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Less Conservative Robust Control of Constrained Linear Systems with Bounded Disturbances

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***Abstract*— This paper develops a novel robust control method for linear systems subject to additive and bounded disturbances. The approach is based on constraint tightening method. The proposed method is computationally efficient in the sense that it does not require solving any online optimization problem, in contrast to several other robust Model Predictive Control approaches proposed in the literature. The algorithm elaborated in this paper guarantees convergence to a minimal disturbance invariant set and the terminal predicted state constraint set is allowed to be larger than the minimal disturbance invariant set.**

I. INTRODUCTION

In this paper, we consider a control problem for constrained discrete time linear systems that are subject to bounded additive disturbances. Our goal is to provide a control method that enforces specified state and input constraints in the presence of disturbances and steers state trajectories to a given target set.

This problem has been studied employing invariant set methods (see [1], [2] and references therein) and using optimization based control strategies such as Model Predictive Control (MPC). MPC is known as an effective method to deal with constraints and uncertainties [3]. One approach to address the aforementioned problem, in the MPC context, relies on the inherent robustness of MPC and on the assumption that the open loop system is sufficiently contractive [4]. Open-loop input control sequence MPC strategies, proposed in [5], in which the control action is taken as the first element of an optimal control sequence, may cause spread of uncertainties over the horizon and therefore may result in a conservative domain of attraction in the presence of disturbances. Therefore, a feedback MPC approach has been proposed in which optimization is performed over feedback policies [6]. However, optimization over arbitrary feedback policies, in the presence of constraints, is especially difficult. Therefore, affine feedback policies were employed where the state feedback gain(s) are calculated off-line and optimization was performed over constant terms [7], [8], [9].

The robust MPC is based on the idea of assuring the robustness of the resulting controlled system by tightening the constraints on states and controls over the prediction horizon has been proposed initially in [10] as well as [11], [7], [12], [13]. The key idea is to retain a suitable margin over the prediction horizon so that feasibility is guaranteed for the future iterations, in the presence of allowable disturbances.

In this paper, we propose a novel robust control method based on the constraint tightening approach. Unlike robust MPC approaches, our proposed method does not involve online optimization to determine the control action. We show that if the set of input and state constraints over the prediction horizon is feasible, then the proposed controller guarantees feasibility for the future iterations. Moreover, it is shown that the minimal invariant set corresponding to the off-line calculated state feedback is an attractor, i.e., all trajectories will converge to this set. Another advantage of the proposed method is that it does not require the terminal constraint set to be contained in the desired target set, which was the usual assumption made in the prior literature, except for [12]. In fact, the terminal constraint set, namely the set to which the final predicted state must belong, can be much larger than the target set. Convergence to the target set is guaranteed as long as it contains the minimal invariant set. Moreover, it is shown that the domain of attraction of the proposed method contains the domain of attraction of the method proposed in [12], and the convergence of the state to the minimal disturbance invariant set is provided without having the explicit knowledge about this set.

II. PROBLEM STATEMENT

Consider linear time-invariant, discrete-time systems described by

$$\begin{aligned} &x^+ = Ax + Bu + w, \\ &x(k) \in \mathbb{R}^n,\ u(k) \in \mathbb{R}^m,\ w(k) \in \mathbb{R}^n \end{aligned} \tag{1}$$

where x, u and w are, respectively, the state, control and disturbance; x^+ denotes the successor state of x and $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers.

We assume that the disturbance w belongs to a polytope W, and control and state are subject to hard constraints, i.e.,

$$u \in \mathbb{U},\ x \in \mathbb{X} \text{ and } w \in W, \tag{2}$$

where $\mathbb{U}$ and W are (convex, compact) polytopes, containing origin in their interior, and $\mathbb{X}$ is a (convex) closed polyhedron. Finally, a target constraint set $\mathbb{X}_t$ is given by

$$\mathbb{X}_t = \{x \in \mathbb{R}^n | Yx \leq q\}, Y \in \mathbb{R}^{r\times n},\ q \in \mathbb{R}^r. \tag{3}$$

Assume $\mathbb{X}_t$ is bounded and $0 \in int(\mathbb{X}_t)$. The control objective is to find u that steers the state into the target set $\mathbb{X}_t$.

Notations: Pontryagin difference of two sets S and T is defined as $S \sim T = \{x | x + t \in S,\ \forall\ t \in T\}$.

