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On the Attitude Estimation of Accelerating Rigid-Bodies Using GPS and IMU Measurements

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***Abstract*—We present two observers that estimate the orientation of a rigid body, which is subjected to linear accelerations and rotational motion, using a global positioning system (GPS) and a body-attached inertial measurement unit (IMU). Unlike some other attitude estimation schemes (which assume that the accelerometer measures the gravity vector, which is not realistic when the rigid body is subject to large linear accelerations), the proposed results belong to the special class of *velocity-aided attitude observers*, which instead use the *true* accelerometer measurements (*i.e.,* the system's apparent acceleration). The linear velocity of the rigid body (obtained from the GPS) is used to obviate the requirement of the linear acceleration (which is assumed unavailable in the inertial frame). The new observers can handle large accelerations of the rigid body which could otherwise destroy the performance of other types of attitude observers which assume that the accelerometer measures the gravity vector.**

I. INTRODUCTION

We address the problem of estimating the attitude, or orientation, of a rigid-body which is subjected to linear and rotational motion. Obtaining solutions for this problem is advantageous for a growing number of fields which seek to identify the relative orientation between a body-fixed frame and a fixed inertial frame of reference. For example, some applications which use orientation measurements (or estimates) to achieve their goals include human-motion-capture based computer animation, inertial-navigation and mobile robotics (wheeled, biped, airborne, satellite,etc.). This particular problem has been the focus of several research groups, who have subsequently produced a number of interesting advancements in this area.

In general, the relative attitude of a body-fixed frame with respect to an inertial frame is estimated (or measured) by using a set of known, inertial-referenced vectors, which are subsequently measured in the body fixed frame. Based upon these measurements there are a variety of methods which have been proposed to solve this problem. One of these methods uses the vector measurements to directly calculate or reconstruct the relative orientation, without the use of a filter or attitude observer. Examples of these types of algorithms can be found in [1], [2], [3] and [4]. Another method, which is known as *complementary filtering*, involves using one of these reconstruction algorithms in addition to measurements of the rigid-body's angular velocity (measured in the body-fixed frame using a gyroscope) in order to improve the accuracy of the estimates when the rigid body is subjected to rotational motion. In fact, a simple and yet practical dynamic IMU-based attitude estimation approach is based on linear complementary filtering [5], where the vector measurements are fused with the angular velocity measurement to recover the orientation of the rigid body for small angular movements. This approach has been extended later on to nonlinear complementary filtering for the attitude estimation from vector measurements (for example see [6], [7], [8] and [9]). A number of observers have also been designed to use the vector measurements directly, and therefore no longer require the use of one of these reconstruction methods. Examples of these types of observers can be found in [6], [8], [10] and [11].

A common characteristic of these attitude estimation schemes is that they depend upon a set of vectors which are known in the inertial frame of reference. However, there are a limited number of sensors which can satisfy this requirement. Two sensors which are most commonly used in this capacity are the accelerometer and magnetometer, which due to advancements in sensor technology, such as Integrated Micro-Electro-Mechanical systems (IMEMs), are small, inexpensive and widely available. The magnetometer is used to provide a body-referenced measurement of the known ambient magnetic field, where in many cases the accelerometer is used to provide a body-referenced measurement of the gravity vector. Unsurprisingly, the performance of observers which are designed by assuming the accelerometer measures only the gravity vector will likely suffer when the rigid body is subjected to significant linear accelerations.

Fortunately, this challenge has motivated several research groups to develop observers which acknowledge the fact that the accelerometer measures the *apparent acceleration* (a term which has been recently adopted in the literature to define the inertial-referenced vector which includes the rigid-body linear acceleration vector and the gravity vector). However, the use of the accelerometer in this fashion is a challenging problem since the apparent acceleration vector is not known in the inertial frame, which violates one of the fundamental requirements of the attitude estimation methods listed above. To address this problem, a new type of observer has been previously proposed which uses the inertial-referenced rigid-body velocity (measured using a GPS). Examples of these observers, which are commonly known as *velocity-aided attitude observers*, can be found in [12], [13] and [14]. In [12] the authors are able to show convergence of the attitude estimates locally around the trajectories of the system attitude.

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In [13] the authors demonstrate similar convergence while also considering sensor imperfections. In [14] the authors show that by making appropriate choices of the observer gains, the attitude estimates are guaranteed to converge for almost all initial conditions.

This paper, contributes a step further towards understanding the mechanisms behinds velocity-aided attitude observers by providing new constructive stability proofs. In fact, we propose two new velocity-aided attitude observers guaranteeing almost global results. The first observer, although different from the observer of [14], provides similar almost global results. As a consequence of this modification, the resulting stability analysis of the observer has been significantly simplified. The new stability analysis has proven to be quite advantageous, since in [15] a new type of position controller for vertical take-off and landing (VTOL) unmanned airborne vehicles (UAVs) has been proposed based upon these results, which otherwise may have been more difficult to achieve. The second proposed observer is referred to as higher order observer (one degree higher than the first proposed observer), which is also shown to guarantee almost global stability results. Simulations results are also provided to demonstrate the performance of the proposed observers.

