Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Robust stabilization for arbitrarily switched linear systems with time-varying delays and uncertainties

Lixian Zhang, Peng Shi, Michael Basin

Abstract— **This paper studies the robust stability and stabilization problems for switched linear discrete-time systems. The parameter uncertainties in the system under consideration are time-varying but norm-bounded, and the time delay is assumed to be time-varying and bounded, which covers the constant and mode-dependent constant delays as special cases. First, sufficient conditions are derived to guarantee the stability of the uncertain system. Then, a control law is designed so that the resulting closed-loop system is stable for all admissible uncertainties. A linear matrix inequality (LMIs) approach, together with a cone complementary linearization algorithm, is proposed to solve the above problems. A numerical example is given to show the potential applicability of the obtained theoretic results.**

I. Introduction

The so-called switched systems, a subclass of the hybrid systems, have attracted considerable attention in the past years. A switched system consists of a family of subsystems described by continuous-time or discrete-time dynamics, and a rule specifying the switching among them. The switching rule in such systems is usually considered to be arbitrary, and if the switching signals are governed by stochastic processes (for instance, Markovian chains), the corresponding system is termed as a jump system. The studies on switched systems are motivated by the fact that many physical systems and man-made systems are often modeled based on a framework exhibiting switching features, see for example, [11], [12], [14]. Some examples include automobile dynamics with a manual gearbox [13], stirred tank reactor [8], wind turbine regulation [15], VSTOL aircraft [20], etc. On recent research progress and other practical applications in the field of switched systems, we refer readers to [16], [26] and the references therein.

Among a large variety of references, one of the focused topics is to find non (less)-conservative conditions to guarantee the stability of switched systems under arbitrary switching signals. Many analytical approaches and techniques regarding this issue have been reported in the literature, see for example, [3], [6], [14], [29], [31]. Using a common quadratic Lyapunov function (CQLF) on all subsystems, the quadratic Lyapunov stability facilitates the analysis and synthesis of switched systems. However, the obtained results within this framework have been recognized to be conservative [11], [14]. A notable extension of CQLF is the multiple Lyapunov functions (MLF) approach [3], by which an individual decrescent Lyapunov function is constructed for each subsystem. A switched quadratic Lyapunov function (SQLF) method proposed in [6], actually belongs to the MLF approach, which applies the poly-quadratic stability technique for polytopic uncertain systems to a class of discrete-time switched control problems. Since the SQLF is required to be decrescent between two adjacent subsystems, it can be viewed as a tradeoff between those conservative methodologies (using a single Lyapunov function) and the others, which are hard to verify numerically (see details on the derivation of stability conditions in [6]).

On another research front line, it has been recognized that time delays and parameter uncertainties, which often occur in many physical processes, are great sources of instability and poor performance. Therefore, much attention has been devoted to the study of various systems with uncertainties and time delays, and a great number of useful results have been reported in the literature on the issues of robust stability, robust H_∞ control, robust H_∞ filtering and so on, by considering different types of delay or different classes of parameter uncertainties. For the analysis and synthesis of time-delay systems, the delay-independent and delay-dependent approaches are developed, and the delay-independent one is generally regarded as being more conservative than the delay-dependent one, since the time-delay information is not used in the stability conditions or controller design [1], [19], [22], [28], [30], [32]. In addition, it is worth mentioning that as another important class of hybrid dynamic systems, Markovian jump linear system (MJLS) with uncertainties and time delays is widely studied over the past decades, see, for example, [1], [4], [5], [10], [23]. For the discrete-time case, the robust stability and H_∞ control results are obtained in [4], [23] for the constant time-delay case and in [1] for the mode-dependent time delay case. For continuous-time MJLS, the time delay is further assumed to be mode-dependent time-varying, as done in [28]. Nevertheless, if the transition probabilities (the switching rule) in MJLS are hard to obtain or the Markovian chain is impossible to model, the analysis and synthesis of the corresponding systems have to resort to the theory of switched systems under the assumption that the switching signal is arbitrary. Up to date, some effort have been made to study switched systems with time delays, see for example [24], [32], however, such existing works are still mainly in continuous-time context or in discrete-time domain though, the delays are considered as constant [21] or mode-dependent [25]. The basic stability problem for discrete-time

Lixian Zhang is with the Mechanical Engineering Department, École Polytechnique de Montreal, P.O. Box 6079, station Centre-ville, Montréal,Qué, Canada H3C 3A7. Peng Shi is with the Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, U.K. Michael Basin is with the Department of Physical and Mathematical Sciences, Autonomous University of Nuevo Leon, San Nicolas de los Garza, Nuevo Leon, Mexico, C.P. 66450 lixian.zhang@polymtl.ca

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

switched systems with time-varying delays and the different controllers design (memory or memoryless) have not been fully addressed yet, with or without parameter uncertainties.

