental matrix of (9).

IV. A Lyapunov type theorem

In this section we give exponential stability conditions for system (9).

For any $t\geq 0$ we denote the restriction of the solution $z(t,\varphi)$ on the interval $[t-h,t)$ by $z_t(\varphi)=\{z(t+\theta,\varphi),\theta\in[-h,0)\}$. When the initial function φ is irrelevant we simply write $z(t)$ and z_t instead of $z(t,\varphi)$ and $z_t(\varphi)$. A simple inspection shows that for $t\in[0,h)$, $z_t(\varphi)$ belongs to $\mathbb{C}^0([-h,0),\mathbb{R}^n)$, and for $t\geq h$, $z_t(\varphi)$ belongs to the space of continuous vector functions $\mathbb{C}([-h,0),\mathbb{R}^n)$.

Theorem 4: System (9) is exponentially stable if there exists a functional $v:\mathbb{C}^0([-h,0),\mathbb{R}^n)\to\mathbb{R}$ such that the following conditions hold:

1) $\alpha_1\int_{-h}^{0}\|\varphi(\theta)\|^2d\theta\leq v(\varphi)\leq\alpha_2\int_{-h}^{0}\|\varphi(\theta)\|^2d\theta$, for some $0<\alpha_1\leq\alpha_2$;
2) $\frac{d}{dt}v(z_t(\varphi))\leq-\beta\int_{-h}^{0}\|z(t+\theta,\varphi)\|^2d\theta$, $t\geq 0$, for some $\beta>0$.

Proof: Given any $\varphi\in\mathbb{C}^0([-h,0),\mathbb{R}^n)$ it follows from the theorem conditions that for $2\alpha=\beta\alpha_2^{-1}>0$ the following inequality holds:

$$\frac{d}{dt}v(z_t(\varphi))+2\alpha v(z_t(\varphi))\leq 0,\quad t\geq 0.$$

Thus, on one hand it follows that

$$\begin{aligned}v(z_t(\varphi))&\leq e^{-2\alpha t}v(\varphi)\\ &\leq\alpha_2e^{-2\alpha t}\int_{-h}^{0}\|\varphi(\theta)\|^2d\theta\\ &\leq h\alpha_2e^{-2\alpha t}\|\varphi\|_h^2,t\geq 0.\end{aligned}$$

On the other hand, one gets

$$\begin{aligned}\|z(t,\varphi)\|^2 &\leq \left(m\int_{-h}^{0}\|z(t+\theta,\varphi)\|d\theta\right)^2\\ &\leq m^2h\int_{-h}^{0}\|z(t+\theta,\varphi)\|^2d\theta,\quad t\geq 0,\end{aligned}$$

where

$$m = \max_{\theta\in[-h,0]} \|F(\theta)\|.$$

We therefore have the following exponential upper bound:

$$\|z(t,\varphi)\| \le \mu \|\varphi\|_h e^{-\alpha t},\ t \ge 0,$$

where

$$\mu = mh\sqrt{\frac{\alpha_2}{\alpha_1}}.$$

■

V. A CONVERSE THEOREM

In this section we present a converse statement to theorem 4. More precisely, for an exponentially stable system (9) we construct a quadratic functional satisfying the conditions of the theorem.

Given positive definite matrices W_0 and W_1, let us define on $\mathbb{C}^0([-h,0),\mathbb{R}^n)$ the following functional:

$$w(\varphi) = \varphi^T(-h)W_0\varphi(-h) + \int_{-h}^0 \varphi^T(\theta)W_1\varphi(\theta)d\theta.$$

Remark 2: For $\beta = \lambda_{\min}(W_1)$ functional $w(\varphi)$ admits the following lower bound:

$$\beta\int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \le w(\varphi). \tag{16}$$

Let system (9) be exponentially stable. We aim at constructing a functional $v : \mathbb{C}^0([-h,0),\mathbb{R}^n) \to \mathbb{R}$, such that along the solutions of (9) the following equality holds:

$$\frac{d}{dt}v(z_t) = -w(z_t),\ t \ge 0. \tag{17}$$

To this end, for a positive definite matrix W, we first define a functional $v_0(\varphi,W)$ that satisfies

$$\frac{d}{dt}v_0(z_t,W) = -z^T(t)Wz(t),\ t \ge 0.$$

The functional is of the form

$$v_0(\varphi,W) = \int_0^\infty z^T(t,\varphi)Wz(t,\varphi)dt. \tag{18}$$

Observe that inequality (11) guarantees the existence of the improper integral at the right-hand side of (18).

