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Empirical approach to robust gramian-based analysis of process interactions in control structure selection

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Abstract— **This paper deals with the estimation of a gramian-based interaction measure from logged process data, and thereby removing the need of creating parametric models prior to the selection of the significant input-output interconnections. Moreover, the resulting confidence regions of the estimates can be used to perform a robust control structure selection.**

The considered interaction measure is the Participation Matrix. Based on previous results, a new unbiased statistic is proposed, and confidence bounds for the estimate are derived.

Examples and a case study are used to illustrate how the method can be applied.

I. Introduction

A critical step in the design of control systems in multivariable processes is the choice of the structure of the controller. Current methods for control structure design include the so called Interaction Measures (IMs), which date back to 1966 when the Relative Gain Array (RGA) was introduced [1]. The IMs help the designer to select a subset of the most significant input-output channels, which will form a reduced model on which the control design will be based. The IM considered in this paper is the Participation Matrix (PM) [2].

The IMs require prior process modeling, which is usually a time consuming task. The complexity of the process modeling increases as the number of process variables increases. When modeling a complex process, the designer usually models only the input-output channels which he/she considers significant. When IMs are applied to such model, the analysis may be biased by the judgment of the designer, since potential input-output interconnections may have been neglected. However, it is not optimal to model the full interconnection matrix since the significant input-output channels often form a sparse subset, and therefore most of the created models are neglected thereafter. Estimating any of the IMs from process data would give useful information on how to select the control structure, and the modeling effort can be focused only on those interconnections which were found to be significant.

Besides the clear advantages of estimating IMs from input-output logged data, only a limited amount of work has been published in this field, i.e. the methods described in [3] and [4] to estimate the RGA and the PM respectively.

Clearly, the estimation of process parameters is affected by process uncertainty. Thus, the validity of the decisions based on control structure methods can not be assessed by only analyzing the nominal process parameters. Recently, the effect of model uncertainties on control structure design methods has received increasingly attention, i.e. the different studies on the sensitivity of the RGA to model uncertainties published in [5] and [6], or the work on the sensitivity on the PM also to model uncertainties in [7] and [8].

The objective of this paper is creating a method for estimating an indicator of the significance of the input-output channels and obtaining confidence bounds on the estimation. The clear benefit of this approach is to be able to take a robust decision on control structure selection from a simple experiment.

The IM here considered is the PM, which uses a gramian-based indicator to quantify the significance of the input-output channels. The work in this paper is based on previous results in [4], where the mentioned indicator is estimated from an estimation of the impulse response of each of the input-output channels. Matching each of the input/output channels with a FIR filter provides non-parametric models which are easy and fast to create. The PM then provides the required information for control structure selection, or in the case of requiring more sophisticated models, the PM will pinpoint the input-output channels on which the designer should focus during the modeling task.

The preliminaries on the PM and its estimation are given in Section II. Section III shows that the statistic used in [4] provides a biased estimation (but asymptotically unbiased), and therefore the first step in the work is to derive an unbiased estimator, as described in Section IV. The second step is to create confidence bounds on the estimation by finding the distribution of the estimator, as described in Section V. These confidence bounds on the estimation will be used to perform a robust control structure selection. In Section VI, the method is applied to a bark boiler and several numerical considerations are given. Finally, the conclusions are given in Section VII. For further details on the conducted research the reader can refer to [9].

II. Preliminaries

A. Controllability and observability gramians

Given a sampled system represented by the quadruple (A,B,C,0) in state-space representation, the controllability (P) and observability (Q) gramians are obtained by solving the following discrete-time Lyapunov equations [10]:

$$APA^T - P + BB^T = 0 \quad ; \quad A^TQA - Q + C^TC = 0$$

P quantifies how hard it is to control the system states from the inputs, and Q quantifies how hard it is to observe

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the process states from the outputs.

Since P and Q depend on the state space realization, the product PQ is formed. PQ is a matrix with non-negative eigenvalues which are independent of the state space realization, and the sum of its eigenvalues equals its trace.

The trace of PQ quantifies the combined abilities of the inputs and outputs to control and observe the process state, or in another words, it quantifies the connection of the input and output spaces via the state space.

