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State Space Parameterization of Stabilizing Multirate Controllers for MIMO Linear Time-Invariant Plants

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Abstract— **In this paper, a parameterization of stabilizing multirate controllers which communicate with given plant via a communication channel is proposed. The communication channel does not have any quantizer and the limitation of the communication rate of the channel is model by a projection operator which is defined with the well-known lifting operator. Explicit forms of stabilizing multirate controllers are derived by solving dual periodically weighted model-matching problems with linear periodically time-varying (LPTV) control. The causality constraint of periodic controllers is automatically satisfied without any additional requirement on direct terms of the controllers. A numerical example with 5-periodic control is also provided.**

I. INTRODUCTION

In recent years, digital control over network has attracted many researches and many ideas for it have been proposed. For example, the relationship between the communication capacity of the network and stability margin has been investigated in [1], [2] and so on. These results deal with a quantization on communication channels and the data rate is defined by required digital bits per sample to achieve closed-loop stability. In this paper, we also focus on a closed-loop stabilization problem for a discrete-time linear time-invariant (LTI) plant via a communication channel. However, the limitation of the communication channel is modeled by a projection operator while any quantization is not considered. And the communication rate of a channel is defined as the ratio of available phases (or time slots) of the communication channel in each period to the period. This type of communication channel can be seen as a time-domain modeling and can be found in [3]–[6]. As stated in [5], if a communication channel is introduced on the controlled input or the measurement output, then the resulting controller becomes a linear periodically time-varying (LPTV) system. In addition, if the data rate of controlled input differs from that of measurement output, then a causality problem arises. Among the preceding researches listed above, parameterization of stabilizing causal LPTV controllers has been investigated in [3]. In the research [3], all stabilizing LPTV controllers for a multirate sampled data plant are parameterized via Youla parameterization [7] under the condition that the Q-parameter satisfy a causality constraint. In this paper, a state space parameterization of stabilizing multirate controllers is provided where no causality constraint is required. The reason why the causality constraint disappears is because we use so-called *dual lifted forms* of causal LPTV systems [8].

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The design procedure in this paper is based on dual periodically weighted model-matching problems [8] and utilizes well-known important results, lifting technique and Youla parameterization. Lifting technique has been widely used for sampled-data control, multirate control and LPTV control (see [4], [9]–[14] and references therein). We use the lifting operator to formulate causal N-periodic systems and to model a communication channel.

The merit of the controller proposed in this paper can be summarized as follows:

1) The communication rate of the controlled input signal or the measurement output signal is reduced and the ratio of the input rate to the output rate can be rational.
2) The requirements for the plant are only two conditions; the first is the existence of the double coprime factorization and the second is a kind of stabilizability condition or detectability condition with respect to the system matrices and the operating period of the controller.
3) The resulting LPTV multirate controllers are parameterized in a state space representation form.

Finally, a numerical example with 5-periodic case for an unstable MIMO plant is demonstrated. The example shows that stabilization via reduced data rate can be achieved by using the method in this paper.

II. PRELIMINARIES

The following notation is used in this paper. The set of integers, the set of complex numbers and the set of real numbers are denoted by $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{R}$, respectively. The set of discrete-time sequences of n-dimension is denoted by l^n. The set of linear causal maps from l^m to l^n is denoted by $\mathbf{L}^{n\times m}$. The set $\mathbf{RL}^{n\times m}$ is a subset of $\mathbf{L}^{n\times m}$, which is represented by matrices whose elements are rational functions of z. The set $\mathbf{RH}_\infty$ is a subset of $\mathbf{RL}$, whose poles are in the open unit disk on $\mathbb{C}$. The symbol $\otimes$ represents the matrix Kronecker product [7].

