2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Backstepping Boundary Stabilization and State Estimation of a 2×2 Linear Hyperbolic System

Rafael Vazquez, Miroslav Krstic and Jean-Michel Coron

Abstract— **We consider the problem of boundary stabilization and state estimation for a 2×2 system of first-order hyperbolic linear PDEs with spatially varying coefficients. First, we design a full-state feedback law with actuation on only one end of the domain and prove exponential stability of the closed-loop system. Then, we construct a collocated boundary observer which only needs measurements on the controlled end and prove convergence of observer estimates. Both results are combined to obtain a collocated output feedback law. The backstepping method is used to obtain both control and observer kernels. The kernels are the solution of a 4×4 system of first-order hyperbolic linear PDEs with spatially varying coefficients of Goursat type, whose well-posedness is shown.**

I. Introduction

In this paper we are concerned with the problem of boundary stabilization and state estimation for a 2×2 system of first-order hyperbolic linear PDEs with spatially varying coefficients. We consider actuation in only one of the boundaries, and measurement in the same boundary.

This problem has been previously considered for 2×2 quasilinear systems [7] and even $n \times n$ quasilinear systems [13], using the explicit evolution of the Riemann invariants along the characteristics. More recently, an approach using control Lyapunov functions has been developed, for 2×2 quasilinear systems [2] and $n \times n$ quasilinear systems [3]. These results use only *static output feedback* (the output being the value of the state at the boundaries). However they do not deal with the same class of systems considered in this work (which includes some extra terms in the equations); with these terms, it has been shown in [1] that there are linear systems for which there are no control Lyapunov functions of the form $\int_0^1 w^T Q(x) w dx$ (see the next sections for notation) which would allow the computation of a static output feedback law to stabilize the system (even when feedback is allowed on both sides of the boundary).

Several other authors have also studied this problem. For instance, the linear case has been analyzed in [23] (using a Lyapunov approach) and in [14] (using a spectral approach). The nonlinear case has been considered by [5] and [8] using a Lyapunov approach, and in [15], [16], and [6] using a Riemann invariants approach.

In this work, we use the backstepping method to design a full-state feedback law (with actuation on only one end of the domain) that makes the closed-loop system exponentially stable. Using the same method we construct a collocated boundary observer with measurements on the controlled end, and prove convergence of observer estimates. Then, both results are put together to obtain a (collocated) *dynamic output feedback* law. The gains of the feedback laws and the observer are obtained as solutions of a 4 x 4 system of first-order hyperbolic linear PDEs with spatially varying coefficients, whose well-posedness is shown in an appendix.

The basis of our designs is the backstepping method [10]; initially developed for parabolic equations, it has been applied to first-order hyperbolic equations [12], delay systems [11], second-order hyperbolic equations [18], fluid flows [20], nonlinear PDE equations [21] and even used for PDE adaptive designs [19].

The result in this paper is related to the result in [12], where the method of backstepping was used to deal with a wave equation with antidamping. There, to eliminate the antidamping, an invertible backstepping transformation with a 2×2 structure was developed. The kernels of the transformations were generated from two coupled second-order hyperbolic PDEs in Goursat form. Similarly, in this paper, we obtain four coupled first-order hyperbolic PDEs in Goursat form which could be transformed to obtain two coupled second-order hyperbolic PDEs; however, it is found that well-posedness is more easily shown in the first-order hyperbolic form.

The paper is organized as follows. In Section II we formulate the problem. In Section III we introduce the target system (which we show exponentially stable), the backstepping transformation that maps the plant into the target system, and the resulting full-state feedback law. We close the section by proving exponential stability of the closed-loop system. In Section IV we present our boundary observer design and prove convergence of observer estimates to the real states. Finally, in Section V we combine the full-state feedback law and the observer to obtain a (collocated) output-feedback law that stabilizes the system using only measurements from the actuated boundary. In Section VI we briefly cover a particular case of the system boundary conditions not covered in the general treatment. We finish in Section VII with some concluding remarks. We also include an appendix with the proof of well-posedness of the kernel equations.

II. Problem statement

Consider the following system

$$
\begin{aligned}
u_t &= -\epsilon_1(x) u_x + c_1(x) v, && (1)\\
v_t &= \epsilon_2(x) v_x + c_2(x) u, && (2)
\end{aligned}
$$

R. Vazquez is with the Department of Aerospace Engineering, Universidad de Sevilla, Camino de los Descubrimiento s.n., 41092 Sevilla, Spain.

M. Krstic is with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093-0411, USA.

J.-M. Coron is with the Laboratoire Jacques-Louis Lions, Univ. Pierre et Marie Curie and Institut Universitaire de France, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France.

evolving in $x \in [0,1]$, $t > 0$, with $\epsilon_1(x), \epsilon_2(x) > 0$ and boundary conditions

$$\begin{aligned} u(0,t) &= qv(0,t), && (3)\\ v(1,t) &= U(t), && (4)\end{aligned}$$

where $U(t)$ is the actuation, which can be chosen as desired. In what follows, we assume $q \neq 0$. The case $q = 0$ is briefly covered in Section VI.