Thus, in this paper, our attention is focused on the study on robust stability and stabilization for switched linear discrete-time systems with both time-delays and parametric uncertainties. The parameter uncertainties are time-varying but norm-bounded, and the time delay is assumed to be time-varying and bounded, which covers the constant and mode-dependent constant delays as special cases. By constructing a SQLF for the underlying system, the robust stability condition is proposed, which is dependent on upper and lower delay bounds. This stability criterion can be formulated in terms of linear matrix inequalities (LMIs) and easily tested using standard numerical software. Based on this, the problem of robust stabilization is solved designing a set of so-called delayed or memoryless state-feedback controllers, which are switched depending on the system modes. Since the obtained existence conditions of desired controllers are not expressed as strict LMI, the cone complementary linearization algorithm is employed to obtain the controllers and a suboptimal upper delay bound such that the studied switched systems can be stabilized for all admissible uncertainties.

Notation: The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n dimensional Euclidean space. In addition, in symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are introduced by symmetry and $diag\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. A symmetric matrix $P > 0$ (≥ 0) means P is positive (semi-positive) definite. I and 0 represent, respectively, identity matrix and zero matrix.

II. Problem Formulation and Preliminaries

Consider a class of uncertain switched linear discrete-time systems with time-varying delay in the state as the follows:

$$
\begin{aligned}
(\Sigma):\ x(k+1) &= (A_i+\Delta A_i(k))x(k) \\
&\quad +(A_{di}+\Delta A_{di}(k))x(k-d(k)) \\
&\quad +(B_i+\Delta B_i(k))u(k) \qquad (1)\\
x(k) &= \phi(k), k=-d_M,-d_M+1,\ldots,0 \qquad (2)
\end{aligned}
$$

where $x(k)\in\mathbb{R}^n$ is the state vector; $u(k)\in\mathbb{R}^l$ is the control input; $\{\phi(k),\ k=-d_M,-d_M+1,\ldots,0\}$ is a given initial condition sequence; i denoting $i(k)$ for simplicity, is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $\mathcal{I}=\{1,\ldots,s\}$, $s>1$ is the number of subsystems. As in [6], we assume that the switching signal i is unknown *a priori*, but its instantaneous value is available in real time. At an arbitrary discrete time k, the switching signal i is dependent on k or $x(k)$, or both, or other switching rules; A_i, A_{di} and B_i are known real constant matrices of appropriate dimensions representing the nominal system for each $i\in\mathcal{I}$. $\Delta A_i(k)$, $\Delta A_{di}(k)$ and $\Delta B_i(k)$ are real-valued time-varying matrix functions representing the time-varying norm-bounded parameter uncertainties, which are assumed to be of the form

$$
\begin{aligned}
&\begin{bmatrix} \Delta A_i(k) & \Delta A_{di}(k) & \Delta B_i(k) \end{bmatrix} \\
=\ & G_i\Delta_i(k)\begin{bmatrix} F_{1i} & F_{2i} & F_{3i} \end{bmatrix},\ \forall i\in\mathcal{I},
\end{aligned}
$$

where $\Delta_i(k)$ is a real uncertain matrix function of k satisfying

$$\Delta_i^T(k)\Delta_i(k)\leq I,$$

and G_i, F_{1i}, F_{2i} and F_{3i} are known real constant matrices for all $i\in\mathcal{I}$. These matrices specify how the uncertain parameters in $\Delta_i(k)$ enter the nominal matrices A_i, A_{di} and B_i.

In system (Σ), the time delay $d(k)$ is assumed to be time-varying and satisfy $d_m\leq d(k)\leq d_M$, where d_m and d_M are constant positive scalars representing the minimum and the maximum delay bounds respectively for any subsystems.

Remark 1: Note that if the minimum and maximum delay bounds in system (Σ) become identical, that is $d_m=d_M=d$, then the time delay becomes constant. Also, if $d(k)$ changes only when system mode is switched, then the time delay becomes mode-dependent constant; thus, the time-varying delay considered here covers the previous two cases.

Remark 2: It should also be mentioned that in continuous-time context, the time delay can be further assumed to be mode-dependent time-varying, as done in [28]. However, the meaning of mode-dependent delay in [28] is actually that the delay derivative varies when the system mode changes, that is, if the delay derivative of each modes is identical, then the delay is mode-independent and merely time-varying. On the contrary, due to the limitation of the classic Lyapunov-Krasovskii technique, the time-delay difference was rarely studied in the discrete-time context and the delay could only be assumed time-varying as a consequence.

Remark 3: In addition, the studied systems with time-varying delays are under arbitrary switching, yet within linear context still. The corresponding extension to nonlinear systems can be resort to the methods and ideas explored in [7], [8] for references. Besides, the specific rules (cyclic or prescribed switching) for the underlying systems can be further considered for less conservatism over the arbitrary switching, in the absence or presence of uncertainties, such as [18] or [17], respectively.