For a causal system $F \in \mathbf{L}$, the l_2 induced norm of F is defined by

$$\|F\|_{\mathrm{ind}} \triangleq \sup_{w\in l_2, w\neq 0} \frac{\|Fw\|_{l_2}}{\|w\|_{l_2}}.$$

And for a system $F \in \mathbf{RH}_\infty$, the $\mathcal{H}^\infty$ norm of F is defined by $\|F(z)\|_\infty \triangleq \sup_{|z|\geq 1} \bar{\sigma}(F(z))$ where $\bar{\sigma}$ denotes the largest singular value [7]. The symbol $\mathcal{S}^n$ denotes the set of all stable square matrices of n-dimension:

$$\mathcal{S}^n \quad \stackrel{\triangle}{=} \quad \{M \in \mathbb{R}^{n\times n} \mid \max_{\lambda\in\Lambda(M)} |\lambda| < 1\}, \tag{1}$$

$$\Lambda(A) \quad \stackrel{\triangle}{=} \quad \{\lambda \in \mathbb{C} \mid \exists v \neq 0 \in \mathbb{C}^n, \quad Av = \lambda v\}. \tag{2}$$

We use the well-known lifting operator W_N [9], [10]:

$$W_N u = \sum_{i=0}^{\infty} z^{-i} \begin{pmatrix} u_{N\cdot i} \\ u_{N\cdot i+1} \\ \vdots \\ u_{N\cdot i+N-1} \end{pmatrix}, \tag{3}$$

which is defined for any $u = u(z) = \sum_{i=0}^{\infty} u_i z^{-i} \in l^r$. The inverse operator W_N^{-1} of W_N is also well defined. The definition of the projection operator Pr^i is given by [8]

$$\mathrm{Pr}^i : l^r \to l^r : W_N^{-1}(e_{i+1}e_{i+1}^T \otimes I_r)W_N, \tag{4}$$

where the symbol $e_i \in \mathbb{R}^N$ denotes the i-th unit vector: $e_{ij} = \delta_{ij}$ $(i, j \in \mathbb{Z}[1, N])$. The projection operator Pr^i satisfies the following properties:

1) $\forall\, i, j \in \mathbb{Z}[0, N-1], \quad \mathrm{Pr}^i\mathrm{Pr}^j = \delta_{ij}\mathrm{Pr}^i$.
2) $\mathrm{Pr}^0 + \mathrm{Pr}^1 + \cdots + \mathrm{Pr}^{N-1} = I$.

The symbol $\mathbb{Z}_N$ is defined by $\mathbb{Z}_N \stackrel{\triangle}{=} \mathbb{Z}[0, N-1]$. Let $\mathcal{I}$ denote a subset of $\mathbb{Z}_N = \mathbb{Z}[0, N-1]$ and let $\#\mathcal{I}$ denote the number of members of $\mathcal{I}$. And for $\mathcal{I}$, its complementary set $\mathcal{I}^{\mathrm{c}}$ is defined by $\mathcal{I}^{\mathrm{c}} \stackrel{\triangle}{=} \mathbb{Z}_N \setminus \mathcal{I}$. For example, when $N = 6$, $\mathcal{I}_1 = \{0, 1, 3\}$ and $\mathcal{I}_2 = \{2, 5\}$ are candidates of $\mathcal{I}$ and $\#\mathcal{I}_1 = 3$, $\#\mathcal{I}_2 = 2$, $\mathcal{I}_1^{\mathrm{c}} = \{2, 4, 5\}$ and $\mathcal{I}_2^{\mathrm{c}} = \{0, 1, 3, 4\}$ hold.

We also use the transformation operator R_{id} defined by

$$R_{\mathrm{id}} \stackrel{\triangle}{=} R \otimes I, \quad R \stackrel{\triangle}{=} \left(\begin{array}{c|c} 0 & I_{N-1} \\ \hline 1 & 0 \end{array}\right). \tag{5}$$

Note that R is an orthogonal matrix (i.e. $RR^T = R^T R = I$).

III. STABILIZATION OVER NETWORK

A. Parameterization of Stabilizing Controllers

In this paper, we deal with the closed-loop illustrated in Fig. 1 where P and K denote a plant and a controller, respectively. The symbol d_1 and d_2 are disturbance signals.

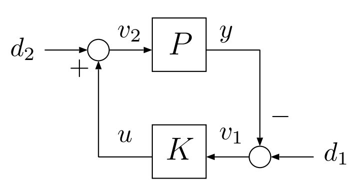

Fig. 1. The standard feedback loop block diagram

The plant $P(z)$ is supposed to be given by an LTI transfer function and to satisfy the following assumption.

Assumption 1: Suppose that $P(z) = (A, B, C, D) \stackrel{\triangle}{=} D + C(zI - A)^{-1}B$ has the following coprime factorization [7].