In (1)–(2), ϵ_1, ϵ_2 are assumed to be positive-valued $\mathcal{C}^1([0,1])$ functions and c_1, c_2 are assumed to be $\mathcal{C}([0,1])$ functions. The initial conditions, denoted as u_0 and v_0, are assumed to belong to $L^2([0,1])$.

Based on the canonical transformation presented in [1], this is the more general form for a one-dimensional 2×2 hyperbolic linear system (without including integral or boundary terms).

Taking into account the signs of the transport coefficients, the variable u represents information that travels from left to right, and v information that travels from right to left. For this coefficients, the system is well posed since u has a boundary condition on the left and v on the right [17].

Our objective is to choose $U(t)$ to ensure that the closed-loop system is globally asymptotically stable. Also we assume that $u(1,t)$ can be measured. First we design a full-state feedback controller, then we design a boundary observer, and finally we formulate an output feedback controller combining both designs.

III. Stabilizing full-state feedback control law

A. Target system

Our approach to design $U(t)$ will be to seek a mapping that transforms (1)–(2) into

$$\begin{aligned} \alpha_t &= -\epsilon_1(x)\alpha_x, && (5)\\ \beta_t &= \epsilon_2(x)\beta_x, && (6)\end{aligned}$$

with boundary conditions

$$\begin{aligned} \alpha(0,t) &= q\beta(0,t), && (7)\\ \beta(1,t) &= 0, && (8)\end{aligned}$$

and then $U(t)$ will be chosen to realize the transformation.

One can find the explicit solution of (5)–(8) as follows. Define

$$\phi_1(x) = \int_0^x \frac{1}{\epsilon_1(\xi)} d\xi,\ \phi_2(x) = \int_0^x \frac{1}{\epsilon_2(\xi)} d\xi, \quad (9)$$

noting that they are monotonically increasing functions of x, and thus invertible. Now, if $\alpha_0(x)$, $\beta_0(x)$ are the initial condition for the states, the solution of the system is:

$$\alpha(x,t) = \begin{cases} \alpha_0\left(\phi_1^{-1}\left(\phi_1(x) - t\right)\right) & t \leq \phi_1(x) \\ q\beta(t - \phi_1(x), 0) & t \geq \phi_1(x) \end{cases} \quad (10)$$

$$\beta(x,t) = \begin{cases} \beta_0\left(\phi_2^{-1}\left(\phi_2(x) + t\right)\right) & t \leq \phi_2(1) - \phi_2(x) \\ 0 & t \geq \phi_2(1) - \phi_2(x) \end{cases} \quad (11)$$

Thus, after $t = t_F$, where

$$t_F = \phi_1(1) + \phi_2(1) = \int_0^1 \left(\frac{1}{\epsilon_1(\xi)} + \frac{1}{\epsilon_2(\xi)} \right) d\xi, \quad (12)$$

one has that $\alpha \equiv \beta \equiv 0$.

It can also be proved that the system is stable in the L^2 sense by using the following Lyapunov function:

$$L = \frac{1}{2} \int_0^1 \left(\frac{2-\xi}{\epsilon_1(\xi)} \alpha^2(\xi,t) + \frac{(1+\xi)2q^2}{\epsilon_2(\xi)} \beta^2(\xi,t) \right) d\xi, \quad (13)$$

since then one has, integrating by parts:

$$\begin{aligned} \dot{L} &= -\alpha^2(1,t) + 4q^2\beta^2(1,t) + 2\alpha^2(0,t) - 2q^2\beta^2(0,t) \\ &\quad - \int_0^1 \left(\alpha^2(\xi,t) + 2q^2\beta^2(\xi,t) \right) d\xi, && (14)\end{aligned}$$

and applying the boundary conditions,

$$\begin{aligned} \dot{L} &= -\alpha^2(1,t) - \int_0^1 \left(\alpha^2(\xi,t) + 2q^2\beta^2(\xi,t) \right) d\xi \\ &\leq -cL, && (15)\end{aligned}$$

with $c = \min_{x\in[0,1]} \{\epsilon_1(x), \epsilon_2(x)\}$.

B. Backstepping transformation and kernel equations

To map the original system (1)–(2) into the target system (5)–(6), we look for a transformation defined as follows:

$$\begin{aligned} \alpha(x,t) &= u(x,t) - \int_0^x K^{uu}(x,\xi)u(\xi,t)d\xi \\ &\quad - \int_0^x K^{uv}(x,\xi)v(\xi,t)d\xi, && (16)\\ \beta(x,t) &= v(x,t) - \int_0^x K^{vu}(x,\xi)u(\xi,t)d\xi \\ &\quad - \int_0^x K^{vv}(x,\xi)v(\xi,t)d\xi. && (17)\end{aligned}$$