B. Participation Matrix (PM)

For a system with n inputs and m outputs, the PM was introduced as [2]:

$$\phi_{ij} = \frac{tr(P_jQ_i)}{\sum_{\substack{i=1\\j=1}}^{m,n} tr(P_jQ_i)} \tag{1}$$

where $tr(P_jQ_i)$ is the trace of the product of the controllability and observability gramians of the (i,j) input-output channel.

The used normalization implies that all the elements in the PM add up to 1, and the element ϕ_{ij} quantifies the relative contribution of the (i,j) input-output channel in the process.

When the PM is used for control structure design, a subset of the most significant input-output channels is selected. A total contribution of the selected channels higher than 0.7 is expected to derive in a satisfactory performance.

The normalization in the computation of the PM involves that a change in the value $tr(P_jQ_i)$ for an unique input-output channel may influence all the elements in the PM. To avoid this dependency, an index array containing the values $tr(P_jQ_i)$ will be analyzed, and denoted by $\tilde{\phi}$:

$$\tilde{\phi}_{ij} = tr(P_jQ_i)$$

In this index array, as in its normalized version, the largest elements identify the most significant input-output channels.

Besides, given a multivariable discrete time system G, the value $tr(P_jQ_i)$ can be computed as [4] :

$$\tilde{\phi}_{ij} = \sum_{k=0}^{N_{ij}} k(h_{ij}(k))^2 \tag{2}$$

where $h_{ij}(k)$ is the true impulse response of the channel (i,j) such that:

$$y_i(\tau \cdot T_s) = \sum_{j=1}^{n}\sum_{k=0}^{N_{ij}} h_{ij}(k) \cdot u_j(\tau \cdot T_s - k \cdot T_s), for \quad \tau = 1, 2, \ldots$$

being T_s the sampling time, and N_{ij} the number of coefficients of the true impulse response of the channel (i,j) until it settles to 0.

III. Biased estimation of the I/O channel significance

The method for the estimation of the PM introduced by [4], uses Equation (2) to estimate $tr(P_jQ_i)$ from an estimation of the impulse response of each input-output channel obtained using linear regression to match FIR filters of selected orders. For selected orders of the FIR filters N_{ij}^{max} larger than the length N_{ij} of the true impulse response, the true response of the output y_i is given by:

$$y_i(\tau \cdot T_s) = \sum_{j=1}^{n}\sum_{k=0}^{N_{ij}^{max}} h_{ij}(k) \cdot u_j(\tau \cdot T_s - k \cdot T_s), for \quad \tau = 1, 2, \ldots$$

with $h_{ij}(k) = 0, \forall k > N_{ij}$.

When the process inputs of a linear system are excited with uncorrelated gaussian noise, and in the presence of additive uncorrelated gaussian noise at the output, the estimated impulse response $\hat{h}_{ij}(k)$ can be obtained using linear regression as described in the Appendix.

The following statistic was used in [4] to estimate $\tilde{\phi}_{ij}$

$$[\tilde{\phi}_{ij}]_B = \sum_{k=0}^{N_{ij}^{max}} k(\hat{h}_{ij}(k))^2 \tag{3}$$

Proposition 1: The statistic in Equation (3) provides a biased estimator for the indicator $tr(P_jQ_i)$. However, the estimator is asymptotically unbiased.

Proof: The estimators for the impulse response can be expressed as $\hat{h}_{ij}(k) = h_{ij}(k) + \nu_{ij}(k)$, where $\nu_{ij}(jk) \sim N(0, \sigma_{ij}^2(k))$, and being $\sigma_{ij}^2(k)$ the variance of $\hat{h}_{ij}(k)$.