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III. Robust control algorithm

Let us suppose $K \in \mathbb{R}^{m\times n}$ is such that $A_K = A+BK$ is stable. Moreover, assume that the minimal robust invariant set F_K for the system $x^+ = A_K x + w$, defined in [14], is such that

$$F_K \subseteq \mathbb{X}_t. \tag{4}$$

For any initial state $x \in \mathbb{X}$, the following control sequence

$$\mathbf{u}^*(x) := \{u_0^*(x), u_1^*(x), \cdots, u_{N-1}^*(x)\}$$

and associated state sequence

$$\mathbf{x}^*(x) := \{x_0^*(x), x_1^*(x), \cdots, x_N^*(x)\}$$

are feasible if they satisfy the set of constraints $C(x)$, defined as follows:

$$\begin{aligned} &x_0^*(x) = x,\\ &x_{i+1}^*(x) = Ax_i^*(x) + Bu_i^*(x),\ i = 0,\cdots,N-1, \end{aligned} \tag{5}$$

$$\begin{aligned} &\mathbb{U}_0 = \mathbb{U},\\ &\mathbb{U}_{i+1} = \mathbb{U}_i \sim KA_K^i W,\ i = 0,\cdots,N-1,\\ &u_i^*(x) \in \mathbb{U}_i,\ i = 0,\cdots,N-1, \end{aligned} \tag{6}$$

$$\begin{aligned} &\mathbb{X}_0 = \mathbb{X},\\ &\mathbb{X}_{i+1} = \mathbb{X}_i \sim A_K^i W,\ i = 0,\cdots,N-1,\\ &x_i^*(x) \in \mathbb{X}_i,\ i = 0,\cdots,N-1,\\ &x_N^*(x) \in \mathbb{X}_f. \end{aligned} \tag{7}$$

Let $\mathbb{X}_f$ be the maximal robust invariant set for the system

$$x^+ = A_K x + w \tag{8}$$

where $w \in A_K^N W$, i.e.,

$$A_K\mathbb{X}_f + A_K^N W \subset \mathbb{X}_f \tag{9}$$

and

$$\begin{aligned} &\mathbb{X}_f \subset \mathbb{X}_N,\\ &K\mathbb{X}_f \subset \mathbb{U}_N, \end{aligned} \tag{10}$$

and for any set S which satisfies conditions (9) and (10), $S \subset \mathbb{X}_f$.

Remark 3.1: The set $\mathbb{X}_f$ can be calculated using the algorithm introduced in [14] to compute the maximal invariant sets with finite number of iterations.

In the sequel, we propose a robust control algorithm that uses $\mathbb{X}_f$ as the terminal constraint set and incorporates the constraint tightening approach to assure iterative feasibility of the control solution. While the use of robust invariant terminal constraint sets has been exploited in many publications [4], [12], [13] and the constraint tightening approach for robust feasibility has been introduced in [7], [15], a novelty of this paper is that no optimization problem is involved in our proposed scheme. Feasible control sequences are generated sequentially and the first element of the sequence is applied at each time instant (similar to MPC). The convergence of the state trajectory to the desired set $\mathbb{X}_t$ is guaranteed, provided that $\mathbb{X}_t$ includes the minimal invariant set corresponding to the system (8) with disturbance set W. This condition can be satisfied using Linear Programming as introduced in Section V. One notable feature of the scheme proposed in this paper is that, unlike other work that address the same problem, such as [13], the desired target set $\mathbb{X}_t$ is not required to include the terminal constraint set $\mathbb{X}_f$. In fact, the set $\mathbb{X}_f$ can be much larger than $\mathbb{X}_t$. This feature makes the robust algorithm considerably less conservative than conventional approaches for which $\mathbb{X}_t$ must include the terminal constraint set. It should be noted that the method proposed in [12] has the same feature for terminal set $\mathbb{X}_f$. However, in Section IV it will be shown that the region of attraction of the method proposed in [12] is contained in the region of attraction of the method proposed in this paper.

Let us assume that for the initial state $x(0)$ that $\mathbf{u}^*(x(0))$ and $\mathbf{x}^*(x(0))$ are control and state sequences satisfying the constraints set of $C(x(0))$. Now we propose the following iterative algorithm, where at each time instant k, the feasible control sequence $\mathbf{u}^*(x(k))$ is constructed using the feasible control and state sequences $\mathbf{u}^*(x(k-1))$ and $\mathbf{x}^*(x(k-1))$, where $x(k)$ is the observed state at the time instant k:

$$\begin{aligned} &u_i^*(x(k)) = u_{i+1}^*(x(k-1)) + K(x_i^*(x(k)) - x_{i+1}^*(x(k-1))),\\ &\text{for } i = 0,\cdots,N-2,\\ &u_{N-1}^*(x(k)) = Kx_{N-1}^*(x(k)); \end{aligned} \tag{11}$$

$$\begin{aligned} &x_0^*(x(k)) = x(k),\\ &x_{i+1}^*(x(k)) = Ax_i^*(x(k)) + Bu_i^*(x(k))\ i = 0,\cdots,N-1. \end{aligned} \tag{12}$$