The objective of this paper is to derive the robust stability conditions and design a stabilizing state-feedback controller for the underlying uncertain switched system (Σ). The controller is considered here to be the following form

$$u(k)=K_{1i}x(k)+K_{2i}x(k-d(k)) \qquad (3)$$

where if $K_{2i}=0$, the stabilizing controller may be called switched memoryless state-feedback controller (SMSFC) and if $K_{2i}\neq 0$, the controller may be called switched delayed state-feedback controller (SDSFC). It is evident that the former is easier to realize, but its performance should not be better than the latter's one, which utilizes the partial information of time delays. Note that the switching signal in the designed controllers is assumed homogeneous with the one in system (Σ).

Before ending this section, we recall the following lemmas which will be used in the proof of our main results.

Lemma 1: [19] Assume that $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$ and $\mathcal{N} \in \mathbb{R}^{n_a \times n_b}$. Then, for any matrices $X \in \mathbb{R}^{n_a \times n_a}$, $Y \in \mathbb{R}^{n_a \times n_b}$ and $R \in \mathbb{R}^{n_b \times n_b}$ satisfying $\begin{bmatrix} X & Y \\ Y^T & R \end{bmatrix} \geq 0$, the following inequality holds

$$-2a^T \mathcal{N} b \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - \mathcal{N} \\ Y^T - \mathcal{N}^T & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Lemma 2: [27] Given appropriately dimensioned matrices Σ_1, Σ_2, Σ_3, with $\Sigma_1^T = \Sigma_1$. Then

$$\Sigma_1 + \Sigma_3 W(k) \Sigma_2 + \Sigma_2^T W^T(k) \Sigma_3^T < 0$$

holds for all $W(k)$, satisfying $W^T(k)W(k) \leq I$, if and only if for some $\epsilon > 0$

$$\Sigma_1 + \epsilon^{-1} \Sigma_3 \Sigma_3^T + \epsilon \Sigma_2^T \Sigma_2 < 0$$

III. Stability and Stabilization for Nominal Systems

In this section, we first consider the nominal switched system given by

$$\begin{aligned} (\mathcal{S}) : x(k+1) &= A_i x(k) + A_{di} x(k - d(k)) + B_i u(k) \\ x(k) &= \phi(k), k = -d_M, -d_M + 1, , 0 \end{aligned} \quad (4)$$

The following theorem presents a sufficient stability condition for system $(\mathcal{S})$.

Theorem 1: The unforced system $(\mathcal{S})$ in (4) with $u(k) \equiv 0$ is asymptotically stable if there exist $n \times n$ matrices $\mathcal{P}_i > 0$, $\mathcal{X}_i > 0$, $\mathcal{Y}_i$, $\forall i \in \mathcal{I}$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$ such that the following LMIs hold $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$,

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \mathcal{P}_j A_i & \mathcal{P}_j A_{di} \\ * & -d_M^{-1} \mathcal{R} & \mathcal{R}(A_i - I) & \mathcal{R} A_{di} \\ * & * & \Lambda_{ij} & -\mathcal{Y}_j \\ * & * & * & -\mathcal{Q} \end{bmatrix} < 0 \quad (5)$$

$$\begin{bmatrix} \mathcal{X}_i & \mathcal{Y}_i \\ \mathcal{Y}_i^T & \mathcal{R} \end{bmatrix} \geq 0 \quad (6)$$

where $\Lambda_{ij} \triangleq -\mathcal{P}_i + d_M \mathcal{X}_j + \mathcal{Y}_j + \mathcal{Y}_j^T + (d_M - d_m + 1)\mathcal{Q}$.

Proof Sketch. Set

$$\begin{aligned} \eta(m) &\triangleq x(m+1) - x(m) \\ &= (A_i - I) x(m) + A_{di} x(m - d(m)) \end{aligned}$$

and one has

$$x(k - d(k)) = x(k) - \sum_{m=k-d(k)}^{k-1} \eta(m) \quad (7)$$

Then, the unforced system $(\mathcal{S})$ in (4) can be transformed into

$$x(k+1) = (A_i + A_{di}) x(k) - A_{di} \sum_{m=k-d(k)}^{k-1} \eta(m) \quad (8)$$