The expected value of the estimator is:

$$\begin{aligned} E\left([\tilde{\phi}_{ij}]_B\right) &= E\left(\sum_{k=0}^{N_{ij}^{max}} k\big(h_{ij}(k) + \nu_{ij}(k)\big)^2\right) \\ &= \sum_{k=0}^{N_{ij}^{max}} k(h_{ij}(k))^2 + \sum_{k=0}^{N_{ij}^{max}} 2kE\left(h_{ij}(k)\nu_{ij}(k)\right) \\ &+ \sum_{k=0}^{N_{ij}^{max}} kE\left(\nu_{ij}^2(k)\right) = \tilde{\phi}_{ij} + \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k) \end{aligned} \tag{4}$$

And therefore is a biased estimator of $\tilde{\phi}_{ij}$. However, $\sigma_{ij}^2(k)$ tend to zero when the number of logged samples tends to infinity. Therefore, the estimator is asymptotically unbiased, and it will converge to the true value of $\tilde{\phi}_{ij}$ for an infinite number of data samples. ■

IV. Unbiased estimation of the I/O channel significance

An unbiased statistic for $\tilde{\phi}_{ij}$ is now introduced for linear systems with uncorrelated gaussian noise both as excitation and additive output noise.

Proposition 2: The statistic

$$[\tilde{\phi}_{ij}]_{UB} = \sum_{k=0}^{N_{ij}^{max}} k(\hat{h}_{ij}(k))^2 - \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k) \tag{5}$$

provides an unbiased estimator of $\tilde{\phi}_{ij}$.

Proof: It follows from Equation (4). ■

The variances of the estimators $\sigma_{ij}^2(k)$ are the diagonal elements of the covariance matrices of the linear regressions. This covariance matrices can be determined knowing the variance of the output disturbance noise, which can be unbiasedly estimated in case of being unknown (see Appendix).

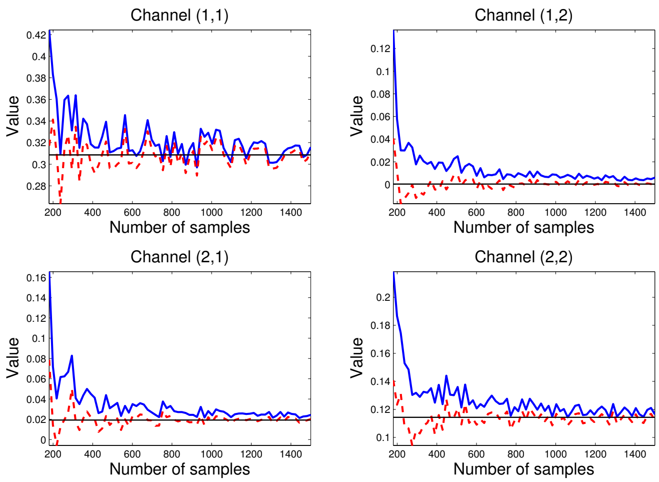

Fig. 1. Estimation of $tr(P_j Q_i)$ for the system in Equation (6) sampled at a rate of 0.1 sec. Biased estimator $[\tilde{\phi}_{ij}]_B$ (solid), unbiased estimator $[\tilde{\phi}_{ij}]_{UB}$ (dashed), and nominal values (solid horizontal line).

Example 1

The following process has been used in [4] to estimate the PM from the statistic $[\tilde{\phi}_{ij}]_B$ in Equation (3).

$$G(s) = \begin{pmatrix} \dfrac{2.231}{s+2.231} & \dfrac{0.1189}{s+3.567} \\ \dfrac{0.5579}{s+2.231} & \dfrac{2.448s+2.567}{s^2+5.192s+5.797} \end{pmatrix} \quad (6)$$

A sampling rate of 0.1 sec was selected. The continuous-time system was excited with a discrete signal and hold by a 'zoh'. The sampling rate of the system has an important impact in the value of the PM of a linear system [11], and the nominal value of $\tilde{\phi}_{ij}$ is that of the discretized system.

Discrete uncorrelated gaussian noise sequences were used for both the excitation and the output additive noise with variances of 4 and 0.04 respectively. The variance of the measurement noise was assumed to be unknown.

The chosen length of the FIR filters was 35 coefficients. Independent experiments with different numbers of logged data samples logged were performed. For the selected length of the FIR filters, at least 179 simulated samples are needed for the degrees of freedom of the residual error in the MISO regressions to be larger than the number of estimated parameters.

From Fig. 1 it can be observed that for more than 179 samples, the new estimator $[\tilde{\phi}_{ij}]_{UB}$ converges to the true value, whilst the estimator $[\tilde{\phi}_{ij}]_B$ needs more samples to converge. When more samples are taken, the variance of both estimators is reduced.