Introducing (16)–(17) into (5)–(6), one obtains the equations that the kernels must satisfy. To simplify the computations, we introduce the following notation:

$$\begin{aligned} K(x,\xi) &= \begin{pmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{pmatrix}, \\ Q_0 &= \begin{pmatrix} 0 & q \\ 0 & 1 \end{pmatrix}, \Sigma(x) = \begin{pmatrix} -\epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) \end{pmatrix}, \\ C(x) &= \begin{pmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{pmatrix}, w(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, \\ \gamma(x,t) &= \begin{pmatrix} \alpha(x,t) \\ \beta(x,t) \end{pmatrix}. \end{aligned}$$

Then the original plant, target system and transformation can be written compactly (omitting dependences in x and t) as

$$w_t = \Sigma w_x + Cw, w(0,t) = Q_0 w(0,t), w(1,t) = \begin{pmatrix} 0 \\ U \end{pmatrix}, \quad (18)$$

$$\gamma_t = \Sigma\gamma_x,\ \gamma(0,t) = Q_0\gamma(0,t),\ \ \gamma(1,t) = 0, \quad (19)$$

$$\gamma = w - \int_0^x K(x,\xi)w(\xi,t)d\xi. \quad (20)$$

Introducing the transformation into the target system, using the plant equations, integrating by parts, and using the boundary conditions, we obtain a set of three matrix equations:

$$\begin{aligned} 0 &= C(x) + \Sigma(x)K(x,x) - K(x,x)\Sigma(x), && (21)\\ 0 &= \Sigma(x)K_x(x,\xi) + K_\xi(x,\xi)\Sigma(\xi) + K(x,\xi)\Sigma'(\xi) \\ &\quad -K(x,\xi)C(\xi), && (22)\\ 0 &= K(x,0)\Sigma(0)Q. && (23)\end{aligned}$$

Expanding these terms, we get the following kernel equations:

$$\begin{aligned}
\epsilon_1(x)K^{uu}_x + \epsilon_1(\xi)K^{uu}_\xi &= -\epsilon_1'(\xi)K^{uu} - c_2(\xi)K^{uv}, && (24)\\
\epsilon_1(x)K^{uv}_x - \epsilon_2(\xi)K^{uv}_\xi &= \epsilon_2'(\xi)K^{uv} - c_1(\xi)K^{uu}, && (25)\\
\epsilon_2(x)K^{vu}_x - \epsilon_1(\xi)K^{vu}_\xi &= \epsilon_1'(\xi)K^{vu} + c_2(\xi)K^{vv}, && (26)\\
\epsilon_2(x)K^{vv}_x + \epsilon_2(\xi)K^{vv}_\xi &= -\epsilon_2'(\xi)K^{vv} + c_1(\xi)K^{vu}, && (27)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
K^{uu}(x,0) &= \frac{\epsilon_2(0)}{q\epsilon_1(0)}K^{uv}(x,0), && (28)\\
K^{uv}(x,x) &= \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (29)\\
K^{vu}(x,x) &= -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (30)\\
K^{vv}(x,0) &= \frac{q\epsilon_1(0)}{\epsilon_2(0)}K^{vu}(x,0). && (31)
\end{aligned}$$

The equations evolve in the triangular domain $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$. Notice that they can be written as two separate 2×2 hyperbolic systems, one for K^{uu} and K^{uv} and another for K^{vu} and K^{vv}.

By Theorem 4 (see the Appendix), one finds that, under the assumptions of Section II, there is a unique solution to (24)–(31), which is in $\mathcal{C}(\mathcal{T})$.

C. The inverse transformation

To study the invertibility of the proposed transformation, we look for a transformation of the the target system (5)–(6) into the original system (1)–(2) as follows:

$$\begin{aligned}
u(x,t) &= \alpha(x,t) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(\xi,t)d\xi\\
&\quad + \int_0^x L^{\alpha\beta}(x,\xi)\beta(\xi,t)d\xi, && (32)\\
v(x,t) &= \beta(x,t) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(\xi,t)d\xi\\
&\quad + \int_0^x L^{\beta\beta}(x,\xi)\beta(\xi,t)d\xi. && (33)
\end{aligned}$$

Introducing (32)–(33) into (1)–(2), one obtains the equations that the kernels must satisfy. Introduce as before the notation

$$L(x,\xi) = \begin{pmatrix} L^{\alpha\alpha}(x,\xi) & L^{\alpha\beta}(x,\xi)\\ L^{\beta\alpha}(x,\xi) & L^{\beta\beta}(x,\xi)\end{pmatrix}. \quad (34)$$