Choose a Lyapunov functional candidate as

$$\begin{aligned} V(k) &= V_1 + V_2 + V_3 + V_4 \\ V_1 &\triangleq x^T(k) \mathcal{P}_i x(k), \\ V_2 &\triangleq \sum_{l=k-d(k)}^{k-1} x^T(l) \mathcal{Q} x(l), \\ V_3 &\triangleq \sum_{n=-d_M+2}^{-d_m+1} \sum_{l=k+n-1}^{k-1} x^T(l) \mathcal{Q} x(l), \\ V_4 &\triangleq \sum_{n=-d_M}^{-1} \sum_{m=k+n}^{k-1} \eta^T(m) \mathcal{R} \eta(m) \end{aligned}$$

where $\mathcal{P}_i$, $\mathcal{Q}$, $\mathcal{R}$ satisfy (5) and (6). Defining $\Delta V \triangleq V(k+1) - V(k)$, together combining with Lemma 1, then the following equality holds along the solution of (8) $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$,

$$\Delta V \leq \lambda^T(k) \Xi \lambda(k) \quad (9)$$

where

$$\begin{aligned} \Xi &= \begin{bmatrix} A_i^T \mathcal{P}_j A_i + \Lambda_{ij} + d_M (A_i - I)^T \mathcal{R} (A_i - I) & -\mathcal{Y}_j + A_i^T \mathcal{P}_j A_{di} + d_M (A_i - I)^T \mathcal{R} A_{di} \\ * & A_{di}^T \mathcal{P}_j A_{di} - \mathcal{Q} + d_M A_{di}^T \mathcal{R} A_{di} \end{bmatrix} \\ \lambda(k) &= \begin{bmatrix} x(k) \\ x(k - d(k)) \end{bmatrix} \end{aligned}$$

By Schur complement [2], inequality (5) ensures $\Delta V < 0$ for all nonzero $x(k)$. Therefore, we can conclude from the standard Lyapunov stability theory that the conditions (5) and (6) ensure the unforced switched system $(\mathcal{S})$ to be asymptotically stable for any time-varying delay $d(k)$ satisfying $d_m \leq d(k) \leq d_M$. This completes the proof. □

Remark 4: It is well known that the reasonable construction of Lyapunov functional is very crucial to derive non (or less)-conservative stability conditions. In the proof of Theorem 1, we apply the SQLF approach proposed in [6] to construct a quadratic Lyapunov functional candidate for switched system $(\mathcal{S})$ using the positive definite matrices $\mathcal{P}_i$, $\mathcal{Q}$ and $\mathcal{R}$. Evidently, the matrices $\mathcal{Q}$ and $\mathcal{R}$ are still the common variables among all subsystem. However, if we further choose common variables $\mathcal{Q}$ and $\mathcal{R}$ as piecewise variables $\mathcal{Q}_i$ and $\mathcal{R}_i$, then the condition will be hard to obtain due to the tight coupling between $\mathcal{Q}$ and $\mathcal{R}$ and time delay terms.

In the following theorem, we extend Theorem 1 to design a stabilizing controller of the form (3) for switched system $(\mathcal{S})$.

Theorem 2: Consider switched system $(\mathcal{S})$ in (4) . A stabilizing state-feedback controller of the form (3) exists if there exist $n \times n$ matrices $\mathcal{J}_i > 0$, $\mathcal{P}_i > 0$, $\mathcal{X}_i > 0$, $\mathcal{Y}_i, \forall i \in \mathcal{I}$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$, $\mathcal{Z} > 0$ and $l \times n$ matrices K_{1i}

and K_{2i} such that (6) and the following conditions hold,

$$\begin{bmatrix} -\mathcal{J}_j & 0 & \Theta_{1i} & \Theta_{2i} \\ * & -d_M^{-1}\mathcal{Z} & \Theta_{1i}-I & \Theta_{2i} \\ * & * & \Lambda_{ij} & -\mathcal{Y}_j \\ * & * & * & -\mathcal{Q} \end{bmatrix} < 0 \quad (10)$$

$$\mathcal{P}_i\mathcal{J}_i = I, \quad \mathcal{R}\mathcal{Z} = I \quad (11)$$

where $\Theta_{1i} \triangleq A_i + B_iK_{1i}$, $\Theta_{2i} \triangleq A_{di} + B_iK_{2i}$ and Λ_{ij} are defined in Theorem 1.

Proof. Consider the corresponding closed-loop system with the control (3), and we replace A_i and A_{di} in (5) with $A_i+B_iK_{1i}$ and $A_{di}+B_iK_{2i}$, respectively. Now performing a congruence transformation to (5) via $diag\{\mathcal{P}_j^{-1}, \mathcal{R}^{-1}, I, I\}$, we have

$$\begin{bmatrix} -\mathcal{P}_j^{-1} & 0 & \Theta_{1i} & \Theta_i \\ * & -d_M^{-1}\mathcal{R}^{-1} & \Theta_{1i}-I & \Theta_i \\ * & * & \Lambda_{ij} & -\mathcal{Y}_j \\ * & * & * & -\mathcal{Q} \end{bmatrix} < 0 \quad (12)$$

Then, the desired result is obtained by defining $\mathcal{J}_i \triangleq \mathcal{P}_i^{-1}, \mathcal{Z} \triangleq \mathcal{R}^{-1}$. □

It should be noted that although the resulting conditions in Theorem 2 are not strict LMI conditions due to (11), we can cope with this nonconvex feasibility problem using the cone complementary linearization algorithm developed in [9], which has been proved to be efficient [19]. First, we transform the nonconvex feasibility problem in Theorem 2 into the following nonlinear minimization problem subject to LMI constraints.