Since at each time instant, the first element of the feasible control sequence is applied as control signal, the robust control law is

$$u(k) = \kappa_N^*(x(k)) := u_0^*(x(k)). \tag{13}$$

Theorem 3.1: Suppose the set of constraints $C(x(0))$ is satisfied with the feasible control and state sequences $\mathbf{u}^*(x(0))$ and $\mathbf{x}^*(x(0))$, respectively. Then the state and input trajectories of the system (1) with the control law

$$u(\cdot) = \kappa_N^*(x(\cdot)) \tag{14}$$

satisfy the input and state constraints (2). Furthermore, the set of constraints $C(x(k))$ is satisfied by control and state sequences $\mathbf{u}^*(x(k))$ and $\mathbf{x}^*(x(k))$, defined by (11) and (12), $\forall k > 0$.

Proof: Assume $\mathbf{u}^*(x)$ and $\mathbf{x}^*(x)$ are feasible control and state sequences for $C(x)$ and x^+ is the successor state defined in (1). Considering the state evolution (12) and control update (11) we have:

$$\begin{aligned} &x_{i+1}^*(x^+) = Ax_i^*(x^+) + Bu_i^*(x^+)\\ &= Ax_i^*(x^+) + Bu_{i+1}^*(x) + BK(x_i^*(x^+) - x_{i+1}^*(x))\\ &= A_K(x_i^*(x^+) - x_{i+1}^*(x)) + x_{i+2}^*(x),\ i = 0,\cdots,N-2 \end{aligned} \tag{15}$$

where the last equality is achieved by adding and subtracting $Ax_{i+1}^*(x)$ and using equation (12). From (12), we have

$$x_0^*(x^+) - x_1^*(x) = x^+ - Ax - Bu_0^*(x) = w \in W, \tag{16}$$

and using (15) it can be easily shown that

$$x_i^*(x^+) - x_{i+1}^*(x) \in A_K^i W,\ i = 0, \cdots, N-1. \tag{17}$$

From (7), (10) and (17) we have

$$x_i^*(x^+) \in \mathbb{X}_{i+1} + A_K^i W = (\mathbb{X}_i \sim A_K^i W) + A_K^i W \subseteq \mathbb{X}_i, \tag{18}$$

for $i = 0, \cdots, N-2$, and since $x_N^*(x) \in \mathbb{X}_f$,

$$x_{N-1}^*(x^+) \in \mathbb{X}_f + A_K^{N-1} W \subseteq \mathbb{X}_N + A_K^{N-1} W \subseteq \mathbb{X}_{N-1}. \tag{19}$$

On the other hand, from equations (11) and (12), we have

$$\begin{aligned} x_N^*(x^+) &= Ax_{N-1}^*(x^+) + Bu_{N-1}^*(x^+) \\ &= (A+BK)x_{N-1}^*(x^+) = A_K x_{N-1}^*(x^+). \end{aligned} \tag{20}$$

From (17), where $i = N-1$, we have

$$x_{N-1}^*(x^+) - x_N^*(x) \in A_K^{N-1} W. \tag{21}$$

Multiplying (21) by A_K and using (20) we have

$$x_N^*(x^+) \in \{A_K x_N^*(x)\} + A_K^N W. \tag{22}$$

Since $x_N^*(x) \in \mathbb{X}_f$ and the set $\mathbb{X}_f$ is a robust invariant set for the system (8) and disturbance set $A_K^N W$,

$$\{A_K x_N^*(x)\} + A_K^N W \subset A_K \mathbb{X}_f + A_K^N W \subset \mathbb{X}_f,$$

and, as the result, $x_N^*(x^+) \in \mathbb{X}_f$. Therefore, $\mathbf{x}^*(x^+)$ satisfies state constraints (7).