Then the inverse transformation can be written compactly as

$$w(x,t) = \gamma(x,t) + \int_0^x L(x,\xi)\gamma(\xi,t)d\xi. \quad (35)$$

Introducing the transformation into the original system we find, proceeding as before, a set of kernel equations:

$$\begin{aligned}
\epsilon_1(x)L^{\alpha\alpha}_x + \epsilon_1(\xi)L^{\alpha\alpha}_\xi &= -\epsilon_1'(\xi)L^{\alpha\alpha} + c_1(x)L^{\beta\alpha}, && (36)\\
\epsilon_1(x)L^{\alpha\beta}_x - \epsilon_2(\xi)L^{\alpha\beta}_\xi &= \epsilon_2'(\xi)L^{\alpha\beta} + c_1(x)L^{\beta\beta}, && (37)\\
\epsilon_2(x)L^{\beta\alpha}_x - \epsilon_1(\xi)L^{\beta\alpha}_\xi &= \epsilon_1'(\xi)L^{\beta\alpha} - c_2(x)L^{\alpha\alpha}, && (38)\\
\epsilon_2(x)L^{\beta\beta}_x + \epsilon_2(\xi)L^{\beta\beta}_\xi &= -\epsilon_2'(\xi)L^{\beta\beta} - c_2(x)L^{\alpha\beta}, && (39)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
L^{\alpha\alpha}(x,0) &= \frac{\epsilon_2(0)}{q\epsilon_1(0)}L^{\alpha\beta}(x,0), && (40)\\
L^{\alpha\beta}(x,x) &= \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (41)\\
L^{\beta\alpha}(x,x) &= -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, && (42)\\
L^{\beta\beta}(x,0) &= \frac{q\epsilon_1(0)}{\epsilon_2(0)}L^{\beta\alpha}(x,0). && (43)
\end{aligned}$$

Again by Theorem 4 (see the Appendix), one finds that there is a unique solution to these equations, which is $\mathcal{C}(\mathcal{T})$.

D. Control law and main result

From the transformation (17) evaluated at $x = 1$, one gets

$$U = \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi. \quad (44)$$

With this control law, we obtain our main state-feedback result, summarized in the following theorem.

Theorem 1: Consider system (1)–(2) with boundary conditions (3)–(4), initial conditions u_0 and v_0, and with control law (44) where the kernels K^{vu} and K^{vv} are obtained from (24)–(31). Then, under the assumptions

$$\epsilon_1, \epsilon_2 \in \mathcal{C}^1([0,1]),\ c_1, c_2 \in \mathcal{C}([0,1]),\ u_0, v_0 \in L^2([0,1]), \quad (45)$$

and $\epsilon_1(x), \epsilon_2(x) > 0$, the equilibrium $u \equiv v \equiv 0$ is exponentially stable in the L^2 sense. Moreover, the equilibrium is reached in finite time $t = t_F$, where t_F is given by (12).

Proof: Since the transformation (16)–(17) is invertible, when applying control law (44) the dynamical behavior of (1)–(2) is the same as the behavior of (5)–(6), which is well-posed from standard results and whose explicit solution we know. Thus, we obtain the explicit solutions of (1)–(2) the system from the direct and inverse transformation, as follows:

$$w(x,t) = \gamma^*(x,t) + \int_0^x L(x,\xi)\gamma^*(\xi,t)d\xi, \quad (46)$$

where $\gamma^*(x,t)$ is the explicit solution of the α, β system, given by (10)–(11), with initial conditions:

$$\gamma_0(x) = w_0(x) - \int_0^x K(x,\xi)w_0(\xi)d\xi. \quad (47)$$

In particular, we know that α and β go to zero in finite time $t = t_F$, therefore u and v also share that property. Finally, since the α, β system is L^2 exponentially stable, we conclude, using the inverse transformation, that the u, v system is also L^2 exponentially stable. ■

IV. A (COLLOCATED) BOUNDARY OBSERVER

A. Observer structure

Next we assume that we can measure $u(x,t)$ at the boundary $x = 1$, and design an observer to estimate both infinite-dimensional states. The estimates are denoted by a hat, and we construct our estimator as a copy of the system with output injection terms, as follows:

$$\begin{aligned}
\hat{u}_t &= -\epsilon_1(x)\hat{u}_x + c_1(x)\hat{v} + p_1(x)\left(u(1,t) - \hat{u}(1,t)\right), && (48)\\
\hat{v}_t &= \epsilon_2(x)\hat{v}_x + c_2(x)\hat{u} + p_2(x)\left(u(1,t) - \hat{u}(1,t)\right), && (49)
\end{aligned}$$

with the following boundary conditions:

$$\hat{u}(0,t) = q\hat{v}(0,t), \quad (50)$$
$$\hat{v}(1,t) = U(t). \quad (51)$$

The terms $p_1(x)$ and $p_2(x)$ are output injection gains to be designed.

Subtracting the estimates from the states, we get the estimation error system (error estimates are denoted by a tilde):

$$\tilde{u}_t = -\epsilon_1(x)\tilde{u}_x + c_1(x)\tilde{v} - p_1(x)\tilde{u}(1,t), \quad (52)$$
$$\tilde{v}_t = \epsilon_2(x)\tilde{v}_x + c_2(x)\tilde{u} - p_2(x)\tilde{u}(1,t), \quad (53)$$

with the following boundary conditions:

$$\tilde{u}(0,t) = q\tilde{v}(0,t), \quad (54)$$
$$\tilde{v}(1,t) = 0. \quad (55)$$