$Minimize\ Tr\left(\sum_{i=1}^{s}(\mathcal{P}_i\mathcal{J}_i) + \mathcal{R}\mathcal{Z}\right)$ subject to (6), (10) and (13)

$$\begin{bmatrix} \mathcal{P}_i & I \\ I & \mathcal{J}_i \end{bmatrix} \geq 0, \begin{bmatrix} \mathcal{Z} & I \\ I & \mathcal{R} \end{bmatrix} \geq 0 \quad (13)$$

Thus, as discussed in [9], if the solution of the above minimization problem is $(s+1)n$, that is,

$$Tr\left(\sum\nolimits_{i=1}^{s}(\mathcal{P}_i\mathcal{J}_i) + \mathcal{R}\mathcal{Z}\right) = (s+1)n$$

then the conditions of Theorem 2 are solvable. Although it is yet not always possible to find the global optimal solution, the proposed nonlinear minimization problem is easier than the original nonconvex feasibility problem. In fact, we can modify Algorithm 1 in [9] to solve the above nonlinear problem as follows:

Algorithm SSC (solving for a stabilizing controller)

1 Find a feasible set $(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_i, \mathcal{Y}_i, K_{1i}, K_{2i}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall i \in \mathcal{I})^0$ satisfying (6), (10) and (13). Set $k = 0$.

2 Solve the following LMI problem

$$Minimize\ Tr\begin{pmatrix} \sum_{i=1}^{s}\left(\mathcal{P}_i\mathcal{J}_i^k + \mathcal{P}_i^k\mathcal{J}_i\right) \\ +\mathcal{R}\mathcal{Z}^k + \mathcal{R}^k\mathcal{Z} \end{pmatrix}$$

subject to (6), (10) and (13).

3 Substitute the obtained matrix variables $(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_i, \mathcal{Y}_i, K_{1i}, K_{2i}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall i \in \mathcal{I})$ into (12). **If** condition (12) is satisfied with

$$\left| Tr\left(\sum_{i=1}^{s}(\mathcal{P}_i\mathcal{J}_i) + \mathcal{R}\mathcal{Z}\right) - (S+1)n \right| < \delta$$

for some sufficiently small scalar $\delta > 0$, then output the feasible solutions $(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_i, \mathcal{Y}_i, K_{1i}, K_{2i}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall i \in \mathcal{I})$, **exit**, **else** Step 4.

4 **If** $k > N$, where N is the maximum number of iterations allowed, **exit**, **else** Step 5.

5 Set $k = k + 1$, $(\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_i, \mathcal{Y}_i, K_{1i}, K_{2i}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall i \in \mathcal{I})^k = (\mathcal{P}_i, \mathcal{J}_i, \mathcal{X}_i, \mathcal{Y}_i, K_{1i}, K_{2i}, \mathcal{R}, \mathcal{Q}, \mathcal{Z}, \forall i \in \mathcal{I})$, and **go to** Step 2.

The above designed algorithm aims to find a feasible solution of desired controller for given d_m and d_M, then, based on this, one can also find the suboptimal maximum delay bound d_M for given d_m when a outside loop procedure is added on.

IV. Robust Stability and Stabilization for Uncertain Switched Systems

In this section, we extend Theorem 1 and Theorem 2 in previous section to obtain the corresponding results for uncertain switched systems (Σ).

A. Robust stability

The following theorem provides the robust stability conditions for uncertain switched systems (Σ) with $u(k) \equiv 0$.

Theorem 3: The unforced switched system (Σ) in (1)-(2) with $u(k) \equiv 0$ is robustly asymptotically stable if there exist $n \times n$ matrices $\mathcal{P}_i > 0, \mathcal{X}_i > 0, \mathcal{Y}_i, \forall i \in \mathcal{I}$, $\mathcal{Q} > 0, \mathcal{Z} > 0$, and scalars $\epsilon_i > 0$ such that (6) and the following LMI hold,$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} -\mathcal{P}_j & 0 & \mathcal{P}_jA_i & \mathcal{P}_jA_{di} & \mathcal{P}_jG_i \\ * & -d_M^{-1}\mathcal{R} & \mathcal{R}(A_i - I) & \mathcal{R}A_{di} & \mathcal{R}G_i \\ * & * & \Lambda_{ij} + \epsilon_iF_{1i}^TF_{1i} & -\mathcal{Y}_j + \epsilon_iF_{1i}^TF_{2i} & 0 \\ * & * & * & -\mathcal{Q} + \epsilon_iF_{2i}^TF_{2i} & 0 \\ * & * & * & * & -\epsilon_iI \end{bmatrix} < 0, \quad (14)$$

where Λ_{ij} are defined in Theorem 1.