Now it remains to show that $\mathbf{u}^*(x^+)$ satisfies control constraints (6). Since $\mathbf{u}^*(x)$ is feasible, from (6) we have

$$u_i^*(x) \in \mathbb{U}_i,\ i = 0, \cdots, N-1. \tag{23}$$

Therefore, control update (11) and (17) imply

$$u_i^*(x^+) \in \mathbb{U}_{i+1} + KA_K^i W. \tag{24}$$

From control set tightening law in (6) and (24), we have

$$u_i^*(x^+) \in (\mathbb{U}_i \sim KA_K^i W) + KA_K^i W \subseteq \mathbb{U}_i$$

where $i = 0, \cdots, N-2$. From (11) and (19), we have

$$\begin{aligned} u_{N-1}^*(x^+) &= Kx_{N-1}^*(x^+) \in K\mathbb{X}_f + KA_K^{N-1} W \\ &\subseteq \mathbb{U}_N + KA_K^{N-1} W \subset \mathbb{U}_{N-1}, \end{aligned}$$

and, as the result, $\mathbf{u}^*(x^+)$ satisfies control constraints (6). ■

To investigate convergence properties of the controller (13), we first introduce the minimal disturbance invariant set for the system (8) with the disturbance set W as it is defined in [14]

Definition 3.1: The disturbance invariant set F_K for the system (8) is minimal if for all closed disturbance invariant sets X such that $A_K X + W \subset X$, it follows that $F_K \subset X$.

The minimal disturbance invariant set F_K is calculated as follows [14]

$$F_K = \sum_{i=0}^{\infty} A_K^i W. \tag{25}$$

We need the following auxiliary results to prove the convergence properties of the proposed controller.

Lemma 3.1: Let $\mathbf{u}^*(x)$ and $\mathbf{x}^*(x)$ be feasible control and state sequences corresponding to state x, and let $\mathbf{u}^*(x^+)$ and $\mathbf{x}^*(x^+)$ be control and state sequences generated by (11) and (12), where x^+ is the successor state defined in (1). Moreover, assume F_K is the minimal disturbance invariant set as defined before. Then

$$d(x_i^*(x^+), A_K^i F_K) \le d(x_{i+1}^*(x), A_K^{i+1} F_K),\ i = 0, \cdots, N\text{–}1. \tag{26}$$

Proof: Equations (16) and (17) imply that for $i = 0, \cdots, N-1$,

$$\exists w \in A_K^i W\ s.t.\ x_i^*(x^+) = x_{i+1}^*(x) + w. \tag{27}$$

Moreover, if $w \in A_K^i W$, from (25) we have

$$A_K^{i+1} F_K + \{w\} \subset A_K^{i+1} F_K + A_K^i W = A_K^i F_K. \tag{28}$$

Therefore, from (27) and (28) we have

$$\begin{aligned} d(x_i^*(x^+), A_K^i F_K)) &\le d(x_{i+1}^*(x) + w, A_K^{i+1} F_K + \{w\}) \\ &= d(x_{i+1}^*(x), A_K^{i+1} F_K) \\ &\text{for } i = 0 \cdots, N\text{–}1. \end{aligned} \tag{29}$$

■

Lemma 3.2: Let $x^+ = A_K x + w,\ w \in A_K^N W$, P be the Lyapunov matrix corresponding to stable matrix A_K, i.e., $P \succ 0$ and $\exists Q \succ 0\ s.t.\ A_K^T P A_K - P = -Q$, and the norm $\|\cdot\|_p$ is defined as $\|x\|_p := \sqrt{x^T P x},\ x \in \mathbb{R}^n$. If the distance is defined in the normed space $(\mathbb{R}^n, \|\cdot\|_p)$ and $\|D\|_p$ is the induced norm of any square matrix $D \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} &\exists\ 0 < \alpha < 1 s.t. \|A_K\|_p \le \alpha \\ \text{and}\quad &d(x^+, A_K^N F_K) \le \alpha d(x, A_K^N F_K) \end{aligned} \tag{30}$$

Proof: Assume $A_K^T P A_K - P = -Q$.

$$\begin{aligned} &\forall x \in S_u = \{x \in \mathbb{R}^n |\ x^T P x = 1\}, \\ &x^T A_K^T P A_K x = 1 - x^T Q x. \end{aligned}$$

Since S_u is compact

$$\begin{aligned} &\exists\ \bar{x} \in S_u\ s.t. \\ &\|A_K\|_p = \sup_{x\in S_u} \sqrt{x^T A_K^T P A_K x} = \sqrt{1 - \bar{x}^T Q \bar{x}} < 1. \end{aligned}$$

The last inequality is due to the fact that $Q \succ 0$.