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{6}$$

$$N(z), M(z), \tilde{M}(z), \tilde{N}(z) \in \mathbf{RH}_\infty, \tag{7}$$

and the following Bezout identity is satisfied:

$$\begin{bmatrix} M & -Y \\ N & X \end{bmatrix} \begin{bmatrix} \tilde{X} & \tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = I, \tag{8}$$

where $X(z), Y(z), \tilde{X}(z), \tilde{Y}(z) \in \mathbf{RH}_\infty$.

Consequently, the Youla parameterization [7] of all stabilizing controllers for $P(z)$ is given by

$$\begin{aligned} K &= (Y - MQ)(X + NQ)^{-1} \\ &= (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}), \end{aligned} \tag{9}$$

where Q is the free design parameter and we restrict Q to the class $\mathcal{Q}_N^{\mathrm{p}}$. The class $\mathcal{Q}_N^{\mathrm{p}}$ represents causal, N-periodic and stable systems [8]:

$$\mathcal{Q}_N^{\mathrm{p}} \stackrel{\triangle}{=} \{Q \mid Q : \text{causal},\ W_N Q W_N^{-1} \in \mathbf{RH}_\infty\}. \tag{10}$$

A feedback transfer function with respect to the pair (P, K) is defined as follows [7]

$$\begin{aligned} H[P, K[Q]] &\stackrel{\triangle}{=} \begin{pmatrix} H_{1,1}[P,Q] & H_{1,2}[P,Q] \\ H_{2,1}[P,Q] & H_{2,2}[P,Q] \end{pmatrix} \\ &= \begin{pmatrix} (I+PK)^{-1} & -(I+PK)^{-1}P \\ K(I+PK)^{-1} & (I+KP)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} X\tilde{M} + NQ\tilde{M} & -X\tilde{N} - NQ\tilde{N} \\ Y\tilde{M} - MQ\tilde{M} & M\tilde{X} + MQ\tilde{N} \end{pmatrix}. \end{aligned} \tag{11}$$

B. Data Rate Reduction of Signal

In this section, a stabilization problem over network is formulated. To model a communication channel, a projection operator $\Pi_{\mathcal{I}}$ is defined for a given integer set $\mathcal{I}$ as

$$\Pi_{\mathcal{I}} \quad \stackrel{\triangle}{=} \quad \sum_{i\in\mathcal{I}} \mathrm{Pr}^i. \tag{12}$$

We use the operator $\Pi_{\mathcal{I}}$ as the communication channel of the controlled input u or the measurement output v_1 in Fig. 1. For example, consider the case with $N = 3$ and $u(z)$ being of one dimension. If we want to reduce the data rate of u to the ratio $2/3$, then we may apply $\Pi_{\{1,2\}}$ to u:

$$\begin{aligned} Wu &= \sum_{i=0}^{\infty} z^{-i} \begin{pmatrix} u_{3i} \\ u_{3i+1} \\ u_{3i+2} \end{pmatrix} \\ \to \quad W\Pi_{\{1,2\}}u &= \sum_{i=0}^{\infty} z^{-i} \begin{pmatrix} 0 \\ u_{3i+1} \\ u_{3i+2} \end{pmatrix}. \end{aligned}$$

In this case, values of the first row of $W\Pi_{\{1,2\}}u$ are always zero and these data does not need to be transfered to the plant. This case is illustrated in Fig. 2.

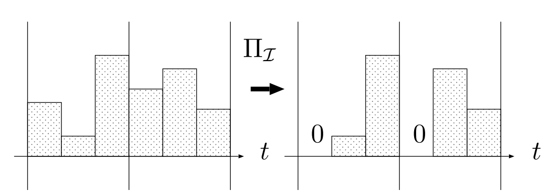

Fig. 2. Data rate reduction of a communication channel

An idea to use the projection operator Pr^i to multirate systems can be found in [13]. However, the procedure in [13] is applicable only for stable plants because it has an an internal model control (IMC) structure.

In the following argument, we extend the idea of the limited channel to general N-periodic case and discuss how to construct stabilizing controllers for unstable MIMO plants via limited communication channels.