Proof Sketch. Replace A_i and A_{di} in (5) with $A + G_i\Delta_i(k)F_{1i}$ and $A_{di} + G_i\Delta_i(k)F_{2i}$, respectively, and use Lemma 2, one can readily conclude that if (6) and the following inequality are satisfied, $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} -\mathcal{P}_j + \epsilon_i^{-1}\mathcal{P}_jG_iG_i^T\mathcal{P}_j & \epsilon_i^{-1}\mathcal{P}_jG_iG_i^T\mathcal{R} & \mathcal{P}_jA_i & \mathcal{P}_jA_{di} \\ * & -d_M^{-1}\mathcal{R}+\epsilon_i^{-1}\mathcal{R}G_iG_i^T\mathcal{R} & \mathcal{R}(A_i - I) & \mathcal{R}A_{di} \\ * & * & \Lambda_{ij} + \epsilon_iF_{1i}^TF_{1i} & -\mathcal{Y}_j + \epsilon_iF_{1i}^TF_{2i} \\ * & * & * & -\mathcal{Q} + \epsilon_iF_{2i}^TF_{2i} \end{bmatrix} < 0 \quad (15)$$

then the underlying system is robustly asymptotically stable. By Schur compliment, (15) implies (14); thus, the proof is completed. □

B. Robust stabilization

The existence conditions of a stabilizing state-feedback controller for uncertain switched system (Σ) are presented in the following theorem.

Theorem 4: Consider uncertain switched system (Σ) in (1)-(2). A robustly stabilizing state-feedback controller of the form (3) exists if there exist matrices $\mathcal{J}_i > 0$, $\mathcal{P}_i > 0$, $\mathcal{X}_i > 0$, $\mathcal{Y}_i, \forall i \in \mathcal{I}$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$, $\mathcal{Z} > 0$, $l \times n$ matrices K_{1i} and K_{2i} and scalars $\epsilon_i > 0$ such that (6), (11) and the following inequality hold,$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} -\mathcal{J}_j + \epsilon_i G_i G_i^T & \epsilon_i G_i G_i^T & \Theta_{1i} & \Theta_{2i} & 0 \\ * & -d_M^{-1}\mathcal{Z} + \epsilon_i G_i G_i^T & \Theta_{1i} - I & \Theta_{2i} & 0 \\ * & * & \Lambda_{ij} & -\mathcal{Y}_j & (F_{1i} + F_{3i}K_{1i})^T \\ * & * & * & -\mathcal{Q} & (F_{2i} + F_{3i}K_{2i})^T \\ * & * & * & * & -\epsilon_i I \end{bmatrix} < 0 \qquad (16)$$

where Θ_{1i} and Θ_{2i} are defined in Theorem 2 and Λ_{ij} is defined in Theorem 1.

Proof. The result is carried out using the techniques employed for proving Theorems 2 and 3. □

Note that if the designed controller of the form (3), where $K_{2i} \neq 0$ (SDSFC) is difficult to realize in practical applications, one can set $K_{2i} = 0$ in (3) and get the corresponding SMSFC. In addition, from (10) and (16), it is evident that the iterative controller design procedure *Algorithm SSC* for nominal switched systems $(\mathcal{S})$ can be easily modified to suit for uncertain system $(\mathcal{S})$ so that a robust controller can be designed.

V. Illustrative Example

In this section, a numerical example is presented to demonstrate the applicability of the obtained theoretic results.

Consider the uncertain switched system (Σ) in (1) and (2) consisting of two uncertain subsystems. For subsystem 1, the dynamics of the system is described as

$$\begin{aligned} &A_1 = \begin{bmatrix} 0.70 & 0 \\ 0.08 & 0.95 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.15 & 0 \\ -0.10 & -0.10 \end{bmatrix}, \\ &B_1 = \begin{bmatrix} -0.70 \\ 1 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}, \\ &\Delta_1(k) = 0.80\sin(k),\ F_{31} = 0.10 \\ &F_{11} = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix},\ F_{21} = \begin{bmatrix} 0 & -0.1 \end{bmatrix}. \end{aligned}$$

For subsystem 2, the dynamics of the system is described as

$$\begin{aligned} &A_2 = \begin{bmatrix} 0.70 & 0 \\ 0.08 & 0.90 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.14 & 0 \\ -0.10 & -0.05 \end{bmatrix}, \\ &B_2 = \begin{bmatrix} 0.80 \\ -0.50 \end{bmatrix},\ G_2 = \begin{bmatrix} 0.05 \\ -0.02 \end{bmatrix}, \\ &\Delta_2(k) = 0.80\sin(k), \\ &F_{12} = \begin{bmatrix} -0.10 & -0.10 \end{bmatrix}, F_{22} = \begin{bmatrix} -0.30 & -0.20 \end{bmatrix} \\ &F_{32} = -0.20. \end{aligned}$$

Suppose the switching signal is generated randomly and a possible case is shown in Figure 1.