B. Backstepping observer transformation

To find the output injection gains that guarantee that the error system decays to zero, we use a backstepping transformation to map the error system into the target system:

$$\tilde{\alpha}_t = -\epsilon_1(x)\tilde{\alpha}_x, \quad (56)$$
$$\tilde{\beta}_t = \epsilon_2(x)\tilde{\beta}_x, \quad (57)$$

with boundary conditions

$$\tilde{\alpha}(0,t) = q\tilde{\beta}(0,t), \quad (58)$$
$$\tilde{\beta}(1,t) = 0. \quad (59)$$

The backstepping transformation is:

$$\tilde{u}(x,t) = \tilde{\alpha}(x,t) - \int_x^1 P^{uu}(x,\xi)\tilde{\alpha}(\xi,t)d\xi - \int_x^1 P^{uv}(x,\xi)\tilde{\beta}(\xi,t)d\xi, \quad (60)$$
$$\tilde{v}(x,t) = \tilde{\beta}(x,t) - \int_x^1 P^{vu}(x,\xi)\tilde{\alpha}(\xi,t)d\xi - \int_x^1 P^{vv}(x,\xi)\tilde{\beta}(\xi,t)d\xi, \quad (61)$$

Introducing (60)–(61) into (52)–(53), one obtains the equations that the kernels must satisfy. Introducing, as before, the following notation:

$$P(x,\xi) = \begin{pmatrix} P^{uu}(x,\xi) & P^{uv}(x,\xi) \\ P^{vu}(x,\xi) & P^{vv}(x,\xi) \end{pmatrix},$$
$$\tilde{w}(x,t) = \begin{pmatrix} \tilde{u}(x,t) \\ \tilde{v}(x,t) \end{pmatrix}, \ \tilde{\gamma}(x,t) = \begin{pmatrix} \tilde{\alpha}(x,t) \\ \tilde{\beta}(x,t) \end{pmatrix},$$
$$P_i(x) = \begin{pmatrix} p_1(x) & 0 \\ p_2(x) & 0 \end{pmatrix}, \ Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (62)$$

Then the original plant, target system and transformation can be written compactly as

$$\tilde{w}_t = \Sigma\tilde{w}_x + C\tilde{w} - P_i\tilde{w}(1,t), \quad (63)$$
$$\tilde{w}(0) = Q_0\tilde{w}(0), \ \tilde{w}(1) = Q_1\tilde{w}(1), \quad (64)$$
$$\tilde{\gamma}_t = \Sigma\tilde{\gamma}_x, \ \tilde{\gamma}(0) = Q_0\tilde{\gamma}(0), \ \tilde{\gamma}(1) = Q_1\tilde{\gamma}(1), \quad (65)$$
$$\tilde{w}(x,t) = \tilde{\gamma}(x,t) - \int_x^1 P(x,\xi)\tilde{\gamma}(\xi,t)d\xi. \quad (66)$$

Introducing the transformation into the estimation error system we find as before the following system of kernel equations:

$$\epsilon_1(x)P^{uu}_x + \epsilon_1(\xi)P^{uu}_\xi = -\epsilon_1'(\xi)P^{uu} - c_1(x)P^{vu}, \quad (67)$$
$$\epsilon_1(x)P^{uv}_x - \epsilon_2(\xi)P^{uv}_\xi = \epsilon_2'(\xi)P^{uv} - c_1(x)P^{vv}, \quad (68)$$
$$\epsilon_2(x)P^{vu}_x - \epsilon_1(\xi)P^{vu}_\xi = \epsilon_1'(\xi)P^{vu} + c_2(x)P^{uu}, \quad (69)$$
$$\epsilon_2(x)P^{vv}_x + \epsilon_2(\xi)P^{vv}_\xi = -\epsilon_2'(\xi)P^{vv} + c_2(x)P^{uv}, \quad (70)$$

with boundary conditions:

$$P^{uu}(0,\xi) = qP^{vu}(0,\xi), \quad (71)$$
$$P^{uv}(x,x) = \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, \quad (72)$$
$$P^{vu}(x,x) = -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, \quad (73)$$
$$P^{vv}(0,\xi) = \frac{1}{q}P^{uv}(0,\xi), \quad (74)$$

and the additional conditions on the output injection kernels:

$$p_1(x) = -\epsilon_1(1)P^{uu}(x,1), \quad (75)$$
$$p_2(x) = -\epsilon_1(1)P^{vu}(x,1). \quad (76)$$

By Theorem 4 (see the Appendix), one finds that there is a unique solution to (67)–(74), which is in $\mathcal{C}(\mathcal{T})$. Thus the output injection gains can be found and are continuous.