Lemma 5: Let system (9) be exponentially stable. Then the functional $v(\varphi)$ satisfying (17) can be written as

$$\begin{aligned} v(\varphi) &= v_0(\varphi, W_0 + hW_1) \\ &+ \int_{-h}^0 \varphi^T(\xi)\left[W_0 + (\xi+h)W_1\right]\varphi(\xi)d\xi. \end{aligned} \tag{19}$$

Proof: Let $z(t)$, $t \ge 0$, be a solution of (9), then

$$\begin{aligned} v(z_t) &= v_0(z_t, W_0 + hW_1) \\ &+ \int_{-h}^0 z^T(t+\xi)\left[W_0 + (\xi+h)W_1\right]z(t+\xi)d\xi. \end{aligned}$$

We then have

$$\begin{aligned} &\frac{d}{dt}v(z_t) = \frac{d}{dt}v_0(z_t, W_0 + hW_1) \\ &+\frac{d}{dt}\int_{-h}^0 z^T(t+\xi)\left[W_0 + (\xi+h)W_1\right]z(t+\xi)d\xi \\ = \quad &-z^T(t)(W_0 + hW_1)z(t) + z^T(t)(W_0 + hW_1)z(t) \\ &-z^T(t-h)W_0z(t-h) \\ &-\int_{-h}^0 z^T(t+\xi)W_1z(t+\xi)d\xi \\ = \quad &-w(z_t). \end{aligned}$$

■

We now construct the functional $v_0(\varphi,W)$. Substituting (15) into (18) we obtain

$$\begin{aligned} &v_0(\varphi,W) = \tilde\varphi^T(0)U(0)\tilde\varphi(0) \\ &-2\tilde\varphi^T(0)\int_{-h}^0 U(-h-\theta)F(-h)\varphi(\theta)d\theta \\ &-2\tilde\varphi^T(0)\int_{-h}^0\int_{-h}^\theta U(\xi-\theta)\dot F(\xi)\varphi(\theta)d\xi d\theta \\ &+\int_{-h}^0\int_{-h}^0 \varphi^T(\theta_1)F^T(-h)U(\theta_1-\theta_2)F(-h)\varphi(\theta_2)d\theta_2d\theta_1 \\ &-\int_{-h}^0\int_{-h}^0 \varphi^T(\theta_1)F^T(-h)K_0^TW\left(\int_{-h-\theta_2}^{\theta_1-\theta_2}K(\xi)d\xi\right)\times \\ &\times F(-h)\varphi(\theta_2)d\theta_2d\theta_1 + 2\int_{-h}^0\int_{-h}^0\int_{-h}^{\theta_2}\varphi^T(\theta_1)F^T(-h)\times \\ &\times U(h+\theta_1+\xi-\theta_2)\dot F(\xi)\varphi(\theta_2)d\xi d\theta_2d\theta_1 \\ &-2\int_{-h}^0\int_{-h}^0\int_{-h}^{\theta_2}\varphi^T(\theta_1)F^T(-h)K_0^TW\times \\ &\times\left(\int_{\xi-\theta_2}^{h+\theta_1+\xi-\theta_2}K(\eta)d\eta\right)\dot F(\xi)\varphi(\theta_2)d\xi d\theta_2d\theta_1 \\ &+\int_{-h}^0\int_{-h}^{\theta_1}\int_{-h}^0\int_{-h}^{\theta_2}\varphi^T(\theta_1)\dot F^T(\xi_1)\times \\ &\times U(\theta_1-\xi_1+\xi_2-\theta_2)\dot F(\xi_2)\varphi(\theta_2)d\xi_2d\theta_2d\xi_1d\theta_1 \\ &-\int_{-h}^0\int_{-h}^{\theta_1}\int_{-h}^0\int_{-h}^{\theta_2}\varphi^T(\theta_1)\dot F^T(\xi_1)K_0^TW\times \\ &\times\left(\int_{\xi_2-\theta_2}^{\theta_1-\xi_1+\xi_2-\theta_2}K(\eta)d\eta\right)\dot F(\xi_2)\varphi(\theta_2)d\xi_2d\theta_2d\xi_1d\theta_1. \end{aligned}$$