II. BACKGROUND

A. Attitude Representation

We make use of two forms of attitude parameterization, namely the direct cosine (rotation) matrix and the unit-quaternion, in order to model the relative orientation from an inertial frame of reference (denoted as $\mathcal{I}$) to a body-fixed frame of reference (denoted as $\mathcal{B}$) which is rigidly attached to the system center-of-gravity (COG) in North-East-Down coordinates. The unit-quaternion which defines the relative orientation of $\mathcal{I}$ to $\mathcal{B}$ is defined as $Q=(\eta,q)$, $\eta\in\mathbb{R}$, $q\in\mathbb{R}^3$ where Q belongs to the set of unit-quaternion defined by

$$Q=(\eta,q)\in\mathbb{Q}:=\{Q\in\mathcal{S}^3,\ \|Q\|=1\}, \qquad (1)$$

where $\mathcal{S}^3$ denotes a three-dimensional sphere [1]. Consequently, the quaternion scalar η and vector q must satisfy $\eta^2+q^{\mathsf{T}}q=1$. We denote R as the rotation matrix which defines the relative orientation of $\mathcal{I}$ to $\mathcal{B}$, where R belongs to the special orthogonal group $R\in SO(3)$ where

$$SO(3):=\{R\in\mathbb{R}^{3\times3}, \det\ R=1, RR^T=R^TR=I\}. \qquad (2)$$

where $I=I_{3\times3}$ is the three-dimensional identity matrix. A well known expression which relates the values of $Q=(\eta,q)$ and R is given by

$$R(\eta,q) \quad = \quad I+2S(q)^2-2\eta S(q), \qquad (3)$$

where $S(\cdot)$ is the *skew-symmetric matrix*

$$S(u)=\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0\end{bmatrix}, \qquad (4)$$

[1]For more details on the unit-quaternion (in addition to other forms of attitude representation) the reader is referred to [16], [17], and [18].

and $u=[u_1,u_2,u_3]^{\mathsf{T}}$. The set $\mathbb{Q}$ forms a group with the quaternion product operation, denoted by $\odot$, with the quaternion inverse defined by $Q^{-1}=(\eta,-q)$ and identity-quaternion $Q=(1,0_{3\times1})$, where $0_{3\times1}\in\mathbb{R}^3$ is a column vector of zeros. Given $Q,P\in\mathbb{Q}$ where $P=(p_0,p)$ the quaternion product is defined by

$$Q\odot P \quad = \quad \left(p_0\eta-q^Tp, \quad \eta p+p_0q+S(q)p\right). \qquad (5)$$

B. System Model

Let r_1 denote the ambient magnetic field vector (assumed constant), and v denote the system velocity of the rigid-body, which are both expressed in $\mathcal{I}$. We consider the following model for the system translational and rotational dynamics:

$$\dot v = ge_3+r_2, \qquad (6)$$

$$\dot Q=\frac{1}{2}Q\odot(0,\omega)=\frac{1}{2}\begin{bmatrix} -q^{\mathsf{T}} \\ \eta I_{3\times3}+S(q)\end{bmatrix}\omega, \qquad (7)$$

where g is the acceleration due to gravity, $e_3=[0,0,1]^{\mathsf{T}}$, $r_2=r_2(t)$ is a time-varying unknown system input which is commonly referred to as the *apparent acceleration* and ω is the angular velocity of the rigid-body expressed in $\mathcal{B}$. The available system outputs are defined as $y=[v,\omega,b_1,b_2]^{\mathsf{T}}$ where $b_1=Rr_1$ and $b_2=Rr_2$ are the outputs of the magnetometer and accelerometer sensors, respectively, which give the coordinates of magnetic field vector and the apparent acceleration in the body-fixed frame $\mathcal{B}$. The system velocity v and the angular velocity ω are measured using a GPS and a gyroscope, respectively. Note that the attitude dynamics can be equivalently expressed in terms of the rotation matrix using $\dot R=-S(\omega)R$.

III. PROBLEM FORMULATION

Let $\hat Q=(\hat\eta,\hat q)$, $\hat\eta\in\mathbb{R}$, $\hat q\in\mathbb{R}^3$, $\hat Q\in\mathbb{Q}$ denote a unit quaternion which is an estimate of the system attitude Q, and let $\hat R=R(\hat\eta,\hat q)$ denote the rotation matrix that corresponds to $\hat Q$, as defined by (3). Our primary objective in this paper is to develop a suitable estimation law for $\dot{\hat Q}$ such that the attitude estimate converges to the actual system position, or $\hat Q\to Q$. To preserve the unit-quaternion properties of the estimate $\hat Q$ we consider observers which are based on the dynamics of the unit quaternion and rotation matrix, for example

$$\dot{\hat Q}=\frac{1}{2}\begin{bmatrix} -\hat q^{\mathsf{T}} \\ \hat\eta I_{3\times3}+S(\hat q)\end{bmatrix}\beta, \qquad \dot{\hat R}=-S(\beta)\hat R, \qquad (8)$$

where $\beta\in\mathbb{R}^3$ is an observer law which is later designed using the system outputs $y=[v,\omega,b_1,b_2]^{\mathsf{T}}$ in addition to the value of the known magnetic field vector r_1. One characteristic of this problem that complicates the design and analysis of the observers, is that the system apparent acceleration r_2 is not known in the inertial frame. However, this challenge can be somewhat simplified by placing some realistic constraints on the value of r_2, which are stated in the following assumptions.

Assumption 1: *There exists positive constants* c_1 *and* c_2 *such that* $\|r_2\|\le c_1$ and $\|\dot r_2\|\le c_2$.