Previous simulation work in this process using $[\tilde{\phi}_{ij}]_B$ is described in [4], where an error in the estimation was observed in the channels with poor S/N. In this example, we identify a bias term contributing to this discrepancy, which can be subtracted to obtain an unbiased estimation.

V. CONFIDENCE BOUNDS ON THE ESTIMATION OF THE I/O CHANNEL SIGNIFICANCE

In this section we derive the distribution of the unbiased estimator $[\tilde{\phi}_{ij}]_{UB}$ introduced in the previous section and apply it to generate confidence bounds on the estimation.

Proposition 3: The estimator in Equation (5) is distributed as a linear combination of noncentral chi-square random variables with one degree of freedom of the form:

$$[\tilde{\phi}_{ij}]_{UB} = \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k) \cdot H_{ij}(k) - \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k)$$
$$H_{ij}(k) \sim \chi_1^2 \left(\frac{\mu_{ij}^2(k)}{\sigma_{ij}^2(k)} \right) \quad (7)$$

where $\mu_{ij}^2(k)$ and $\sigma_{ij}^2(k)$ are the means and variances of the estimators $\hat{h}_{ij}(k)$ of the impulse response.

Proof: The independence of $\hat{h}_{ij}$ has been ensured by using an uncorrelated excitation signal. The coefficients of the estimated impulse response $\hat{h}_{ij}$ are independently normally distributed random variables with mean $\mu_{ij}(k)$ and variance $\sigma_{ij}^2(k)$. The parameters $\mu_{ij}(k)$ are the coefficients estimated in the linear regressions and $\sigma_{ij}^2(k)$ are the diagonal elements of the covariance matrices from the linear regressions (see Appendix).

Introducing the random variables $\tilde{h}_{ij}(k) = \hat{h}_{ij}(k)/\sigma_{ij}(k)$ in Equation (5) we obtain:

$$[\tilde{\phi}_{ij}]_{UB} = \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k) \cdot (\tilde{h}_{ij}(k))^2 - \sum_{k=0}^{N_{ij}^{max}} k \cdot \sigma_{ij}^2(k)$$
$$\tilde{h}_{ij}(k) \sim N\left(\frac{\mu_{ij}(k)}{\sigma_{ij}(k)}, 1 \right) \quad (8)$$

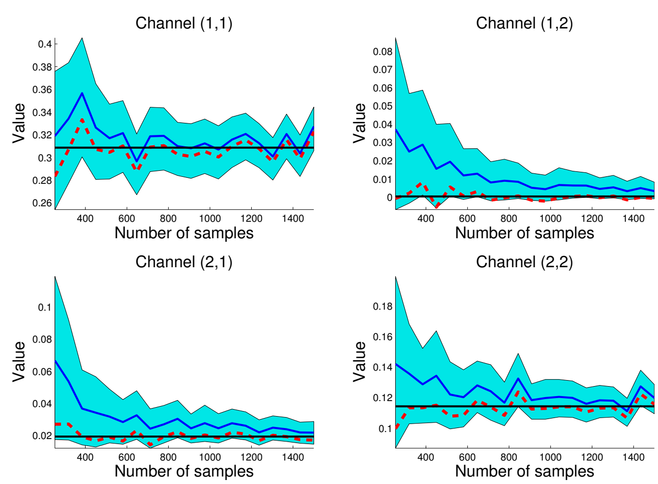

Fig. 2. Estimation of $tr(P_j Q_i)$ for the system in Equation (6) sampled at a rate of 0.1 sec. Biased estimator $[\tilde{\phi}_{ij}]_B$ (solid), unbiased estimator $[\tilde{\phi}_{ij}]_{UB}$ (dashed), 99% confidence region (shaded area) and nominal value (horizontal solid line).