Here $\tilde\varphi(0) = \int_{-h}^0 F(\theta)\varphi(\theta)d\theta$, and matrix function

$$U(\tau) = \int_0^\infty K^T(t)WK(t+\tau)dt,\ \tau\in[-h,h]. \tag{20}$$

The exponential stability of (9) guarantees the existence of the improper integral in (20).

Lemma 6: Let system (9) be exponentially stable. Then there exist constants $0 < \alpha_1 \le \alpha_2$, such that functional (19) satisfies the inequalities

$$\alpha_1\int_{-h}^0 \|\varphi(\theta)\|^2 d\theta \le v(\varphi) \le \alpha_2\int_{-h}^0 \|\varphi(\theta)\|^2 d\theta. \tag{21}$$

For the sake of brevity, we omit here the detailed proof of this lemma, and only give the explicit bounds for constants α_1 and α_2.

Thus, inequalities (21) hold for $0 < \alpha_1 \le \lambda_{\min}(W_0)$ and

$$\begin{aligned} &\alpha_2 \ge hu_0\left((m+\|F(-h)\|)^2 + r^2 + 2rm + 2\|F(-h)\|r\right) \\ &+\lambda_{\max}(W_0+hW_1) + \left(1 + mhe^{mh}\right)h^2\|K_0\|\times \\ &\times\left(\left\|F^T(-h)K_0^TW\right\|\|F(-h)\|\right. \\ &\left.+2\left\|F^T(-h)K_0^TW\right\|r + \left\|K_0^TW\right\|r^2\right), \end{aligned}$$

where $u_0 = \max_{\tau\in[-h,h]}\|U(\tau)\|$ and $r = \int_{-h}^0 \left\|\dot F(\theta)\right\| d\theta$.

We thus obtain the following converse Lyapunov-Krasovskii theorem for (9).

Theorem 7: Let system (9) be exponentially stable. Then for any given positive definite matrices W_0 and W_1 there exist positive constants α_1, α_2 and β such functional (19) satisfies the conditions of theorem 4.

Proof: Given any positive definite matrices W_0 and W_1, the exponential stability of (9) implies that functional

$v(\varphi)$ is defined by (19) and (20). Then, the proof follows from lemma 6 and remark 2. ∎

In order to compute $v_0(\varphi, W)$ one has to know matrix $U(\tau)$ for $\tau \in [-h, h]$. Expression (20) defines $U(\tau)$ as an improper integral. This is not convenient from a practical point of view. Hence we derive some important properties of $U(\tau)$ that will provide an alternative way to compute it.

Lemma 8: Matrix $U(\tau)$ satisfies

$$U(\tau) = \int_{-h}^{0} U(\tau+\theta)F(\theta)d\theta, \quad \tau \in [0, h]. \tag{22}$$

Proof: Substituting (14) into (20) we obtain (22). ∎

Lemma 9: Let $W^T = W$, then matrix $U(\tau)$ satisfies the symmetry property

$$U(\tau) = K_0^T W \int_0^{\tau} K(\xi)d\xi + U^T(-\tau), \tau \in [0, h]. \tag{23}$$

Proof: The statement can be easily verified by direct calculations. ∎

Corollary 10: Matrix $U(0)$ is symmetric, $U^T(0) = U(0)$.