***Assumption** 2: Given two positive constants, γ_1 and γ_2, there exists a positive constant $c_w(\gamma_1,\gamma_2)$ such that $c_w < \lambda_{\min}(W)$ where*

$$W = -\gamma_1 S(r_1)^2 - \gamma_2 S(r_2)^2. \tag{9}$$

The second assumption is satisfied if the system apparent acceleration r_2 is non-vanishing and is not collinear to the magnetic field vector r_1. This follows from the fact that the eigenvalues of the negative semi-definite matrix $S(r_i)^2$, $r_i \in \mathbb{R}^3$, are given by $\lambda(S(r_i)^2) = (-\|r_i\|, -\|r_i\|, 0)$, where the null-space of $S(r_i)^2$ is collinear to the vector r_i. Therefore, if both r_1 and r_2 are non-zero and are not-collinear, then $W = W^T$ is positive definite. In the case where $r_2 = 0$, the system velocity dynamics become $\dot{v} = ge_3$ (which corresponds to the rigid body being in a free-fall state) which is not likely under normal circumstances.

In addition to the restrictions on the apparent acceleration r_2, we place the following additional constraints on the system angular and linear velocity signals (observer inputs).

***Assumption** 3: There exists a positive constants c_3 and c_4 such that the system angular velocity ω and system linear velocity v are bounded such that $\|\omega\| < c_3$ and $\|v\| < c_4$.*

Although the third assumption is not needed to show convergence of the attitude estimates to the actual system attitude, we include this requirement in order to ensure boundedness of all signals involved with the attitude observers (internal stability).

To characterize the attitude error we use the quaternion $\tilde{Q} = (\tilde{\eta}, \tilde{q}) \in \mathbb{Q}$ and the rotation matrix $\tilde{R} = R(\tilde{\eta}, \tilde{q}) \in SO(3)$ where

$$\tilde{Q} = Q \odot \hat{Q}^{-1}, \quad \tilde{R} = \hat{R}^{\mathsf{T}} R. \tag{10}$$

In light of (7) and (8) the attitude error is governed by the following dynamic equations

$$\begin{aligned}
\dot{\tilde{Q}} &= \frac{1}{2}\begin{bmatrix} -\tilde{q}^{\mathsf{T}} \\ \tilde{\eta} I + S(\tilde{q}) \end{bmatrix} \tilde{\omega}, && (11)\\
\dot{\tilde{R}} &= -S(\tilde{\omega})\tilde{R}, && (12)\\
\tilde{\omega} &= \hat{R}^{\mathsf{T}}(\omega - \beta). && (13)
\end{aligned}$$

Given this representation for the attitude error, the primary objective of $\hat{Q} \to Q$ is equivalent to $\tilde{q} \to 0$ and $\tilde{\eta} \to \pm 1$. Note that there are two (physically identical) equilibria that correspond to the two desired values of $\tilde{\eta}$. This is due to the fact that the map from the unit-quaternion space to the real-space is non-injective (two-to-one map). For more details regarding this well known topological obstruction the reader is referred to [19] and [20]. Based upon this formulation we now define the observers in the following two sections.

IV. Observer 1

We define the adaptive state $\hat{v} \in \mathbb{R}^3$, and the error function $\tilde{v} = v - \hat{v}$, which is subsequently used to define the following observer

$$\begin{aligned}
\dot{\hat{Q}} &= \frac{1}{2}\begin{bmatrix} -\hat{q}^{\mathsf{T}} \\ \hat{\eta} I + S(\hat{q}) \end{bmatrix} (\omega + \sigma), && (14)\\
\sigma &= -\gamma_1 S(\hat{R} r_1) b_1 - \gamma_2 k_1 S(\hat{R}\tilde{v}) b_2, && (15)\\
\dot{\hat{v}} &= k_1 \tilde{v} + g e_3 + \hat{R}^{\mathsf{T}} b_2 + \frac{1}{k_1}\hat{R}^{\mathsf{T}} S(\sigma) b_2, && (16)
\end{aligned}$$

where $k_1, \gamma_1, \gamma_2 > 0$, $\hat{R} = R(\hat{\eta}, \hat{q})$ is defined using (3) and $S(\cdot)$ is the skew-symmetric matrix defined by (4). Using the definition of the attitude error as defined by (10), we propose the following theorem:

***Theorem** 1:* Consider the system (6)-(7), coupled with the observer given by (14)-(16) where Assumptions 1-3 are satisfied. Then for all initial conditions $\tilde{\eta}(t_0) \neq 0$, there exists a constant $\kappa_1 > 0$ such that for all $k_1 > \kappa_1$, there exists a constant ρ, $0 < \rho < |\tilde{\eta}(t_0)|$, such that $|\tilde{\eta}(t)| \geq \rho$ for all $t \geq t_0$, all of the observer signals are bounded and the signals $\tilde{v}$ and $\tilde{q}$ converge exponentially to zero.

Proof: The first step in the proof is to study the dynamics of the attitude error (in terms of the quaternion scalar $\tilde{\eta}$). To this end let us define the following error signal:

$$\tilde{r}_2 = k_1 \tilde{v} - (I - \tilde{R}) r_2. \tag{17}$$

Using the definition of the attitude error, in addition to the definition of $\tilde{\omega}$ from (13) as well as (8) and (14), one can find $\tilde{\omega} = -\hat{R}^{\mathsf{T}}\sigma$. Therefore, using (15) and the definition of the error signal defined by (17), the derivative of the attitude error in terms of the quaternion scalar $\tilde{\eta}$ is given by $\dot{\tilde{\eta}} = -\frac{1}{2}\tilde{q}^{\mathsf{T}}(S(r_1)\tilde{R}r_1 + \gamma_2 S(r_2)\tilde{R}r_2 + \gamma_2 S(\tilde{r}_2)\tilde{R}r_2)$ where we used the fact that $S(\hat{R}u) = \hat{R}S(u)\hat{R}^{\mathsf{T}}$, and $S(u)u = 0$, where $u \in \mathbb{R}^3$. Due to the definition of the rotation matrix (3), in addition to the properties $S(u)^2 = uu^{\mathsf{T}} - u^{\mathsf{T}}uI$ and $S(u)v = -S(v)u$, $u, v \in \mathbb{R}^3$, one can further show that $\tilde{q}^{\mathsf{T}} S(r_i)\tilde{R}r_i = 2\tilde{\eta}\tilde{q}^{\mathsf{T}} S(r_i)^2 \tilde{q}$ where $r_i \in \mathbb{R}^3$. Therefore, the derivative of $\tilde{\eta}$ is found to be