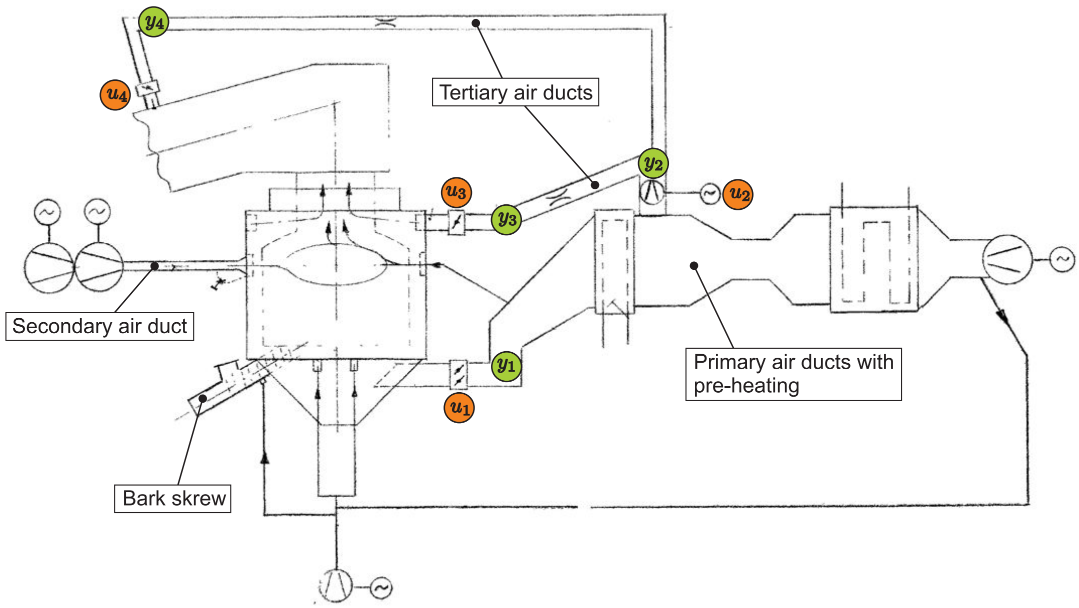

Fig. 3. Sketch of a bark boiler with air system.

The square of a normally distributed variable with mean μ and unit variance, follows a non-central chi-square distribution $\chi_1^2(\lambda)$ with one degree of freedom, and noncentrality parameter $\lambda = \mu^2$. Therefore, the variables $H_{ij}(k) = \tilde{h}_{ij}^2(k)$ are distributed as $H_{ij}(k) \sim \chi_1^2\left(\frac{\mu_{ij}^2(k)}{\sigma_{ij}^2(k)}\right)$. Substituting $\tilde{h}_{ij}^2$ by $H_{ij}(k)$ in Equation (8) we get Equation (7). ■

To apply this proposition, the Cumulative Distribution Functions (CDFs) of linear combinations of independent non-central chi-square random variables have to be computed. An algorithm for this was proposed in [12]. The author distributes the implemented algorithm in FORTRAN and C++ versions. In the work described in this paper, the algorithm was ported to Matlab code.

Example 2

The system in Equation (6) was excited as described in Example 1. The obtained data was used to create asymmetric 99% confidence bounds (0.1% from the left, 0.9% from the right) on the estimation of $[\tilde{\phi}_{ij}]_{UB}$, and the result is depicted in Fig. 2.

This example shows how for a short number of logged samples, the uncertainty in the estimation is too large, and the only robust conclusion that we can take is that the input-output channel with higher significance is the channel $(1, 1)$. The uncertainty is reduced as the number of logged samples increases, and it becomes clear that the most important input-output channels are the diagonal ones, being able to take the robust decision of using a decentralized controller.

VI. Case study: Bark boiler

A bark boiler is used in the pulp and paper industry to burn rest products from debarking of wood in order to produce steam. If there is an over production of steam which is not used within the production processes, electrical power is produced from remaining steam. Nowadays, these boilers are operated with hard environmental constraints on the composition of the flue gases, resulting in trade-off between optimal steam production at low cost while producing minimal exhaust gases like CO, CO_2 and NO_x.

In this case study, the air control system of a bark boiler at SCA Obbola, Sweden is analyzed. A simple sketch of the boiler is given in Fig. 3. There, the primary, secondary and tertiary air ducts are indicated. The tertiary air is tapped from the primary air after the first heater using an extra fan to achieve a desired flow. Subsequently, the air flow is split in an upper an lower part to be supplied in the exhaust gas duct. A good control performance of the air system is a prerequisite to achieve a stable operation of the bark boiler.