Lemma 11: Matrix $U(\tau)$ satisfies

$$\int_{-h}^{0} \dot{U}(\theta)F(\theta)d\theta + \left[\int_{-h}^{0} \dot{U}(\theta)F(\theta)d\theta\right]^T = -K^T(0)WK(0). \tag{24}$$

Proof: Let us compute

$$\frac{d}{dt}\left[K^T(t)WK(t)\right] = \left(\int_{-h}^{0} \dot{K}(t+\theta)F(\theta)d\theta\right)^T WK(t) + K^T(t)W\left(\int_{-h}^{0} \dot{K}(t+\theta)F(\theta)d\theta\right).$$

Integrating this equality with respect to t from 0 to ∞ we obtain (24). ∎

Lemma 8 defines $U(\tau)$ as a solution of (22). In order to compute such a solution one needs to know the corresponding initial condition. The initial condition is not given explicitly. On the other hand, the symmetry property (23) along with (24) serve as a replacement of the unknown initial condition. Indeed, a piecewise linear approximation of matrix $U(\tau)$ for $\tau \in [-h, h]$ can be computed from equations (22)-(24).

Despite the fact that there are still no efficient algorithms to check the positivity of functionals of the form (19), and that functionals cannot be directly used for the stability analysis of (9), they can be directly applied to derive exponential estimates for solutions and robust stability conditions for perturbed systems. Such applications will be presented in the next two sections.

VI. Exponential estimates for the solutions

In this section, we show how functionals (19) can be used to obtain exponential estimates for the solutions of a given exponentially stable integral delay system (9).

Given positive definite matrices W_0 and W_1, based on lemma 6 and remark 2 we compute positive constants α_1, α_2 and β. Then the functional $v(\varphi)$ defined by (19) and (20) satisfies conditions 1 and 2 of theorem 4. From this theorem

$$\|z(t, \varphi)\| \leq hm\sqrt{\frac{\alpha_2}{\alpha_1}}\,\|\varphi\|_h\, e^{-\alpha t}, \; t \geq 0,$$

where $\alpha = \frac{\beta}{2\alpha_2}$, for all solutions $z(t, \varphi), \varphi \in \mathbb{C}^0([-h, 0), \mathbb{R}^n)$, of (9).

Now we illustrate the result by an example. Let us consider the following integral delay system:

$$z(t) = G\int_{-h}^{0} z(t+\theta)d\theta. \tag{25}$$

System (25) is a particular case of (9), where $F(\theta) = G$.

Let $h = 1$ and

$$G = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}.$$

As all eigenvalues of G lie in the open domain Γ whose boundary in the complex plane is described by

$$\partial\Gamma = \left\{ \frac{\omega \sin(\omega)}{2\left[1-\cos(\omega)\right]} + i\frac{\omega}{2} \;\middle|\; \omega \in (-2\pi, 2\pi) \right\},$$

system (25) is exponentially stable, see [5].

In this case functional (19) looks as

$$\begin{aligned} v(\varphi) &= \left(G\int_{-h}^{0}\varphi(\theta)d\theta\right)^T U(0)\left(G\int_{-h}^{0}\varphi(\theta)d\theta\right) \\ &-2\left(G\int_{-h}^{0}\varphi(\theta)d\theta\right)^T \int_{-h}^{0} U(-h-\tau)G\varphi(\tau)d\tau \\ &+\int_{-h}^{0}\int_{-h}^{0}\varphi^T(\theta_1)G^T U(\theta_1-\theta_2)G\varphi(\theta_2)d\theta_1 d\theta_2 \\ &+\int_{-h}^{0}\varphi^T(\theta)\left[W_0 + (\theta+h)W_1\right]\varphi(\theta)\,d\theta \\ &-\int_{-h}^{0}\int_{-h}^{0}\varphi^T(\theta_1)G^T K_0^T W \times \\ &\times\left(\int_{-h-\theta_2}^{\theta_1-\theta_2} K(\theta)d\theta\right)G\varphi(\theta_2)d\theta_1 d\theta_2. \end{aligned} \tag{26}$$

Let us assume that $W_0 = W_1 = I$. A piecewise linear approximation of matrix $U(\tau)$ is given on Fig. 1. From the computed values of $U(\tau)$ on $[-1, 1]$ we get $u_0 = 0.1752$.