C. Inverse observer transformation

The inverse observer transformation is defined as:

$$\tilde{\alpha}(x,t) = \tilde{u}(x,t) + \int_x^1 R^{\alpha\alpha}(x,\xi)\tilde{u}(\xi,t)d\xi + \int_x^1 R^{\alpha\beta}(x,\xi)\tilde{v}(\xi,t)d\xi, \quad (77)$$
$$\tilde{\beta}(x,t) = \tilde{v}(x,t) + \int_x^1 R^{\beta\alpha}(x,\xi)\tilde{u}(\xi,t)d\xi + \int_x^1 R^{\beta\beta}(x,\xi)\tilde{v}(\xi,t)d\xi, \quad (78)$$

As before, defining:

$$R(x,\xi) = \begin{pmatrix} R^{\alpha\alpha}(x,\xi) & R^{\alpha\beta}(x,\xi) \\ R^{\beta\alpha}(x,\xi) & R^{\beta\beta}(x,\xi) \end{pmatrix}, \quad (79)$$

the transformation is written as

$$\tilde{\gamma}(x,t) = \tilde{w}(x,t) + \int_x^1 R(x,\xi)\tilde{w}(\xi,t)d\xi. \quad (80)$$

Introducing the transformation into the estimation error system we find as before the following equations:

$$\epsilon_1(x)R^{\alpha\alpha}_x + \epsilon_1(\xi)R^{\alpha\alpha}_\xi = -\epsilon_1'(\xi)R^{\alpha\alpha} - c_2(\xi)R^{\alpha\beta}, \quad (81)$$
$$\epsilon_1(x)R^{\alpha\beta}_x - \epsilon_2(\xi)R^{\alpha\beta}_\xi = \epsilon_2'(\xi)R^{\alpha\beta} - c_1(\xi)R^{\alpha\alpha}, \quad (82)$$
$$\epsilon_2(x)R^{\beta\alpha}_x - \epsilon_1(\xi)R^{\beta\alpha}_\xi = \epsilon_1'(\xi)R^{\beta\alpha} + c_2(\xi)R^{\beta\beta}, \quad (83)$$
$$\epsilon_2(x)R^{\beta\beta}_x + \epsilon_2(\xi)R^{\beta\beta}_\xi = -\epsilon_2'(\xi)R^{\beta\beta} + c_1(\xi)R^{\beta\alpha}, \quad (84)$$

with boundary conditions:

$$R^{\alpha\alpha}(x,0) = qR^{\beta\alpha}(x,0), \quad (85)$$

$$R^{\alpha\beta}(x,x) = \frac{c_1(x)}{\epsilon_1(x)+\epsilon_2(x)}, \quad (86)$$

$$R^{\beta\alpha}(x,x) = -\frac{c_2(x)}{\epsilon_1(x)+\epsilon_2(x)}, \quad (87)$$

$$R^{\beta\beta}(x,0) = \frac{1}{q}R^{\alpha\beta}(x,0). \quad (88)$$

By Theorem 4 (see the Appendix), one finds that there is a unique solution to (81)–(88), which is in $\mathcal{C}(\mathcal{T})$. Thus the output injection gains can be found and are continuous.

D. Main observer result

From what has been shown, exponential stability of the estimation error system follows, which implies that the state estimates go to the real values as time grows. In fact, they do in finite time. This is summarized in the following theorem, whose proof we skip since it is identical to the proof of Theorem 1.

Theorem 2: Consider system (52)–(53) with boundary conditions (54)–(55) and initial conditions $\tilde{u}_0$ and $\tilde{v}_0$, with output injection kernels given by (75)–(76), where P^{uu} and P^{vu} are obtained from (67)–(74). Under the assumptions of Theorem 1, the equilibrium $\tilde{u} \equiv \tilde{v} \equiv 0$ is exponentially stable in the L^2 sense. Moreover, the equilibrium is reached in finite time $t = t_F$, where t_F is given by (12). This implies that $\|\hat{u}(\cdot,t)-u(\cdot,t)\|_{L^2(0,1)} \to 0$ and $\|\hat{v}(\cdot,t)-v(\cdot,t)\|_{L^2(0,1)} \to 0$ as $t \to t_F$.

V. Collocated output feedback control

Combining the full state feedback law and the observer estimates, we propose a feedback law

$$U = \int_0^1 K^{vu}(1,\xi)\hat{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\hat{v}(\xi,t)d\xi, \quad (89)$$

where $\hat{u}$ and $\hat{v}$ are computed from

$$\begin{aligned} \hat{u}_t &= -\epsilon_1(x)\hat{u}_x + c_1(x)\hat{v} \\ &\quad -\epsilon_1(x)P^{uu}(x,1)\left(u(1,t)-\hat{u}(1,t)\right), \quad (90) \\ \hat{v}_t &= \epsilon_2(x)\hat{v}_x + c_2(x)\hat{u} \\ &\quad -\epsilon_1(x)P^{vu}(x,1)\left(u(1,t)-\hat{u}(1,t)\right), \quad (91) \\ \hat{u}(0,t) &= q\hat{v}(0,t),\ \hat{v}(1,t)=U, \quad (92) \end{aligned}$$

and where the kernels K and P are obtained from their respective kernel equations. The following result follows from standard arguments, combining Theorems 1 and 2.

Theorem 3: Consider system (1)–(2) with boundary conditions (3)–(4), initial conditions u_0 and v_0, and with control law (89)–(92) where the kernels K^{vu} and K^{vv} are obtained from (24)–(31) and the kernels P^{uu} and P^{vu} from (67)–(74). Under the assumptions of Theorem 1, the equilibrium $u \equiv v \equiv 0$ of is exponentially stable in the L^2 sense. Moreover, the equilibrium is reached in finite time $t = 2t_F$, where t_F is given by (12).