A. Process models

In order to acquire measurement data, experiments were performed at the bark boiler around a specified working point. Thereafter, the logged data was used to estimate models using the prediction error method for approximating parameters of first and second order processes models with time delays. These models were derived as multiple-input single-output models for each output.

The resulting process model for the complete air system is given as a 4×4 transfer function matrix:

$$G = \begin{bmatrix} \frac{0.301e^{-0.838s}}{1.120s+1} & \frac{-0.025e^{-0.527s}}{0.2283s+1} & \frac{-0.014e^{-30s}}{68.69s+1} & \frac{-0.004e^{-0.178s}}{0.2633s+1} \\ \frac{1.199e^{-27.8s}}{280.2s+1} & \frac{0.076}{2.542s+1} & \frac{0.004e^{-0.615s}}{16.31s+1} & \frac{0.611}{145.2s+1} \\ \frac{(0.258s+0.708)e^{-28.6s}}{48.2s^2+280.4s+1} & \frac{(0.016s+0.045)e^{-0.814s}}{0.438s^2+2.714s+1} & \frac{(0.282s+0.022)e^{-4.48}}{11.97s^2+14.57s+1} & \frac{(-0.009s+0.002)e^{-0.814s}}{s+0.007} \\ \frac{-0.035e^{-28.5s}}{602.3s^2+282.3s+1} & \frac{-0.002e^{-0.704s}}{5.465s^2+4.692s+1} & \frac{-0.003e^{-2.45}}{61.96s+1} & \frac{-0.5476}{2.689s+1} \end{bmatrix}$$