Moreover,

$$d(x^+, A_K^N F_K) = d(A_K x + w, A_K^{N+1} F_K + A_K^N W).$$

Since $w \in A_K^N W$,

$$\begin{aligned} &d(A_K x + w, A_K^{N+1} F_K + A_K^N W) \\ &< d(A_K x + w, A_K^{N+1} F_K + \{w\}) \\ &= d(A_K x, A_K^{N+1} F_K). \end{aligned} \tag{31}$$

According to definition of distance in the normed space

$$\begin{aligned} d(A_K x, A_K^{N+1} F_K) &= \inf_{w \in A_K^N F_K} \|A_K x - A_K w\|_p \\ &\le \|A_K\|_p \inf_{w \in A_K^N F_K} \|x - w\|_p \\ &= \|A_K\|_p d(x, A_K^N F_K). \end{aligned} \tag{32}$$

From (31) and (32) we have

$$d(x^+, A_K^N F_K) \le \|A_K\|_p d(x, A_K^N F_K), \tag{33}$$

and the proof is complete. ■

Theorem 3.2: If for an initial state $x(0)$, there exist control and state sequences satisfying the set of constraints $C(x(0))$, then the set F_K is robustly attractive (all trajectories converge to F_K despite disturbances) for the controlled uncertain system

$$x^+ = Ax + B\kappa_N^*(x) + w, \tag{34}$$

where $w \in W$. Furthermore, the region of attraction is

$$R = \{x \in \mathbb{R}^n |\ C(x) \text{ is feasible}\}.$$

Proof: Let us define the cost $J(x, \mathbf{u}^*(x))$ as follows

$$J(x, \mathbf{u}^*(x)) := \sum_{i=0}^{N} d(x_i^*(x), A_K^i F_K), \tag{35}$$

where $x_i^*(x),\ i = 0, \cdots, N$ is defined in (12). If x^+ is the successor state defined in (1), according to Lemma 3.1

$$\sum_{i=0}^{N-1} d(x_i^*(x^+), A_K^i F_K) \le \sum_{i=1}^{N} d(x_i^*(x), A_K^i F_K). \tag{36}$$

From definition (35) and inequality (36) and the fact that $x = x_0^*(x)$ we have

$$\begin{aligned} J(x^+, \mathbf{u}^*(x^+)) - J(x, \mathbf{u}^*(x)) &\le d(x_N^*(x^+), A_K^N F_K) \\ &- d(x, F_K). \end{aligned} \tag{37}$$

Substituting x and x^+ with $x(k)$ and $x(k+1)$, respectively, and making summation over the inequality (37) from time instant 0 to $M < \infty$, we obtain

$$\begin{aligned} \sum_{k=0}^{M} d(x(k), F_K) &\le \sum_{k=1}^{M+1} d(x_N^*(x(k)), A_K^N F_K) \\ &+ J(x(0), \mathbf{u}^*(x(0))) - J(x(M+1), \mathbf{u}^*(x(M+1))) \\ &\le J(x(0), \mathbf{u}^*(x(0))) + \sum_{k=1}^{M+1} d(x_N^*(x(k)), A_K^N F_K). \end{aligned} \tag{38}$$

On the other hand, from inequality (30) we obtain

$$d(x_N^*(x(k)), A_K^N F_K) \le \alpha^k d(x_N^*(x(0)), A_K^N F_K). \tag{39}$$

Therefore, for all integer M, we have

$$\sum_{k=1}^{M+1} d(x_N^*(x(k)), A_K^N F_K) \le \frac{\alpha}{1-\alpha} d(x_N^*(x(0)), A_K^N F_K). \tag{40}$$

From (38) and (40), we have

$$\begin{aligned} &\sum_{k=0}^{M} d(x(k), F_K) \\ &\le J(x(0), \mathbf{u}^*(x(0))) + \frac{\alpha}{1-\alpha} d(x_N^*(x(0)), A_K^N F_K). \end{aligned} \tag{41}$$

Since $\|x(0)\|_p < \infty$ and the set $\mathbb{U}$ is compact, the right hand side of the above inequality is bounded. Therefore, the sequence $\{V_M := \sum_{k=0}^{M} d(x(k), F_K)\}$ is bounded and non-decreasing in $\mathbb{R}^n$. Hence, $\{V_M\}$ is convergent and, as the result, we have

$$d(x(M), F_K) = V_{M+1} - V_M \to 0,\ as\ M \to \infty. \tag{42}$$

■

Remark 3.2: The important feature of the proposed method is that the attraction to F_K is achieved without involving any optimization or minimal robust invariant set approximation, while in the MPC based method [7], [12] attraction to F_K is achieved by solving online an optimization problem.