C. Problem Formulation with Projection Operator

When we take an arbitrary set $\mathcal{I} \subset \mathbb{Z}_N$, a multirate stabilization problem with limited controlled input data rate is defined as follows

P1. Find a stabilizing controller K such that

$$\begin{aligned} H[P,K] &\in \mathcal{Q}_N^{\mathrm{p}}, &(13)\\ \Pi_{\mathcal{I}} K &= K. &(14)\end{aligned}$$

The condition (14) means that the controlled input u satisfies $\Pi u = u$ and consequently the communication rate of the input u can be reduced. In this case, the feedback loop in Fig. 1 is equivalent to the feedback loop in Fig. 3.

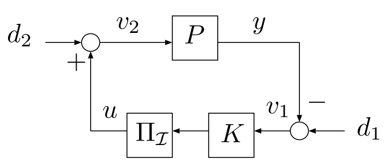

Fig. 3. Feedback loop with control data rate constraint

The ratio of communication data rate of input and that of output is expressed by

$$r = \frac{\#\mathcal{I}}{N}. \qquad (15)$$

Hence, there is a trade-off relationship between the value $\#\mathcal{I}$ and the internal stability of the closed-loop.

Similarly, a stabilization problem with limited measurement communication rate is defined as

P2. Find a stabilizing controller K such that

$$\begin{aligned} H[P,K] &\in \mathcal{Q}_N^{\mathrm{p}}, &(16)\\ K\Pi_{\mathcal{I}} &= K. &(17)\end{aligned}$$

In this case, the feedback loop in Fig. 1 is equivalent to the feedback loop in Fig. 4.

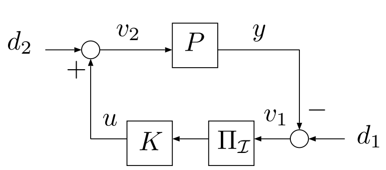

Fig. 4. Feedback loop with measurement data rate constraint

IV. DESIGN PROCEDURE

The problems P1 and P2 can be solved via dual periodically weighted model-matching problems [8] and the following theorem plays a key role.

Theorem 1: [8] For any $Q \in \mathcal{Q}_N^{\mathrm{p}}$, there always exist LTI transfer functions $Q^{\mathrm{I}}(z) \in \mathbf{RH}_\infty$ and $Q^{\mathrm{O}}(z) \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} T_1(z) - T_2(z)QT_3(z) &= \sum_{j=0}^{N-1} T_j^{\mathrm{c}}(z)\mathrm{Pr}^j &(18)\\ &= \sum_{j=0}^{N-1} \mathrm{Pr}^j T_{N-1-j}^{\mathrm{r}}(z) &(19)\end{aligned}$$

holds where $T_i^{\mathrm{c}}(z)$ and $T_i^{\mathrm{r}}(z)$ are defined by

$$\begin{aligned} T_i^{\mathrm{c}}(z) &\triangleq T_1(z) - T_2(z)Q^{\mathrm{I}}(z)R_{\mathrm{id}}^{-i}\mathrm{rrs}\ T_3(z), &(20)\\ T_i^{\mathrm{r}}(z) &\triangleq T_1(z) - \mathrm{ccs}\ T_2(z)R_{\mathrm{id}}^{i}Q^{\mathrm{O}}(z)T_3(z). &(21)\end{aligned}$$

And for an LTI system $G(z) = (A,B,C,D)$, the operators ccs and rrs are defined by

$$\begin{aligned} \mathrm{ccs}\ G(z) &\triangleq \left[\begin{array}{cccc} D & 0 & \cdots & 0 \end{array}\right]\\ &\quad + C(z^N I - A^N)^{-1}\left[\begin{array}{cc} A^{N-1}B & z^{N-1}B \end{array}\right.\\ &\qquad \left.\begin{array}{ccc} z^{N-2}AB & \cdots & zA^{N-2}B \end{array}\right],\\ \mathrm{rrs}\ G(z) &\triangleq \left(\mathrm{ccs}\ G^T(z)\right)^T. \end{aligned}$$

The new free parameters $Q^{\mathrm{I}} \in \mathbf{RH}_\infty$ and $Q^{\mathrm{O}} \in \mathbf{RH}_\infty$ are unique representations that we call *dual lifted forms* [8] of the original parameter $Q \in \mathcal{Q}_N^{\mathrm{p}}$ and no causality constraint is necessary to the parameters Q^{I} and Q^{O}. As a result, to solve P1 problem or P2 problem becomes to design appropriate parameter $Q^{\mathrm{O}}(z)$ or $Q^{\mathrm{I}}(z)$, respectively. The procedure to construct the original $Q \in \mathcal{Q}_N^{\mathrm{p}}$ from $Q^{\mathrm{O}}(z)$ and $Q^{\mathrm{I}}(z)$ is stated completely in [8].