$$\dot{\tilde{\eta}} = \tilde{\eta}\tilde{q}^{\mathsf{T}} W \tilde{q} + \gamma_2 \tilde{q}^{\mathsf{T}} S(\tilde{R}r_2)\tilde{r}_2/2, \tag{18}$$

where W is the matrix defined by (9). We now focus our attention to study the dynamics of the error signals $\tilde{v}$ and $\tilde{r}_2$. In light of (6) and (16), the derivative of $\tilde{v}$ is given by

$$\dot{\tilde{v}} = -\tilde{r}_2 - k_1^{-1}\hat{R}^{\mathsf{T}} S(\sigma) b_2. \tag{19}$$

Using this result, in addition to (12), the derivative of $\tilde{r}_2$ is subsequently found to be

$$\dot{\tilde{r}}_2 = -k_1 \tilde{r}_2 - (I - \tilde{R})\dot{r}_2. \tag{20}$$

We now consider the following Lyapunov function candidate:

$$\mathcal{V} = \frac{\gamma}{2}\tilde{r}_2^{\mathsf{T}}\tilde{r}_2 + \gamma_q \tilde{q}^{\mathsf{T}}\tilde{q} = \frac{\gamma}{2}\tilde{r}_2^{\mathsf{T}}\tilde{r}_2 + \gamma_q(1 - \tilde{\eta}^2), \tag{21}$$

where $\gamma, \gamma_q > 0$. Using the expressions (18)-(20) the derivative of $\mathcal{V}$ is found to be

$$\begin{aligned}
\dot{\mathcal{V}} = & -\gamma k_1 \tilde{r}_2^{\mathsf{T}}\tilde{r}_2 - 2\gamma_q \tilde{\eta}^2 \tilde{q}^{\mathsf{T}} W \tilde{q} \\
& -\gamma_2\gamma_q \tilde{\eta}\tilde{q}^{\mathsf{T}} S(\tilde{R}r_2)\tilde{r}_2 - \gamma \tilde{r}_2^{\mathsf{T}}(I - \tilde{R})\dot{r}_2 \qquad (22)
\end{aligned}$$

To determine the upper bound for the derivative of $\mathcal{V}$, in light of Assumption 1, we first use Young's inequality to find

$$\gamma_2\gamma_q\tilde{\eta}\tilde{q}^{\mathsf{T}}S(\tilde{R}r_2)\tilde{r}_2 \leq \frac{\epsilon_1\gamma_2\gamma_q c_1^2}{2}\tilde{r}_2^{\mathsf{T}}\tilde{r}_2 + \frac{\gamma_2\gamma_q}{2\epsilon_1}\tilde{\eta}^2\tilde{q}^{\mathsf{T}}\tilde{q} \tag{23}$$

$$\begin{aligned}\gamma\tilde{r}_2^{\mathsf{T}}(I-\tilde{R})\dot{r}_2 &= 2\gamma\tilde{r}_2^{\mathsf{T}}\left(S(\tilde{q})-\tilde{\eta}I\right)S(\dot{r}_2)\tilde{q}\\ &\leq \epsilon_2\gamma c_2^2\tilde{r}_2^{\mathsf{T}}\tilde{r}_2 + \frac{\gamma}{\epsilon_2}\tilde{q}^{\mathsf{T}}\tilde{q}.\end{aligned} \tag{24}$$

where we used the fact that $\|S(\tilde{q})-\tilde{\eta}I\| \leq 1$. Consequently, we find the following bound for the derivative of $\mathcal{V}$

$$\begin{aligned}\dot{\mathcal{V}} \leq\ & -\gamma\left(k_1 - \epsilon_1\gamma_2\gamma_q c_1^2/(2\gamma) - \epsilon_2 c_2^2\right)\tilde{r}_2^{\mathsf{T}}\tilde{r}_2,\\ & -\tilde{\eta}^2\tilde{q}^{\mathsf{T}}\tilde{q}\left(2\gamma_q c_w - \gamma_2\gamma_q/(2\epsilon_1) - \gamma/(\epsilon_2\tilde{\eta}^2)\right).\end{aligned} \tag{25}$$

In the case where $\tilde{\eta} = 0$ we cannot make any claims on stability since (25) may be positive. Therefore, we are forced to exclude the initial condition $\tilde{\eta}(t_0) = 0$. We then define a desired minimum bound for the quaternion scalar, which we denote as ρ. The choice of ρ must therefore satisfy

$$0 < \rho < |\tilde{\eta}(t_0)| \leq 1. \tag{26}$$

We subsequently choose a value for the gain γ as follows

$$\gamma = \bar{\gamma}(\|\tilde{r}_2(t_0)\|^2 + \delta)^{-1}, \tag{27}$$

where we also place the additional constraints $\delta > 0$ and

$$0 < \bar{\gamma} < 2\gamma_q\left(\tilde{\eta}(t_0)^2 - \rho^2\right). \tag{28}$$