Remark 3.3: The proposed robust control method is based on tightening constraints, at each time instance over the prediction horizon, by $A_K^i W$, similar to [4], [13]. However, the advantage of the proposed method is that it does not require the final constraint set $\mathbb{X}_f$ to be a subset of the desired target set $\mathbb{X}_t$. In fact, the target set $\mathbb{X}_t$ is only required to contain the minimal robust invariant set F_K, i.e., $F_K \subset \mathbb{X}_t$, in order to be attractive.

IV. Comparison of Regions of Attraction

In this section we compare the region of attraction R achieved by the algorithm proposed in Section III with the MPC based methods in which the tightened sets are constructed via Pontriagin subtraction of a minimal robust invariant set from state constraint set $\mathbb{X}$ and its multiplication by K from control constraint set $\mathbb{U}$. One such MPC-based constraint tightening method proposed in [12] is as follows:

$$\begin{aligned} &\mathbb{U}_i^* = \mathbb{U} \sim KF_K \\ &\mathbb{X}_i^* = \mathbb{X} \sim F_K,\ i = 1, \cdots, N-1 \\ &\mathbb{X}_f^* \subset \mathbb{X} \sim F_K, \end{aligned} \tag{43}$$

where $\mathbb{X}_f^*$ satisfies the following property

$$A_K \mathbb{X}_f^* \subset \mathbb{X}_f^*,\ \mathbb{X}_f^* \subset \mathbb{X} \sim F_K,\ K\mathbb{X}_f^* \subset \mathbb{U} \sim KF_K. \tag{44}$$

If $\mathbb{X}_i$ and $\mathbb{U}_i$ and $\mathbb{X}_f$ are replaced by $\mathbb{X}_i^*$ and $\mathbb{U}_i^*$ and $\mathbb{X}_f^*$ in (6) and (7), we have a new set of constrains which is called $\bar{C}(x)$.

In [12], a controller is proposed which provides the following domain of attraction:

$$\bar{R} = \{x | \exists\ x_0\ s.t.\ x \in x_0 + F_K,\ \bar{C}(x_0) \text{ is feasible}\}. \tag{45}$$

Here, we show that $\bar{R}$ is contained in the region of attraction of the proposed controller, namely, $\bar{R} \subset R$.

Let us assume $x \in \bar{R}$ and x_0 and $\mathbf{u} = \{u_0, \cdots, u_{N-1}\}$ are associated state and control sequence. Assume $\mathbf{x} = \{x_0, x_1, \cdots, x_N\}$ is the state trajectory produced by initial state x_0 and control sequence $\mathbf{u}$ subject to nominal dynamics $x^+ = Ax + Bu$. Now let us define control and state sequences $\mathbf{u}^* = \{u_0^*, \cdots, u_{N-1}^*\}$ and $\mathbf{x}^* = \{x, x_1^*, \cdots, x_N^*\}$ as follows

$$\begin{aligned} &x_0^* = x, \\ &u_i^* = u_i + K(x_i^* - x_i), \\ &x_{i+1}^* = Ax_i^* + Bu_i^*,\ i = 0, \cdots, N-1. \end{aligned} \tag{46}$$

From (46) we have

$$\begin{aligned} x^*_{i+1} - x_{i+1} &= A_K(x^*_i - x_i), \\ u^*_i - u_i &= K(x^*_i - x_i). \end{aligned} \tag{47}$$

Let $w = x - x_0 \in F_K$. Then

$$\begin{aligned} u^*_i &= u_i + KA^i_K F_K, \\ x^*_i &= x_i + A^i_K F_K, \ i = 0, \cdots, N-1. \end{aligned} \tag{48}$$

Since $u_i \in \mathbb{U} \sim KF_k$ and $x_i \in \mathbb{X} \sim F_k$, we have

$$\begin{aligned} u^*_i &\in (\mathbb{U} \sim KF_K) + KA^i_K F_K \\ &= (\mathbb{U} \sim K\sum_{j=0}^{\infty} A^j_K W) + K\sum_{j=i}^{\infty} A^j_K W \\ &\subset \mathbb{U} \sim K\sum_{j=i}^{i-1} A^j_K W \subset \mathbb{U}_i, \\ i &= 1, \cdots, N-1, \end{aligned} \tag{49}$$

and $u^*_0 \in \mathbb{U}$. With the same argument, we have

$$x^*_i \in \mathbb{X}_i, \ i = 0, \cdots, N-1. \tag{50}$$

Moreover, at $i = N$ we have

$$x^*_N \in x_N + A^N_K F_K. \tag{51}$$

Since $x_N \in \mathbb{X}^*_f$, we have $x^*_N \in \mathbb{X}^*_f + A^N_K F_K$. In addition

$$A_K(\mathbb{X}^*_f + A^N_K F_K) + A^N_K F_K \subset \mathbb{X}^*_f + A^N_K F_K.$$