Firstly, we check the robust stability of the above uncertain switched system with $u(k) \equiv 0$. Assume that the minimum bound of time-varying delay $d(k)$ is $d_m = 2$, then, using Theorem 3, it is found that $d_M = 5$, which means that the above system is asymptotically stable for $2 \leq d(k) \leq 5$.

Furthermore, based on the conditions in Theorem 4 and *Algorithm SSC*, choosing $\epsilon_1 = \epsilon_2 = 0.1$, we obtain the maximum delay bound $d_M = 7$ by SMSFC and $d_M = 12$ by SDSFC, respectively, which implies that the admissible delay bound in the evaluated system is increased upon applying such controllers. Moreover, the better performance of the SDSFC is demonstrated. In addition, applying the SMSFC and SDSFC and assuming that the delay varies randomly in $2 \leq d(k) \leq 7$ and $2 \leq d(k) \leq 12$, respectively, we obtain the control trajectories and the state responses of the corresponding closed-loop systems in Figures 2 and 3, respectively, for given initial condition $x = [-0.5\ 0.3]^T$. It is clearly observed from the curves that the obtained controller robustly stabilizes the switched system against variations of uncertain parameters under the randomly generated switching signals.

VI. Conclusions

In this paper, the robust stability and stabilization problems are studied for switched linear discrete-time systems with both bounded time-varying delays and norm-bounded time-varying uncertainties. A switched quadratic Lyapunov function is constructed for the underlying system and the robust stability criterion dependent on delay bounds is derived via LMI formulation, which can be easily tested using standard numerical software. Furthermore, the robust stabilization problem is also solved by designing a set of so-called switched delayed or memoryless state-feedback controllers. A cone complementary linearization algorithm is employed to obtain the controllers and a suboptimal upper delay bound such that the underlying switched systems can be stabilized for all admissible uncertainties. A numerical example is included to show the effectiveness of the developed approach.

References

[1] E. K. Boukas and Z. K. Liu. Robust stability and stabilizability of jump linear uncertain systems with mode-dependent time delays. *IEEE Trans. Automat. Control*, 46(12):1918–1924, 2001.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM, Philadelphia, PA, 1994.

[3] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automat. Control*, 43(4):475–782, 1998.

[4] Y. Y. Cao and J. Lam. Stochastic stabilizability and H_∞ control for discrete-time jump linear systems with time delay. *J. Franklin Inst.*, 336(8):1263–1281, 1999.

[5] W. H. Chen, J. X. Xu, and Z. H. Guan. Guaranteed cost control for uncertain Markovian jump systems with mode-dependent time-delays. *IEEE Trans. Automat. Control*, 48(12):2270–2277, 2003.

[6] J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Trans. Automat. Control*, 47(11):1883–1887, 2002.

[7] N. H. El-Farra and P. D. Christofides. Bounded robust control of constrained multivariable nonlinear processes. *Chemical Engineering Science*, 58(13):3025–3047, 2003.

[8] N. H. El-Farra, P. Mhaskar, and P. D. Christofides. Output feedback control of switched nonlinear systems using multiple Lyapunov functions. *Systems & Control Letters*, 54(12):1163–1182, 2005.

[9] L. El Ghaoui, F. Oustry, and M. Ait Rami. A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Trans. Automat. Control*, 42(8):1171–1176, 1997.

[10] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck. Stochastic stability properties of jump linear systems. *IEEE Trans. Automat. Control*, 37(1):38–52, 1992.

[11] G. Ferrari-Trecate, F. A. Cuzzola, D. Mignone, and M. Morari. Analysis of discrete-time piecewise affine and hybrid systems. *Automatica*, 38(12):2139–2146, 2002.

[12] J. Imura. Well-posedness analysis of switch-driven piecewise affine systems. *IEEE Trans. Automat. Control*, 48(11):1926–1935, 2003.

[13] K. H. Johansson, J. Lygeros, and S. Sastry. Modelling hybrid systems. In *H. Unbehauen, Ed., Encyclopedia of life support systems (EOLSS)*, 1999.

[14] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Automat. Control*, 43(4):555–559, 1998.

[15] D. J. Leith, R. N. Shorten, W. E. Leithead, O. Mason, and P. Curran. Issues in the design of switched linear control systems: a benchmark study. *Int. J. Adaptive Control & Signal Processing*, 17(2):103–118, 2003.