From remark 2 we have that for $\beta = 1$, inequality (16) holds. Simple calculations derived from lemma 6 show that for $\alpha_2 = 2.7611 \times 10^3$ and $\alpha_1 = 1$, inequalities (21) hold.

So, solutions of (25) satisfy the following exponential estimate:

$$\|z(t, \varphi)\| \leq \mu\,\|\varphi\|_1\, e^{-\alpha t}, \; t \geq 0, \tag{27}$$

where

$$\mu \approx 245.43 \text{ and } \alpha \approx 1.8 \times 10^{-4}.$$

VII. Robust stability conditions

In this section we show how functionals (19) can be used to obtain robust stability conditions for perturbed integral systems by investigating the robust stability of the exponentially stable integral delay system (25) introduced in section VI. Thus, let us assume that the nominal system (25) is exponentially stable and consider the perturbed system

$$y(t) = \int_{-h}^{0}\left[G + \Delta\right]y(t+\theta)d\theta, \tag{28}$$

where Δ is an unknown matrix satisfying

$$\|\Delta\| \leq \rho. \tag{29}$$

Our goal is to find an upper bound on ρ such that the perturbed system (28) remains exponentially stable for all perturbations Δ satisfying (29).

To derive such upper bound we will use functional (26) computed for the nominal system (25).

The time derivative of (26) along solutions of (28) is

$$\frac{d}{dt}v(y_t) = -w(y_t) + \eta^T(y_t)W\eta(y_t) \tag{30}$$
$$+2\eta^T(y_t)\int_{-h}^{0}\left[G^T\left(U^T(\tau) - U(-\tau-h) + K_0^T W K_0 \tau\right) + U(-h)G - U(0)G + K_0^T W\right] Gy(t+\theta)d\theta,$$

where

$$\eta(y_t) = \Delta\int_{-h}^{0} y(t+\theta)d\theta.$$

The following inequalities can be easily derived:

$$2\eta^T(y_t)\int_{-h}^{0}\left[G^T\left(U^T(\tau) - U(-\tau-h) + K_0^T W K_0 \tau\right) + U(-h)G - U(0)G + K_0^T W\right] Gy(t+\theta)d\theta$$
$$\leq 2h\left[2\left\|G\right\|^2 u_0 + \left\|G^T K_0^T W K_0 G\right\| h + \right.$$
$$\left. + \left\|\left[U(-h)G - U(0)G + K_0^T W\right] G\right\|\right] \rho\int_{-h}^{0}\left\|y(t+\theta)\right\|^2 d\theta,$$

$$\eta^T(y_t)W\eta(y_t) \leq h\rho^2\left\|W\right\|\int_{-h}^{0}\left\|y(t+\theta)\right\|^2 d\theta.$$

As a consequence, we get the following upper bound:

$$\frac{dv(y_t)}{dt} \leq -w(y_t) + \left(a\rho^2 + b\rho\right)\int_{-h}^{0}\left\|y(t+\tau)\right\|^2 d\tau,$$

where $a = h\left\|W\right\|$ and

$$b = 2h\left[2\left\|G\right\|^2 u_0 + \left\|G^T K_0^T W K_0 G\right\| h + \left\|\left[U(-h)G - U(0)G + K_0^T W\right] G\right\|\right].$$