If we consider the result (25) we can conclude that there exists constants $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ such that for all $\epsilon_1 > \bar{\epsilon}_1$ and $\epsilon_2 > \bar{\epsilon}_2$ the following inequality is satisfied

$$2\gamma_q c_w > \gamma_2\gamma_q/(2\epsilon_1) + \gamma/(\epsilon_2\rho^2). \tag{29}$$

Furthermore, there exists a gain $\kappa_1(\epsilon_1,\epsilon_2)$ such that for all $k_1 > \kappa_1$ the following inequality is also satisfied

$$k_1 > \epsilon_1\gamma_2\gamma_q c_1^2/(2\gamma) + \epsilon_2 c_2^2. \tag{30}$$

Note that the gain k_1 approaches infinity as $\rho \to |\tilde{\eta}(t_0)|$, therefore it is reasonable to take ρ a sufficient distance away from the initial condition. As a result of the choices on the gains, a sufficient condition for $\dot{\mathcal{V}} \leq 0$ is $|\tilde{\eta}(t)| \geq \rho$. We now wish to show that $|\tilde{\eta}(t)| \geq \rho$ for all $t > t_0$. To this effect we first recall that $|\tilde{\eta}(t_0)| > \rho$. Suppose there exists a time t_1 such that for all $t_0 \leq t < t_1$, $|\tilde{\eta}(t)| \geq \rho$, and $|\tilde{\eta}(t_1)| < \rho$. Due to the choice of the gain γ from (27) the expression of the Lyapunov function can be written as

$$\mathcal{V}(t) = \frac{\bar{\gamma}}{2}\frac{\|\tilde{r}_2(t)\|^2}{\|\tilde{r}_2(t_0)\|^2 + \delta} + \gamma_q\left(1 - \tilde{\eta}(t)^2\right). \tag{31}$$

Therefore, in light of the choice of $\bar{\gamma}$ given by (28) it can easily be seen that

$$\mathcal{V}(t_0) < \gamma_q\left(1-\rho^2\right). \tag{32}$$

Since $|\tilde{\eta}(t_1)| < \rho$ then we can also conclude

$$\mathcal{V}(t_1) > \gamma_q\left(1-\tilde{\eta}(t_1)^2\right) > \gamma_q\left(1-\rho^2\right). \tag{33}$$

Therefore, we have shown that $\mathcal{V}(t_1) > \mathcal{V}(t_0)$, which is a contradiction since $\dot{\mathcal{V}} \leq 0$ for $t_0 \leq t < t_1$. Therefore, we conclude that $|\tilde{\eta}(t)| \geq \rho$ for all $t \geq t_0$, and

$$\dot{\mathcal{V}} \leq -\delta_r\tilde{r}_2^{\mathsf{T}}\tilde{r}_2 - \delta_q\tilde{q}^{\mathsf{T}}\tilde{q}, \tag{34}$$

where $\delta_r = \gamma(k_1 - \epsilon_1\gamma_2\gamma_q c_1^2/(2\gamma) - \epsilon_2 c_2^2)$ and $\delta_q = \rho^2(2\gamma_q c_w - \gamma_2\gamma_q/(2\epsilon_1) - \gamma/(\epsilon_2\rho^2))$. Therefore, since δ_r and δ_q are positive (due to the choices of the gains), $\dot{\mathcal{V}} \leq 0$ which implies that $\tilde{r}_2$ is bounded. Since r_2 is bounded (due to Assumption 1), it follows from the definition of $\tilde{r}_2$ that the error function $\tilde{v}$ is bounded, and therefore the signal σ is bounded. In light of Assumption 3, it follows that the adaptive state $\hat{v}$ and the observer input $\beta = \omega + \sigma$ are bounded. Due to the definition of the Lyapunov function, we can further see that $\dot{\mathcal{V}} \leq -\epsilon_v\mathcal{V}$, where $\epsilon_v = \min(\delta_r,\delta_q)/\max(\gamma/2,\gamma_q)$ which implies that the states $\tilde{r}_2$ and $\tilde{q}$ converge exponentially to zero. ■

V. Observer 2

For the second observer we define two adaptive estimates, denoted by $\hat{v},\psi \in \mathbb{R}^3$, and the error function $\tilde{v} = v - \hat{v}$. We consider the following observer:

$$\dot{\hat{Q}} = \frac{1}{2}\begin{bmatrix} -\hat{q}^{\mathsf{T}} \\ \hat{\eta}I + S(\hat{q})\end{bmatrix}(\omega+\sigma), \tag{35}$$

$$\sigma = -\gamma_1 S(\hat{R}r_1)b_1 - \gamma_2 S(\hat{R}\hat{r}_2)b_2, \tag{36}$$

$$\hat{r}_2 = k_2\psi + k_3\tilde{v}, \tag{37}$$

$$\dot{\psi} = -k_4\psi + \frac{1}{k_2}\hat{R}^{\mathsf{T}}S(b_2)\sigma - k_5\tilde{v}, \tag{38}$$

$$\dot{\hat{v}} = k_1\tilde{v} + ge_3 + \hat{R}^{\mathsf{T}}b_2 + k_6\psi, \tag{39}$$

where $k_1,k_3,\gamma_1,\gamma_2 > 0$, $\hat{R} = R(\hat{\eta},\hat{q})$ as defined by (3) and $S(\cdot)$ is the skew-symmetric matrix defined by (4). Using the expressions for the attitude error defined by (10) we now state the following theorem.