VI. The case of small or zero values of q

If the coefficient q is zero in (3), the method presented in the paper is not valid since (31) would require the value of one of the control kernels to be infinity in the boundary of the domain $\mathcal{T}$. Similarly, if the coefficient is close to zero one still gets very large values for the kernels close to the boundary.

The method can be modified to accommodate zero or small values of q by setting a slightly different target system (5)–(6), as follows:

$$\alpha_t = -\epsilon_1(x)\alpha_x + g(x)\beta(0,t), \quad (93)$$

$$\beta_t = \epsilon_2(x)\beta_x, \quad (94)$$

where $g(x)$ is to be obtained from the method; regardless of the value of $g(x)$, this is a cascade system which is still L^2 exponentially stable by standard arguments.

The kernel equations resulting from the transformation are still the same (24)–(27), with the same boundary conditions (29)–(31) for K^{uv} , K^{vu} and K^{vv} (which reduces to $K^{vv}(x,0) = 0$ when $q = 0$), but one obtains an *undetermined* boundary conditions for K^{uu}:

$$K^{uu}(x,0) = h(x), \quad (95)$$

where $h(x)$ can be chosen as desired. After $h(x)$ has been chosen and the kernels have been computed, one must set $g(x) = q\epsilon_1(0)K^{uu}(x,0) - \epsilon_2(0)K^{uv}(x,0)$.

Invertibility of the transformation follows as before, thus one obtains a result equivalent to Theorem 1. The non-uniqueness in (95) gives the designer some freedom in shaping the input function $g(x)$ from β to α.

To design a boundary observer for small or zero q, it is necessary to use measurements of v from the $x = 0$ end. Then, the boundary condition (50) is modified to

$$\hat{u}(0,t) = q\hat{v}(0,t) + \hat{q}(\hat{v}(0,t) - v(0,t)), \quad (96)$$

where $\hat{q}$ can be chosen as desired. Thus boundary condition (54) is changed to

$$\tilde{u}(0,t) = (q+\hat{q})\tilde{v}(0,t), \quad (97)$$

which implies that q is substituted by $q + \hat{q}$ in the observer kernel equations, a value that can be chosen to avoid large boundary conditions of the kernels. A equivalent result to Theorem 2 then follows.

To avoid the need of using measurements from the two boundaries, one might design instead an anti-collocated observer that requires only measurements of v at the uncontrolled end. We skip the details for lack of space.

VII. Concluding remarks

In this work, we have solved the problem of boundary stabilization and state estimation for a 2×2 system of first-order hyperbolic linear PDEs with spatially varying coefficients. We have shown L^2 exponential stability of the state (and of the error system, for the boundary observer).

The method can also be applied if the system is not linear but *quasilinear*, obtaining a *local* result; the details can be found in [22]. There, it is shown that to guarantee well-posedness of the closed-loop system one must obtain H^2 stability [2], which requires the control kernels to be at least twice differentiable. The results of Theorem 5 guarantee this degree of differentiability if the coefficients of the system are smooth enough. The quasilinear case is of interest since several relevant physical systems are described by 2×2 systems of first-order hyperbolic quasilinear PDEs, such as open channels, transmission lines, gas flow pipelines or road traffic models.

REFERENCES

[1] G. Bastin and J.-M. Coron, "Further Results on boundary feedback stabilisation of 2×2 hyperbolic systems over a bounded interval," *Proceedings of 8th NOLCOS*, pp. 1081–1085, 2010.

[2] J.-M. Coron, B. d'Andrea-Novel and G. Bastin, "A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws," *IEEE Transactions on Automatic Control*, vol. 52, pp. 2–11, 2006.

[3] J.-M. Coron, G. Bastin and B. d'Andrea-Novel, "Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems," *SIAM Journal of Control and Optimization*, vol. 47, pp. 1460–1498, 2008.

[4] A. Diagne, G. Bastin and J.-M. Coron, "Lyapunov exponential stability of linear hyperbolic systems of balance laws," to appear in *Proceedings of the 18th IFAC World Congress*, 2011.

[5] M. Dick, M. Gugat and G. Leugering, "Classical solutions and feedback stabilisation for the gas flow in a sequence of pipes," *Networks and heterogeneous media*, vol. 5, pp. 691–709, 2010.

[6] V. Dos Santos and C. Prieur, "Boundary control of open channels with numerical and experimental validations," *IEEE Trans. Control Syst. Tech.*, vol. 16, pp. 1252–1264, 2008.

[7] J-M. Greenberg and T-t. Li, "The effect of boundary damping for the quasilinear wave equations," *Journal of Differential Equations*, vol. 52, pp. 66–75, 1984.

[8] M. Gugat and M. Herty, "Existence of classical solutions and feedback stabilisation for the flow in gas networks," *ESAIM Control Optimisation and Calculus of Variations*, vol. 17, pp. 28–51, 2011.