Therefore $\mathbb{X}^*_f + A^N_K F_K$ is robust positive invariant for the system $x^+ = Ax + Bu + w, \ w \in A^N_K W$. Since $\mathbb{X}_f$ is the maximal robust invariant set for the same system, we have $\mathbb{X}^*_f + A^N_K F_K \subset \mathbb{X}_f$.

Therefore, $\mathbf{u}^*$ and $\mathbf{x}^*$ satisfy the set of constraints $C(x)$, hence $x \in R$ and $\bar{R} \subseteq R$.

Remark 4.1: Given the region of attraction $\bar{R}$, the method proposed in [12] can be modified such that the iterative optimization is eliminated using suboptimal strategies, e.g. [16].

Remark 4.2: While we show that $\bar{R} \subseteq R$ in general, we can also show numerically that $\bar{R} \neq R$ for an example given in Section VI in Figure 3.

V. Calculating the feedback gain K

In order for the target set $\mathbb{X}_t$ to be attractive for the controlled system, according to Theorem 3.2, it is sufficient to have set inclusion $F_K \subset \mathbb{X}_t$. If $\mathbb{X}_t$ is robust positive invariant for the system (8) with $w \in W$, based on definition of F_K, the $F_K \subset \mathbb{X}_t$ is satisfied. In this section we introduce a Linear Programming algorithm to compute a feedback gain K which makes $\mathbb{X}_t$ robust invariant for the system (8).

Our objective is to find K such that

$$\begin{aligned} (A + BK)\mathbb{X}_t &\subset \mathbb{X}_t \sim W, \\ K\mathbb{X}_t &\subset \mathbb{U}. \end{aligned} \tag{52}$$

Since W is non-empty and $0 \in int(W)$, it can be easily shown that (52) implies that $A + BK$ is stable.

Let y_i be the ith row of matrix Y in (3), $i = 1, \cdots, r$. Then $\mathbb{X}_t \sim W$ can be written as [14]

$$\mathbb{X}_t \sim W = \{z \in \mathbb{R}^n | y_i z \leq q_i - h_W(y_i), \ i = 1, \cdots, r\}, \tag{53}$$

where $h_S(\eta)$ is the support function of a bounded set S at $\eta \in \mathbb{R}^n$ (a row vector) defined as follows

$$h_S(\eta) := \sup_{s \in S} \ \eta s. \tag{54}$$

If $\mathbb{X}_t$ has vertices $\{g_1, \cdots, g_{n_v}\}$, then the first condition in (52) can be written as

$$\begin{aligned} &y_i(A + BK)g_j \leq q_i - h_W(y_i), \ i \in \{1, \cdots, r\}, \\ &j \in \{1, \cdots, n_v\}. \end{aligned} \tag{55}$$

Moreover, if

$$\mathbb{U} = \{u \in \mathbb{R}^m | t_i u \leq 1, \ i = 1, \cdots, M\}, t_i \in \mathbb{R}^m$$

then the second condition in (52) can be written as

$$t_i K g_j \leq 1, \ i \in \{1, \cdots, M\}, \ j \in \{1, \cdots, n_v\}. \tag{56}$$

Let us consider the following Linear Programming problem:

$$\begin{aligned} &\max_{K,\beta} \beta \\ &\text{subject to:} \\ &y_i(A + BK)g_j \leq q_i - \beta h_W(y_i), \ i \in \{1, \cdots, r\}, \\ &j \in \{1, \cdots, n_v\} \\ &t_i K g_j \leq 1, \ i \in \{1, \cdots, M\}, \ j \in \{1, \cdots, n_v\}. \end{aligned} \tag{57}$$

If this Linear Programming is feasible with β_m as the solution, then $\mathbb{X}_t$ is robust invariant for disturbance set $\beta_m W$. If $\beta_m \geq 1$, then constraints (55) and (56) are feasible and K_m is the desired feedback gain.

Remark 5.1: If $\mathbb{X}_t$ is robust invariant for disturbance set βW, larger β results in a more contractive matrix A_K and therefore constraint tightening will be less conservative.