[16] D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magzine*, 19(5):59–70, 1999.

[17] P. Mhaskar, N. H. El-Farra, and P. D. Christofides. Robust predictive control of switched systems: Satisfying uncertain schedules subject to state and control constraints. *Int. J. Adaptive Control & Signal Processing*. 2008, to appear.

[18] P. Mhaskar, N. H. El-Farra, and P. D. Christofides. Predictive control of switched nonlinear systems with scheduled mode transitions. *IEEE Trans. Automat. Control*, 50(11):1670–1680, 2005.

[19] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *Int. J. Control*, 74(14):1447–1455, 2001.

[20] M. Oishi and C. Tomlin. Switched nonlinear control of a VSTOL aircraft. In *Proc. 38th Conf. Decision Control*, pages 2685–2690, Phoenix, AZ, 1999.

[21] V. N. Phat. Robust stability and stabilizability of uncertain linear hybrid systems with state delays. *IEEE Trans. Circuits and Systems (II)*, 52(2):94–98, 2005.

[22] J. P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[23] P. Shi, E. K. Boukas, and R. K. Agarwal. Kalman filtering for continuous-time uncertain systems with Markovian jumping parameters. *IEEE Trans. Automat. Control*, 44(8):1592–1597, 1999.

[24] X. Sun, J. Zhao, and D. Hill. Stability and L_2-gain analysis for switched delay systems: A delay-dependent method. *Automatica*, 42(10):1769–1774, 2006.

[25] Y. Sun, L. Wang, and G. Xie. Delay-dependent robust stability and H_∞ control for uncertain discrete-time switched systems with mode-dependent time delays. *Applied Mathematics and Computation*, 187(2):1228–1237, 2007.

[26] Z. Sun and S. S. Ge. Analysis and synthesis of switched linear control systems. *Automatica*, 41(2):181–195, 2005.

[27] L. Xie, M. Fu, and C. E. De Souza. H_∞ control and quadratic stabilization of systems with parameter uncertainty via output feedback. *IEEE Trans. Automat. Control*, 37:1253–1256, 1992.

[28] S. Xu, T. Chen, and J. Lam. Robust H_∞ filtering for uncertain Markovian jump systems with mode-dependent time-delays. *IEEE Trans. Automat. Control*, 48(5):900–907, 2003.

[29] L. Zhang, P. Shi, E. Boukas, and C. Wang. H_∞ control of switched linear discrete-time systems with polytopic uncertainties. *Optimal Control Applications and Methods*, 27(5):273–291, 2006.

[30] L. Zhang, P. Shi, E. Boukas, and C. Wang. Robust l_2-l_∞ filtering for switched linear discrete time-delay systems with polytopic uncertainties. *IET Control Theory Appl.*, 1(3):722–730, 2007.

[31] L. Zhang, P. Shi, C. Wang, and H. Gao. Robust H_∞ filtering for switched linear discrete-time systems with polytopic uncertainties. *Int. J. Adaptive Control & Signal Processing*, 20(6):291–304, 2006.

[32] L. Zhang, C. Wang, and H. Gao. Delay-dependent stability and stabilization of a class of linear switched time-varying delay systems. *J. of Systems Engineering & Electronics*, 18(2):320–326, 2007.

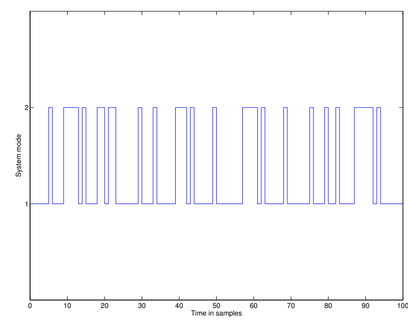

Fig. 1. Switching signal

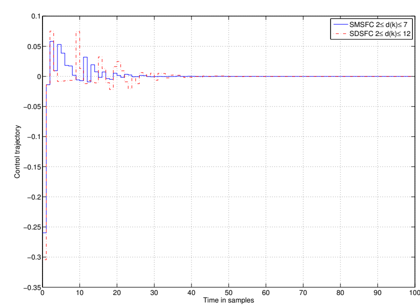

Fig. 2. Control trajectories of two different controllers

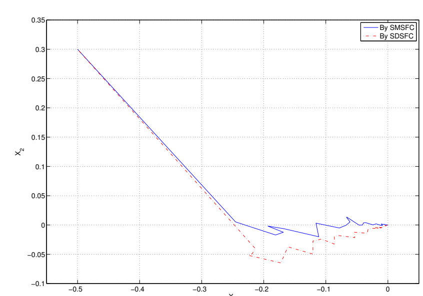

Fig. 3. State responses of the closed-loop systems by two different controllers