Theorem 2: Consider the system (6)-(7), coupled with the observer given by (35)-(39) where Assumptions 1-3 are satisfied, and we choose the following values for the gains k_5 and k_6

$$k_5 = \frac{k_3(k_3-k_1)}{k_2} + \frac{k_4-k_3}{k_2k_3\gamma_r}, \quad k_6 = \frac{k_2(k_3-k_4)}{k_3}, \tag{40}$$

where $\gamma_r > 0$. Then for all initial conditions $\tilde{\eta}(t_0) \neq 0$, there exists gains $\kappa_1(\tilde{\eta}(t_0)) > k_3 - k_4 > 0$ and $\kappa_3(\tilde{\eta}(t_0)) > 0$ such that for all $k_1 > \kappa_1$ and $k_3 > \kappa_3$, there exists a constant ρ, $0 < \rho < |\tilde{\eta}(t_0)|$, such that $|\tilde{\eta}(t)| \geq \rho$ for all $t \geq t_0$, all the signals are bounded, and the signals $\tilde{v}$ and $\tilde{q}$ converge exponentially to zero.

Proof: We begin the proof by studying the derivative of the attitude error in terms of the quaternion scalar $\tilde{\eta}$. This requires we first define the following error signal

$$\tilde{r}_2 = k_2\psi + k_3\tilde{v} + (\tilde{R}-I)r_2 \tag{41}$$

Similar to the previous observer, in light of (8), (13) and (35) the expression for $\tilde{\omega}$ is given by $\tilde{\omega} = -\hat{R}^{\mathsf{T}}\sigma$. Therefore, using (11) and (36) the derivative of $\tilde{\eta}$ is found to be

$$\dot{\tilde{\eta}} = \tilde{\eta}\tilde{q}^{\mathsf{T}}W\tilde{q} + \frac{\gamma_2}{2}\tilde{q}^{\mathsf{T}}S(\tilde{R}r_2)\tilde{r}_2 \tag{42}$$

At this point we wish to study the derivatives of the error functions $\tilde{v}$ and $\tilde{r}_2$. In light of (6) and (39) in addition to the definition of the error signal $\tilde{r}_2$ defined by (41), the derivative of $\tilde{v}$ is given by

$$\dot{\tilde{v}} = \alpha_1\tilde{v} + \alpha_2\tilde{r}_2 + \alpha_3(\tilde{R} - I)r_2 \quad (43)$$

where $\alpha_1 = -k_1 + k_3k_6/k_2$, $\alpha_2 = -k_6/k_2$ and $\alpha_3 = k_6/k_2 - 1$. Using this result, in addition to (12), (13) and (38), we find the derivative of $\tilde{r}_2$ to be

$$\dot{\tilde{r}}_2 = \alpha_4\tilde{r}_2 + \alpha_5\tilde{v} + \alpha_6(\tilde{R} - I)r_2 + (\tilde{R} - I)\dot{r}_2 \quad (44)$$

where $\alpha_4 = -k_4 - k_3k_6/k_2$, $\alpha_5 = k_3k_4 + k_3^2k_6/k_2 - k_2k_5 - k_1k_3$ and $\alpha_6 = k_4 - k_3 + k_3k_6/k_2$. Note that in light of the choices of k_5 and k_6 from (40), the coefficients α_i are subsequently found to be

$$\begin{aligned}
&\alpha_1 = -k_1 + k_3 - k_4\\
&\alpha_2 = -k_6/k_2 = k_4/k_3 - 1\\
&\alpha_3 = (k_3 - k_4)/k_3 - 1 = -k_4/k_3\\
&\alpha_4 = -k_4 + k_4 - k_3 = -k_3\\
&\alpha_5 = k_3k_4 + k_3(k_3 - k_4) - (k_3(k_3 - k_1)\\
&\quad +(k_4 - k_3)/(k_3\gamma_r)) - k_1k_3\\
&\quad = 1/\gamma_r(1 - k_4/k_3)\\
&\alpha_6 = k_4 - k_3 + k_3k_6/k_2 = 0
\end{aligned} \quad (45)$$

Consequently, one can see that $\alpha_2 + \gamma_r\alpha_5 = 0$ and $\alpha_6 = 0$. Now, let us consider the following Lyapunov function candidate:

$$\mathcal{V} = \frac{\gamma}{2}(\tilde{v}^\mathsf{T}\tilde{v} + \gamma_r\tilde{r}_2^\mathsf{T}\tilde{r}_2) + \gamma_q(1 - \tilde{\eta}^2) \quad (46)$$

where $\gamma, \gamma_r > 0$. In light of the expressions for the derivatives of $\tilde{v}$, $\tilde{r}_2$ and $\tilde{\eta}$ from (43), (44) and (42), respectively, in addition to the fact that $\alpha_2 + \gamma_r\alpha_5 = 0$ and $\alpha_6 = 0$, the derivative of $\mathcal{V}$ is given by

$$\begin{aligned}
\dot{\mathcal{V}} = &\gamma\alpha_1\tilde{v}^\mathsf{T}\tilde{v} + \gamma\gamma_r\alpha_4\tilde{r}_2^\mathsf{T}\tilde{r}_2 + \gamma\alpha_3\tilde{v}^\mathsf{T}(\tilde{R} - I)r_2\\
&-2\gamma_q\tilde{\eta}^2\tilde{q}^\mathsf{T}W\tilde{q} - \gamma_2\gamma_q\tilde{\eta}\tilde{q}^\mathsf{T}S(\tilde{R}r_2)\tilde{r}_2
\end{aligned} \quad (47)$$