As shown in the following lemma, the Q-parameter of the controller K for problem P1 can be obtained explicitly.

Lemma 1: There exists a stabilizing controller K which satisfies (13) and (14) if there exists $(\mathrm{ccs}\ M)_{\mathcal{I}_0}^{\#}(z) \in \mathbf{RH}_\infty$ such that

$$\forall j \in \mathcal{I}_0 \quad \mathrm{ccs}\ MR_{\mathrm{id}}^{j}(\mathrm{ccs}\ M)_{\mathcal{I}_0}^{\#} = I. \qquad (22)$$

where $\mathcal{I}_0 \triangleq \Phi_N(\mathcal{I}^{\mathrm{c}})$ and the set $\Phi_N(\mathcal{I}')$ for any $\mathcal{I}' \subset \mathbb{Z}_N$ is defined by

$$\Phi_N(\mathcal{I}') \triangleq \{i \in \mathbb{Z}_N \mid N-1-i \in \mathcal{I}'\}. \qquad (23)$$

Proof: From the definition of $\Pi_{\mathcal{I}}$ and the property of Pr^i, $\Pi_{\mathcal{I}}K = K$ is equivalent to $(I - \Pi_{\mathcal{I}})K = \sum_{i\in\mathbb{Z}[0,N-1]\setminus\mathcal{I}} \mathrm{Pr}^i K = \Pi_{\mathcal{I}^c}K = 0$. From Theorem 1 and the parameterized expression of the controller (9), there exists $Q^{\mathrm{O}}(z) \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} \mathrm{Pr}^i K &= \mathrm{Pr}^i(Y - MQ)(X + NQ)^{-1}\\ &= \mathrm{Pr}^i(Y - \mathrm{ccs}\ MR_{\mathrm{id}}^{N-1-i}Q^{\mathrm{O}})(X + NQ)^{-1}. \end{aligned} \qquad (24)$$

Therefore, if we select $Q^{\mathrm{O}}(z)$ as

$$Q^{\mathrm{O}*}(z) = (\mathrm{ccs}\ M)_{\mathcal{I}_0}^{\#}Y(z), \qquad (25)$$

then $\mathrm{Pr}^i K = 0$ holds for any $i \in \mathcal{I}^c$. ■

Similarly, problem P2 can be solved as follows.

Lemma 2: There exists a stabilizing controller K which satisfies (16) and (17) if there exists $(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}^c}(z) \in \mathbf{RH}_\infty$ such that

$$\forall j \in \mathcal{I}^c, \quad (\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}^c} R_{\mathrm{id}}^{-j} \mathrm{rrs}\ \tilde{M} = I. \tag{26}$$

Proof: Similar to Lemma 1, it is true that $K\Pi_{\mathcal{I}} = K$ is equivalent to $K(I - \Pi_{\mathcal{I}}) = K\Pi_{\mathcal{I}^c} = 0$. From Theorem 1 and the parameterized expression of the controller (9), there exists $Q^{\mathrm{I}}(z) \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} K\mathrm{Pr}^i &= (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})\mathrm{Pr}^i \\ &= (\tilde{X} + Q\tilde{N})^{-1}(\tilde{Y} - Q^{\mathrm{I}} R_{\mathrm{id}}^{-i} \mathrm{rrs}\ \tilde{M})\mathrm{Pr}^i. \end{aligned} \tag{27}$$