Then the perturbed system (28) remains exponentially stable for all perturbations Δ satisfying (29) if ρ is such that the following inequality holds:

$$\lambda_{\min}(W_1) - \left(a\rho^2 + b\rho\right) > 0. \tag{31}$$

As in section VI, let us take $W_0 = W_1 = I$. From the computed values of $U(\tau)$, see Fig. 1, we get

$$U(-1) = \begin{pmatrix} -0.1451 & -0.0653 \\ -0.0590 & 0.0026 \end{pmatrix},$$
$$U(0) = \begin{pmatrix} 0.0752 & -0.0513 \\ -0.0524 & 0.1412 \end{pmatrix}$$

and $u_0 = 0.1752$. Direct calculations from (31) show that the perturbed system (28) remains exponentially stable for all perturbations Δ satisfying (29) if

$$\rho < 0.0362.$$

VIII. CONCLUSIONS

In this paper, we studied the exponential stability of integral delay systems which appear in several stability problems of time-delay systems. Lyapunov type necessary and sufficient conditions for the exponential stability are given. A constructive procedure for computing quadratic Lyapunov-Krasovskii functionals for a given exponentially stable system is provided. It is shown that the functionals depend on matrix function $U(\tau)$ which satisfies a matrix integral delay equation. Some important properties of this matrix function that allow its computation are explicitly given. The proposed functionals are used to obtain exponential estimates of the system solutions as well as robust stability conditions for perturbed systems.

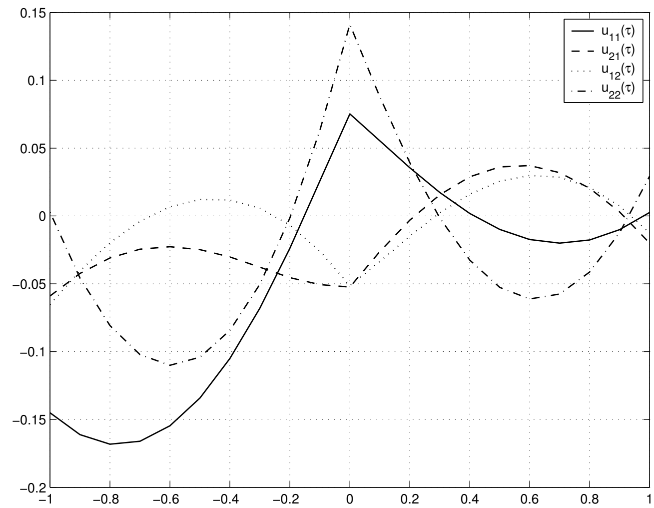

Fig. 1. Components of a piecewise linear approximation of matrix $U(\tau)$

REFERENCES

[1] R. Bellman and K.L. Cooke, *Differential-Difference Equations*, Academic Press, New York, 1963.
[2] K. Engelborghs, M. Dambrine and D. Roose, Limitations of a class of stabilization methods for delay systems, *IEEE. Trans. Automat. Contr.* Vol. 46, pp. 336-339, 2001.
[3] K. Gu and S.-I. Niculescu, Additional dynamics introduced in model system transformation, *IEEE Trans. Automat. Contr.*, 45, pp. 572-575, 2000.
[4] J.K. Hale , S.M. Verduyn Lunel . *Introduction to functional differential equations*. New York: Springer-Verlag, 1993.
[5] V.L. Kharitonov and D. Melchor-Aguilar, On delay dependent stability conditions, *Syst. Control Lett.*, 40, pp. 71-76, 2000.
[6] V.L. Kharitonov and D. Melchor-Aguilar, Additional dynamics for general class of time delay systems, *IEEE. Trans. Automat. Contr.*, 48, 6, pp. 1060-1064, 2003.
[7] A.Z. Manitus and A.W. Olbrot, Finite Spectrum Assignment Problem for Systems with Delay, *IEEE. Trans. Automat. Contr.*, 24, pp. 541-553, 1979.
[8] S. Mondié, M. Dambrine and O. Santos, Approximation of control laws with distributed delays: A necessary condition for stability, *Kybernetica,* Vol. 38. No. 5, pp. 541-551, 2002.
[9] S.-I. Niculescu, *Delay effects on stability. A robust control approach,*Springer-Verlag: Heidelberg, LNCIS, vol. **269**, 2001.
[10] V. Van Assche , M. Dambrine , J.-F. Lafay , and J.-P. Richard, "Some problems arising in the implementation of distributed-delay control law," *in Proc. 39th IEEE Conf. on Decision and Contr.*, Phoenix, Az., 1999.