In order to find an upper bound for the derivative of $\mathcal{V}$, in light of Assumption 1 and the expressions for the coefficients of α_i given by (45), we use Young's inequality to find the bounds of the following cross terms:

$$\gamma\alpha_3\tilde{v}^\mathsf{T}(R_e - I)r_2 \leq \epsilon_1\frac{\gamma c_1^2k_4^2}{k_3^2}\tilde{v}^\mathsf{T}\tilde{v} + \frac{\gamma}{\epsilon_1}\tilde{q}^\mathsf{T}\tilde{q} \quad (48)$$

$$\gamma\gamma_r\tilde{r}_2^\mathsf{T}(\tilde{R} - I)\dot{r}_2 \leq \epsilon_2\gamma\gamma_r c_2^2\tilde{r}_2^\mathsf{T}\tilde{r}_2 + \frac{\gamma\gamma_r}{\epsilon_2}\tilde{q}^\mathsf{T}\tilde{q} \quad (49)$$

$$\gamma_2\gamma_q\tilde{\eta}\tilde{q}^\mathsf{T}S(\tilde{R}r_2)\tilde{r}_2 \leq \epsilon_3\frac{\gamma_q c_1^2\gamma_2^2}{2}\tilde{r}_2^\mathsf{T}\tilde{r}_2 + \frac{\gamma_q}{2\epsilon_3}\tilde{\eta}^2\tilde{q}^\mathsf{T}\tilde{q} \quad (50)$$

Therefore, the derivative of $\mathcal{V}$ is bounded by

$$\begin{aligned}
\dot{\mathcal{V}} \leq &-\gamma\left(k_1 + k_4 - k_3 - \epsilon_1\frac{c_1^2k_4^2}{k_3^2}\right)\tilde{v}^\mathsf{T}\tilde{v}\\
&-\gamma\gamma_r\left(k_3 - \epsilon_2c_2^2 - \epsilon_3\frac{\gamma_q c_1^2\gamma_2^2}{2\gamma\gamma_r}\right)\tilde{r}_2^\mathsf{T}\tilde{r}_2\\
&-\gamma_q\left(2c_w - \frac{1}{2\epsilon_3} - \frac{\gamma}{\gamma_q\tilde{\eta}^2}\left(\frac{1}{\epsilon_1} + \frac{\gamma_r}{\epsilon_2}\right)\right)\tilde{\eta}^2\tilde{q}^\mathsf{T}\tilde{q}
\end{aligned} \quad (51)$$

Similar to the previous observer, we need to show that $\tilde{\eta} \neq 0$. Therefore, we choose a desired minimum bound for the value of $|\tilde{\eta}(t)|$, which we denote as ρ, which must satisfy $0 < \rho < |\tilde{\eta}(t_0)| \leq 1$. We also choose the following value for γ:

$$\gamma = \bar{\gamma}(\|\tilde{r}_2(t_0)\|^2 + \delta)^{-1} \quad (52)$$

where $\delta > 0$ and $\bar{\gamma} < 2\gamma_q(\tilde{\eta}(t_0)^2 - \rho^2)$. Subsequently, we specify the following values for the remaining gains. The gain k_4 is arbitrarily chosen [2], and γ_r is chosen such that $\gamma_r > 0$. There exist strictly positive constants $\bar{\epsilon}_i, i = 1, 2, 3$ such that for $\epsilon_i > \bar{\epsilon}_i$ we have $2c_w > 1/(2\epsilon_3) + \gamma(1/\epsilon_1 + \gamma_r/\epsilon_2)/(\gamma_q\rho^2)$. Based on the values of ϵ_1 and ϵ_2, we then choose $k_3 > \kappa_3$ and $k_1 > \kappa_1$ where $\kappa_3 := \epsilon_2c_2^2 + \epsilon_3\gamma_q c_1^2\gamma_2^2/(2\gamma\gamma_r)$. and $\kappa_1(\epsilon_1) := k_3 - k_4 + \epsilon_1c_1^2k_4^2/k_3^2$. As a result of these choices for gains, it is clear that $\dot{\mathcal{V}}(t_0) \leq 0$. Furthermore, using similar arguments to the proof of the previous observer, we can show that if there exists a time t_1 such that $|\tilde{\eta}(t_1)| < \rho$, then $V(t_1) > V(t_0)$ which is not possible since $\dot{V}(t) \leq 0$ for $t_0 \leq t < t_1$. Therefore,

$$\begin{aligned}
&\dot{\mathcal{V}} \leq -\delta_r\tilde{r}_2^\mathsf{T}\tilde{r}_2 - \delta_v\tilde{v}^\mathsf{T}\tilde{v} - \delta_q\tilde{q}^\mathsf{T}\tilde{q},\\
&\delta_r = \gamma\gamma_r\left(k_3 - \epsilon_2c_2^2 - \epsilon_3\gamma_q c_1^2\gamma_2^2/(2\gamma\gamma_r)\right),\\
&\delta_v = \gamma\left(k_1 + k_4 - k_3 - \epsilon_1c_1^2k_4^2/k_3^2\right),\\
&\delta_q = \gamma_q\rho^2\left(2c_w - 1/(2\epsilon_3) - \gamma/(\gamma_q\rho^2)\left(1/\epsilon_1 + \gamma_r/\epsilon_2\right)\right),
\end{aligned}$$

where δ_r, δ_v and δ_q are positive constants due to the choices of the observer gains. This implies that $\tilde{v}$ and $\tilde{r}_2$ are bounded. Since $\tilde{v}$ is bounded and v is bounded due to Assumption 3, then $\hat{v}$ is bounded. The bound of the signal ψ follows from the fact that $\tilde{r}_2$ and r_2 are bounded. Therefore, the signal σ is bounded, and the observer input $\beta = \omega + \sigma$ is bounded due to the bound of ω. Also, in light of the definition of the Lyapunov function we obtain $\dot{\mathcal{V}} \leq -\epsilon_v\mathcal{V}$, $\epsilon_v = \min(\delta_v, \delta_r, \delta_q)/\max(\gamma/2, \gamma\gamma_r/2, \gamma_q)$ which implies that $\mathcal{V}$ and therefore $\tilde{r}_2$, $\tilde{v}$ and $\tilde{q}$ converge exponentially to zero. ■