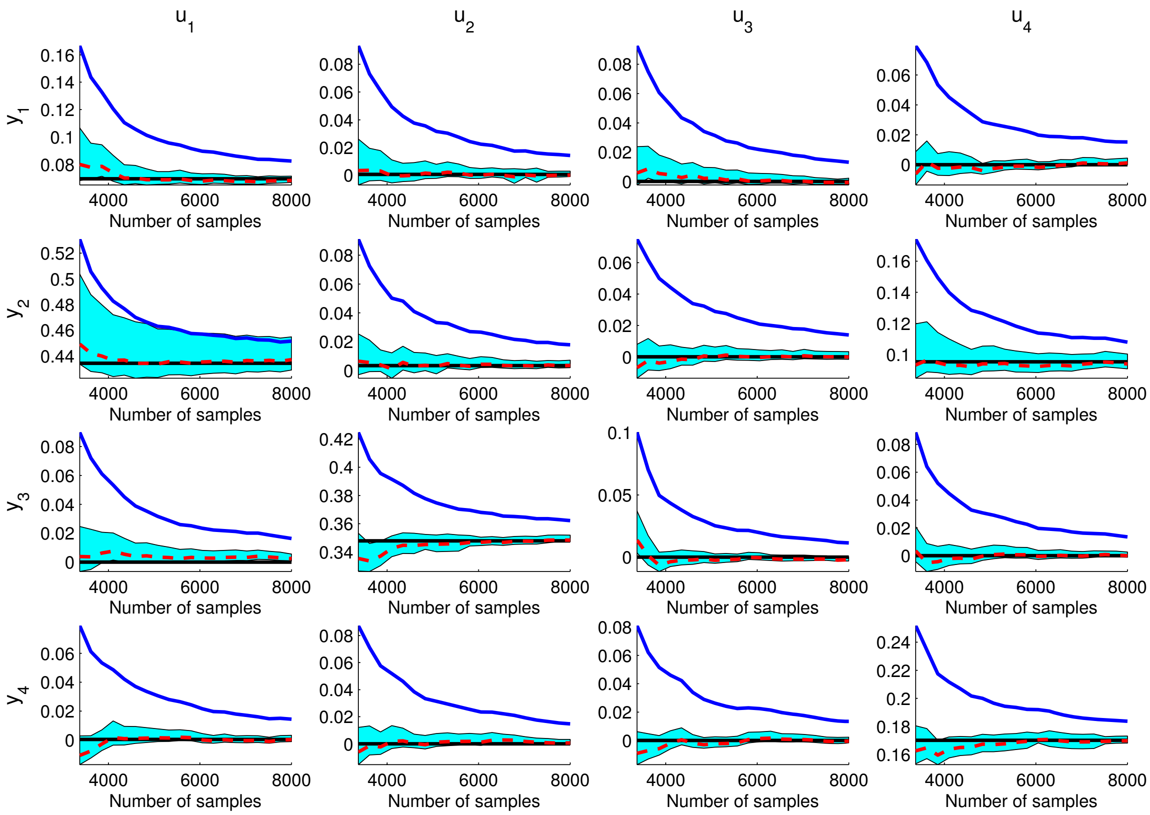

Fig. 4. Estimated $tr(P_jQ_i)$ for the air system of the bark boiler with respect to the number of logged samples for a single experiment with a sampling rate of 3 sec. Biased estimation (solid), unbiased estimation (dashed), 99.5% confidence region (shaded area) and nominal value (horizontal solid line).

B. Estimating $tr(P_jQ_i)$ for the bark boiler

The linear process models were simulated in order to obtain an estimate for $tr(P_jQ_i)$. The variances of the input excitation and the output noise were 16 and 0.01 respectively. The variance of the measurement noise was assumed to be unknown. A sampling rate of 3 sec was considered appropriate to capture the process dynamics.

It has been observed from the step response that the settling time of the fastest channel is around 20 sec whilst the slowest channel shows a setting time around 1500 sec.

For convenience to the final user of the algorithm, it was of desire to choose the same length of the FIR filter used to match the impulse response for each of the channels. The selected length of the filter was 500 coefficients, which allows to capture the impulse response of the slowest channels.

This means that for the fastest channels, a large part of the coefficients of the estimated impulse response which should be 0 are estimated as different to 0 due to the noise. This effect increases the variance of the estimated value of $tr(P_jQ_i)$ (and the bias if the biased estimator is used).

First attempts to create confidence bounds on the estimation gave incoherent results due to numerical errors. The main source of these errors was found to be the algorithm for computing the CDF of the linear combination of the non-central chi-square random variables. The author of the function reports possible inaccurate results when a few random variables with few degrees of freedom are dominating in the linear combination. This is clearly the case when the length of the FIR filters have been selected too large and then many of the estimated coefficients are distributed around 0. For dealing with this limitation, the coefficients of the impulse responses were separated in a set including the significant coefficients, and a set including the not significant coefficients. Only the significant coefficients are then passed to the function for computing the CDF. We therefore decompose $[\tilde{\phi}_{ij}]_{UB}$ as:

$$[\tilde{\phi}_{ij}]_{UB} = \underbrace{\sum_{k\in\mathbf{k_{sig}}} k\cdot\sigma_{ij}^2(k)\cdot H_{ij}(k)}_{S_{1a}} + \underbrace{\sum_{k\in\mathbf{k_{unsig}}} k\cdot(\hat{h}_{ij}(k))^2}_{S_{1b}} - \underbrace{\sum_{k=0}^{N_{ij}^{max}} k\cdot\sigma_{ij}^2(k)}_{S_2}; \qquad H_{ij}(k)\sim\chi_1^2\left(\frac{\mu_{ij}^2(k)}{\sigma_{ij}^2(k)}\right)$$

with

$$\mathbf{k_{sig}} = \{k : h_{ij}(k) \text{ is a significant coefficient}\}$$
$$\mathbf{k_{unsig}} = \{k : h_{ij}(k) \text{ is not a significant coefficient}\}$$

For computing the confidence bounds on the estimation of $\tilde{\phi}_{ij}$, it is then advised to first compute the confidence bounds on S_{1a} and then add the quantities S_{1b} and S_2.

The classification of the $h_{ij}(k)$ coefficients, has been done by applying statistical t-tests with the following hypothesis:

H_o : $\hat{h}_{ij}(k)$ comes from a normal distribution with 0 mean.

H_1 : $\hat{h}_{ij}(k)$ comes from a normal distribution with mean different than 0.