[9] R. P. Holten, "Generalized Goursat problem," *Pacific Journal of Mathematics*, vol. 12, pp. 207–224, 1962.

[10] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, SIAM, 2008

[11] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

[12] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *System and Control Letters*, vol. 57, pp. 750–758, 2008.

[13] T-t. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Wiley, 1994.

[14] X. Litrico and V. Fromion, "Boundary control of hyperbolic conservation laws using a frequency domain approach," *Automatica*, vol. 45, pp. 647–656, 2009.

[15] C. Prieur, "Control of systems of conservation laws with boundary errors," *Networks and Heterogeneous Media*, vol. 4, pp. 393–407, 2009.

[16] C. Prieur, J. Winkin and G. Bastin, "Robust boundary control of systems of conservation laws," *Mathematics of Control, Signals, and Systems*, vol. 20, pp. 173–197, 2008.

[17] D. L. Russell, " Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions," *SIAM Review*, Vol. 20(4), pp. 639–739, 1978.

[18] A. Smyshlyaev, E. Cerpa and M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping," *SIAM Journal of Control and Optimization*, vol. 48, pp. 4014–4031, 2010.

[19] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

[20] R. Vazquez and M. Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flow*, Birkhauser, 2008.

[21] R. Vazquez and M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities — Part I: Design," *Automatica*, vol. 44, pp. 2778–2790, 2008.

[22] R. Vazquez, J.-M. Coron, M. Krstic and G. Bastin, "Local Exponential H^2 Stabilization of a 2×2 Quasilinear Hyperbolic System using Backstepping," submitted to *50th IEEE CDC and ECC*, 2011.

[23] C.Z. Xu and G. Sallet, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems," *ESAIM Control Optimisation and Calculus of Variations*, vol. 7, pp. 421–442, 2002.

APPENDIX

A. Well-posedness of the kernel equations

We show well-posedness of the following hyperbolic 4×4 system, which is generic enough to contain all the kernel equation systems that appear in the paper:

$$\epsilon_1(x)F^1_x + \epsilon_1(\xi)F^1_\xi = g_1(x,\xi) + \sum_{i=1}^{4} C_{1i}(x,\xi)F^i(x,\xi), \quad (98)$$

$$\epsilon_1(x)F^2_x - \epsilon_2(\xi)F^2_\xi = g_2(x,\xi) + \sum_{i=1}^{4} C_{2i}(x,\xi)F^i(x,\xi), \quad (99)$$

$$\epsilon_2(x)F^3_x - \epsilon_1(\xi)F^3_\xi = g_3(x,\xi) + \sum_{i=1}^{4} C_{3i}(x,\xi)F^i(x,\xi), \quad (100)$$

$$\epsilon_2(x)F^4_x + \epsilon_2(\xi)F^4_\xi = g_4(x,\xi) + \sum_{i=1}^{4} C_{4i}(x,\xi)F^i(x,\xi), \quad (101)$$

evolving in the domain $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$, with boundary conditions:

$$F^1(x,0) = h_1(x) + q_1(x)F^2(x,0) + q_2(x)F^3(x,0), \quad (102)$$

$$F^2(x,x) = h_2(x), \quad F^3(x,x) = h_3(x), \quad (103)$$

$$F^4(x,0) = h_4(x) + q_3(x)F^2(x,0) + q_4(x)F^3(x,0). \quad (104)$$

This type of system has been called "generalized Goursat problem" by some authors [9]. However the boundaries of the domain $\mathcal{T}$ are characteristic for (98) and (101), thus the general results derived in [9] cannot be applied. The following theorems discusses existence, uniqueness and smoothness of solutions to the equations.

Theorem 4: Consider the hyperbolic system (98)–(104). Under the assumptions

$$q_i, h_i \in \mathcal{C}([0,1]),\ g_i, C_{ji} \in \mathcal{C}(\mathcal{T}),\ i,j = 1,2,3,4 \quad (105)$$

and $\epsilon_1, \epsilon_2 \in \mathcal{C}([0,1])$ with $\epsilon_1(x), \epsilon_2(x) > 0$, there exists a unique $\mathcal{C}(\mathcal{T})$ solution F^i, $i = 1,2,3,4$.

Theorem 5: Consider the hyperbolic system (98)–(104). Under the assumptions of Theorem 4, and the additional assumptions

$$\epsilon_i, q_i, h_i \in \mathcal{C}^N([0,1]),\ g_i, C_{ji} \in \mathcal{C}^N(\mathcal{T}), \quad (106)$$

there exists a unique $\mathcal{C}^N(\mathcal{T})$ solution F^i, $i = 1,2,3,4$.

We skip the proof of the theorems due to page limitation. The steps used in the proof are transforming the equations into integral equations (using the method of characteristics) and solving them using a successive approximation method, which results in a series solution. Thus existence and uniqueness is proven. To prove smoothness of solutions, one takes derivatives in (98)–(101) with respect to x and ξ finding new hyperbolic systems of equations which are formally similar to (98)–(101). Applying Theorem 4 to this system one finds solutions for these derivatives. Iterating this procedure, the required degree of smoothness can be shown.