VI. Simulations

Simulations were performed to test the performance of the two proposed attitude observers. For both simulations the trajectory of the rigid-body position was specified as $p(t) = (4\sin(0.5t + 0.5), 3\sin(1.25t + 0.5), \sin(0.5t + 0.5))$, from which the rigid-body velocity v and apparent acceleration r_2 were obtained. The angular velocity of the rigid body was chosen as $\omega = (\sin(0.1t), 0.2\sin(0.2t + \pi), 0.1\sin(0.3t + \pi/3))$. The inertial-referenced ambient magnetic field vector was chosen as $r_1 = [0.18, 0, 0.54]^\mathsf{T}G$. Figure 1 shows the signal r_2 as a result of the chosen position trajectory, which clearly demonstrates large deviations from the non-accelerating condition $r_2 = [0, 0, -g]$. Figure 2 gives the simulation results for the first observer, and Figure 3 give the results for the second observer.

The following initial conditions were used for both simulations: $\hat{v}(t_0) = [0, 1, 0]^\mathsf{T}$, $\hat{\eta}(t_0) = 1$, $\hat{q} = [0, 0, 0]^\mathsf{T}$, $\eta(t_0) = 0$ and $q(t_0) = [1, 0, 0]^\mathsf{T}$. This choice corresponds to a value of the attitude error $\tilde{\eta} = 0$ and $\tilde{q} = [1, 0, 0]^\mathsf{T}$, which is the worst

[2] Although the gain k_4 can be chosen to take any value, from a practical standpoint this gain should be chosen to be positive since this introduces a *leakage term* in the dynamics of ψ, and can improve the performance of the observer in the presence of noise and other disturbances. For more information on this leakage term, in addition to other practical tools in the area of adaptive control, the reader is referred to [21].

case scenario where the stability of the proposed observers is not guaranteed according to our proof. This initial conditions have been selected on purpose to show that the proposed observers did not fail even in this extreme case.

The following gains were used for observer 1: $k_1 = \gamma_1 = \gamma_2 = 1$. For the second observer the following gains were used: $k_1 = 10$, $\gamma_1 = \gamma_2 = 5$, $k_2 = k_3 = 2$ and $k_4 = 1$. The gains k_5 and k_6 were chosen to satisfy (40).

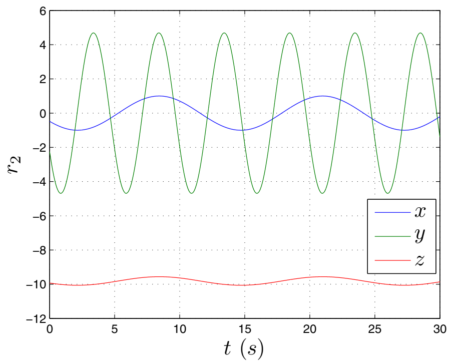

Fig. 1: System Apparent Accel. $r_2(t)$ (m/s^2)

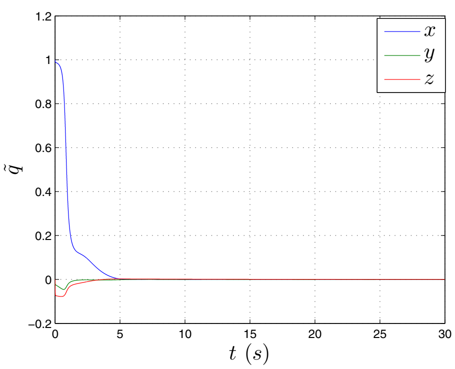

Fig. 2: Observer 1: Attitude Error $\tilde{q}$

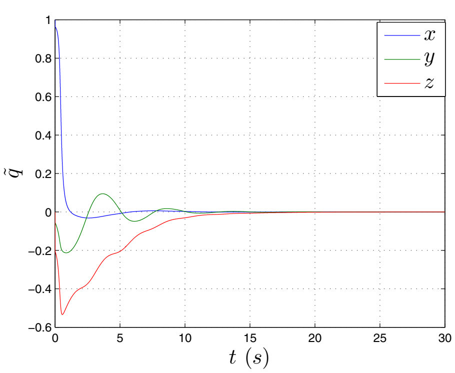

Fig. 3: Observer 2: Attitude Error $\tilde{q}$

VII. Conclusion

Two observers have been proposed which use an accelerometer, magnetometer, gyroscope and GPS to estimate the orientation of a rigid body, subjected to relatively large linear accelerations. In the earlier dynamic IMU-based attitude estimation versions, accelerometers are assumed to measure only the gravity vector, which can lead to poor performance in the case where the rigid body is subject to significant linear accelerations. The proposed observers, although different from the observer of [14], provides similar almost global results. A new stability analysis approach, that looks promising for future developments in this field, has been adopted.

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