VI. Numerical example

In this section, we consider the following constrained linear model

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + w. \tag{58}$$

The state and control constraint sets are

$$\mathbb{X} = \{[x_1 \ x_2] | x_2 \leq 2\}$$

and

$$\mathbb{U} = \{u | u \leq 1\},$$

respectively. The disturbance set is $W = \{w | \|w\|_\infty \leq 0.1\}$ and the target constraint set is $\mathbb{X}_t = \{x | \|x\|_\infty \leq 0.3\}$. The feedback gain $K = [-0.528 \ \ -1.13]$ is designed such that $\mathbb{X}_t$ contains the minimal invariant set corresponding to the system $x^+ = A_K x + w, \ w \in W$. The terminal constraint set $\mathbb{X}_f$ which is the maximal robust invariant set for the system $x^+ = Ax + Bu + w, \ w \in A^N_K W$, satisfying (10), is shown in Figure 1. Moreover, the constrained sets $\mathbb{X}, \mathbb{X}_1, \cdots, \mathbb{X}_N$ are parallel lines in Figure 1. The length of horizon is $N = 10$.

Figure 1 shows the state trajectory for initial state $x_0 = [-20\ 1]'$. It can be seen that the state trajectory is steered to the target set (because $F_K \subset \mathbb{X}_t$), while the terminal constraint set $\mathbb{X}_f$ is much larger than $\mathbb{X}_t$. Figure 2 shows the control signal corresponding to the state trajectory depicted in Figure 1.

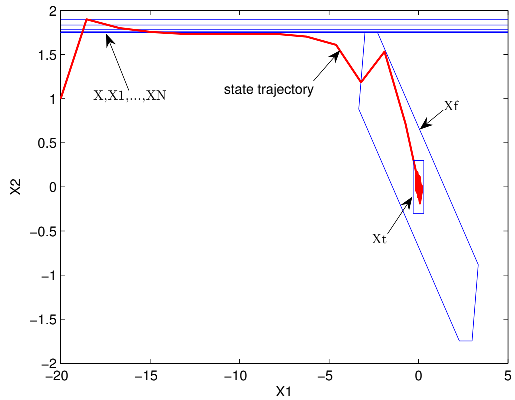

Fig. 1. Trajectories for the example in Section V.

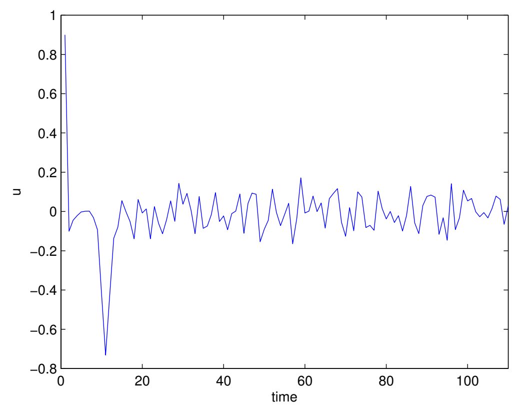

Fig. 2. Control signal for the example in Section V.

Remark 6.1: For the case where the additive disturbance is $W = \{w \| \|w\|_\infty \leq 0.2\}$ the region off attractions introduced in [12] and the one introduced in this paper are compared in Figure 3. It can be seen that, as prooved in Section IV, the region of attraction of the method introduced in [12], blue line, is subset of the one introduced in this paper, red one.

VII. Conclusion

This paper presented a robust controller for constrained linear systems with bounded disturbances. The novel feature of the robust controller is that the control action is a linear combination of known data at each sampling time and therefore it is highly computationally effective. The proposed controller guarantees convergence of state trajectory to a minimal invariant set of the desired system while explicit specification or approximation of such set is not required.

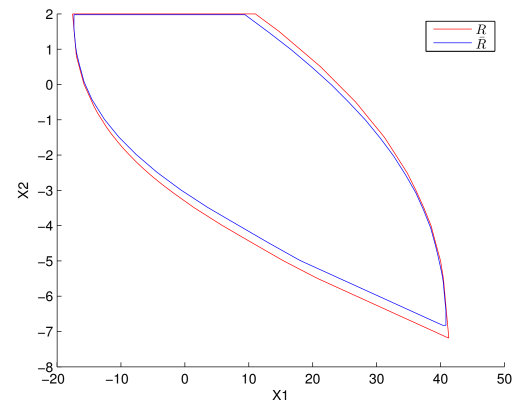

Fig. 3. Comparison of region of attractions introduced in Section IV.

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