For a P-value of the test lower than 0.05, the null hypothesis was rejected and the coefficients were considered significant.

The confidence intervals were selected to be at 99.5% (0.01% from the left and 0.04% from the right), and the result is depicted in Fig. 4.

C. Analysis of results on the bark boiler

In Fig. 4, it can be seen that the unbiased estimate of $tr(PQ)$ is converging rapidly to the nominal value and that it stays within the confidence region. Whereas the biased estimator yields large deviations for most channels of the process, besides (u_1, y_2).

From inspection of nominal values the following control scheme could be suggested:

- Use u_1 primarily to control the tertiary air flow y_2.
- Use u_2 to control the lower tertiary air flow y_3.
- Use u_4 to control the upper tertiary air flow y_4
- Do not use u_3 for control purposes.

Additionally, it can be seen that there is an affect from u_1 to y_1, which should be dealt with by either a feedforward control or by integrating into the control loop for y_2 using a cascade.

VII. Conclusions

The benefit of estimating IMs from process data is to be able to identify a subset of the most important input-output interconnections of a multivariable system. This subset will form a reduced system on which future modeling efforts will be placed and control design will be based.

However, the uncertainty in this estimation can lead to an erroneous selection, and being able to create confidence bounds on the estimation will allow to take robust decisions on the control structure to be selected.

The results in this paper start from an estimator for a gramian-based IM which was previously introduced in [4]. The estimator is proved to be biased (but asymptotically unbiased). The bias is positive, and becomes significant in the channels with poor S/N. This channels may then be erroneously considered as significant. This bias can be subtracted in order to create an unbiased estimation.

The probability distribution of the unbiased estimator is found, allowing to create the sought confidence intervals in the estimation. These intervals can be used to perform a robust decision on the control structure to be used.

Two examples have been used to illustrate the usefulness of the method, and some numerical issues are discussed in order to facilitate the implementation.

Appendix
Estimation of the impulse response

This appendix describes some statistical properties of the estimation by linear regression of the impulse response of a linear system under gaussian noise excitation and in the presence of additive gaussian noise at the output [13].

Denote by N_s the number of logged samples of the output y. Collect the input (u) and output histories in vectors of the form:

$$U_i(kT_s) = \begin{pmatrix} u_1(kT_s) \\ u_1(kT_s - T_s) \\ u_1(kT_s - 2T_s) \\ \vdots \\ u_1(kT_s - N_{i1}^{max} T_s) \\ u_2(kT_s) \\ u_2(kT_s - T_s) \\ \vdots \\ u_n(kT_s - N_{in}^{max} T_s) \end{pmatrix}; \quad \boldsymbol{\Phi}_i = \begin{pmatrix} U_i(T_s)^T \\ U_i(2T_s)^T \\ \vdots \\ U_i(N_sT_s)^T \end{pmatrix} \quad Y_i = \begin{pmatrix} y_i(T_s) \\ y_i(2T_s) \\ \vdots \\ y_i(N_sT_s) \end{pmatrix}$$

The impulse response of the output y_i is then estimated as:

$$\hat{\boldsymbol{H}}_i = (\boldsymbol{\Phi}_i^T \boldsymbol{\Phi}_i)^{-1} \boldsymbol{\Phi}_i^T Y_i \quad ; \quad Cov(\hat{\boldsymbol{H}}_i) = \sigma_{y_i}^2 (\boldsymbol{\Phi}_i^T \boldsymbol{\Phi}_i)^{-1}$$

where

$$\hat{\boldsymbol{H}}_i = (\hat{h}_{i1}(1), \ldots, \hat{h}_{i1}(N_{i1}^{max}), \hat{h}_{i2}(1), \ldots, \hat{h}_{in}(N_{in}^{max}))^T$$

and $\sigma_{y_i}^2$ is the noise variance at the output y_i. If this variance is unknown, it can be estimated using the following unbiased statistic:

$$\hat{\sigma}_{y_i}^2 = \frac{1}{N_s - d_i} (Y_i - \boldsymbol{\Phi}_i \hat{\boldsymbol{H}}_i)^2$$

where d_i is the dimension of $\boldsymbol{\Phi}_i$.

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