Therefore, if we select $Q^{\mathrm{I}}(z)$ as

$$Q^{\mathrm{I}*}(z) = \tilde{Y}(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}^c}(z), \tag{28}$$

then $K\mathrm{Pr}^i = 0$ holds for any $i \in \mathcal{I}^c$. ■

V. CONSTRUCTION OF EXPLICIT SOLUTION

A. Key Lemma and Theorem

As we have seen so far, design methods of $(\mathrm{ccs}(\cdot))^{\#}_{\mathcal{I}}$ and $(\mathrm{rrs}(\cdot))^{\#}_{\mathcal{I}}$ are necessary to solve P1 and P2 problems. However, because the equation

$$\begin{aligned} ((\mathrm{ccs}\ G^T)R_{\mathrm{id}}^i)^T &= (R_{\mathrm{id}}^T)^i (\mathrm{ccs}\ G^T)^T \\ &= R_{\mathrm{id}}^{-i} \mathrm{rrs}\ G \end{aligned}$$

holds, $(\mathrm{rrs}\ G)^{\#}_{\mathcal{I}^c}(z) = ((\mathrm{ccs}\ G^T)^{\#}_{\mathcal{I}^c})^T(z)$ is satisfied and consequently it is sufficient to discuss how to construct $(\mathrm{ccs}\ G)^{\#}_{\mathcal{I}'}(z)$.

It can be verified by calculation that for a system $G(z) = (A, B, C, D)$ the following equation:

$$\mathrm{ccs}\ G(z) = (R \otimes A, I_N \otimes B, e_2^T \otimes C, e_1^T \otimes D) \tag{29}$$

holds. A useful lemma is derived from this reformulation.

Lemma 3: Suppose that an LTI system $G(z)$ is given by $G(z) = (A, B, C, D)$ and there exists a matrix F such that

$$R \otimes A + (I_N \otimes B)F \in \mathcal{S}. \tag{30}$$

Then, the following system $S_G[F, L](z)$:

$$S_G[F, L](z) \triangleq \left(\begin{array}{c|c} R \otimes A + (I \otimes B)F & (I \otimes B)L \\ \hline F & L \end{array}\right) \tag{31}$$

yields

$$\begin{aligned} &\mathrm{ccs}\ G(z) R_{\mathrm{id}}^i S_G[F, L](z) \\ &= \left(\begin{array}{c|c} R \otimes A + (I \otimes B)F & (I \otimes B)L \\ \hline (e_1^T R^{i+1}) \otimes C + DF_i & DL_i \end{array}\right) \end{aligned} \tag{32}$$

for $i \in \mathbb{Z}_N$ where F_j and L_j are defined by

$$F = \begin{pmatrix} F_0 \\ \vdots \\ F_{N-1} \end{pmatrix}, \quad L = \begin{pmatrix} L_0 \\ \vdots \\ L_{N-1} \end{pmatrix}. \tag{33}$$

Proof: The equation can be verified by calculation, combined with the equation:

$$\begin{aligned} (\mathrm{ccs}\ G)R_{\mathrm{id}}^i &= (e_1^T \otimes D)R_{\mathrm{id}}^i \\ &\quad + (e_2^T \otimes C)(zI - R \otimes A)^{-1}(I \otimes B)R_{\mathrm{id}}^i \\ &= (e_1^T R^i) \otimes D \\ &\quad + (e_2^T R^i \otimes C)(zI - R \otimes A)^{-1}(I \otimes B) \\ &= \left(\begin{array}{c|c} R \otimes A & I \otimes B \\ \hline (e_1^T R^{i+1}) \otimes C & (e_1^T R^i) \otimes D \end{array}\right). \end{aligned}$$

Note that $e_i^T R^n = e_{i+n(\mathrm{mod} N)}^T$ is satisfied. ■

From Lemma 3, a sufficient condition of the existence of $(\mathrm{ccs}\ G)^{\#}_{\mathcal{I}}$ can be derived as follows.

Theorem 2: For a given LTI system $G(z) = (A, B, C, D)$ and for any subset of integers $\mathcal{I} \subset \mathbb{Z}_N$, let B^a denote the following matrix:

$$B^a \triangleq \left(e_{k_1+1} \otimes B \quad \cdots \quad e_{k_m+1} \otimes B \right), \tag{34}$$

where $m = \#\mathcal{I}^c$ and integers $k_1, k_2, \cdots, k_m$ are all elements of $\mathcal{I}^c$. If the following conditions:

1) The D matrix is right invertible. ($\exists D^{\#}$ s.t. $DD^{\#} = I$.)
2) There exists a matrix F^c such that

$$R \otimes (A - BD^{\#}C) + B^a F^c \in \mathcal{S}, \tag{35}$$

are satisfied, then there exists $(\mathrm{ccs}\ G)^{\#}_{\mathcal{I}} \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} &\mathrm{ccs}\ G R_{\mathrm{id}}^j (\mathrm{ccs}\ G)^{\#}_{\mathcal{I}} - I \\ &= \begin{cases} 0 & (\text{if } j \in \mathcal{I}) \\ U_j^c[G, F^c, L^c](z) \in \mathbf{RH}_\infty & (\text{if } j \in \mathcal{I}^c) \end{cases}, \end{aligned} \tag{36}$$

where $U_j^c[G, F^c, L^c](z)$ is given by

$$\begin{aligned} &U_j^c[G, F^c, L^c](z) \\ &\triangleq \left(\begin{array}{c|c} R \otimes (A - BD^{\#}C) + B^a F^c & B^b + B^a L^c \\ \hline DF_j^c & DL_j^c - I \end{array}\right), \\ &B^b \triangleq \sum_{j \in \mathcal{I}} e_{j+1} \otimes (BD^{\#}), \end{aligned} \tag{37}$$

where F^c and L^c are free parameters of $(\mathrm{ccs}\ G)^{\#}_{\mathcal{I}}(z)$ and they satisfy the equation:

$$F^c = \begin{pmatrix} F_{k_1}^c \\ \vdots \\ F_{k_m}^c \end{pmatrix}, \quad L^c = \begin{pmatrix} L_{k_1}^c \\ \vdots \\ L_{k_m}^c \end{pmatrix}. \tag{38}$$

Proof: If F_j and L_j in (33) are selected as

$$\begin{aligned} F_j &\triangleq \begin{cases} -(e_1^T R^{j+1}) \otimes (D^{\#}C) & (\text{if } j \in \mathcal{I}) \\ -(e_1^T R^{j+1}) \otimes (D^{\#}C) + F_j^c & (\text{if } j \in \mathcal{I}^c) \end{cases}, \\ L_j &\triangleq \begin{cases} D^{\#} & (\text{if } j \in \mathcal{I}) \\ L_j^c & (\text{if } j \in \mathcal{I}^c) \end{cases} \end{aligned}$$

and (35) is satisfied, then (36) holds from Lemma 3. ■

B. Application to the Coprime Factors M and $\tilde{M}$

In this section, we apply the result in the last section to the coprime factors M and $\tilde{M}$. As is shown in [7], M and $\tilde{M}$ have the following state space realizations.

$$M(z) = \left(\begin{array}{c|c} A+BK_p & B \\ \hline K_p & I \end{array}\right), \tag{39}$$

$$\tilde{M}(z) = \left(\begin{array}{c|c} A+H_pC & H_p \\ \hline C & I \end{array}\right), \tag{40}$$

where K_p is a stabilizing feedback gain and H_p is a stabilizing observer gain. Note that because the D-terms of M and $\tilde{M}$ are identity, Theorem 2 is applicable to M and $\tilde{M}$. The existence of $(\mathrm{ccs}\ M)^{\#}_{\mathcal{I}_0}$ can be stated as follows.

Corollary 1: If the pair $(R \otimes A, B^a)$ is stabilizable where B^a is defined by (34), then there exists an LTI system $(\mathrm{ccs}\ M)^{\#}_{\mathcal{I}}(z) \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} &\mathrm{ccs}\ MR^j_{\mathrm{id}}(\mathrm{ccs}\ M)^{\#}_{\mathcal{I}} - I \\ &= \left\{ \begin{array}{ll} 0 & (\text{if } j \in \mathcal{I}) \\ U^{\mathrm{c}}_j[M, F^{\mathrm{c}}, L^{\mathrm{c}}](z) \in \mathbf{RH}_\infty & (\text{if } j \in \mathcal{I}^{\mathrm{c}}) \end{array} \right. , \end{aligned} \tag{41}$$

where F^{c} and L^{c} are free parameters of $(\mathrm{ccs}\ M)^{\#}_{\mathcal{I}}(z)$ and F^{c} must satisfy $R \otimes A + B^a F^{\mathrm{c}} \in \mathcal{S}$.

The existence of $(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}^{\mathrm{c}}}$ can be also shown as follows.

Corollary 2: If the pair $(C^a, R^T \otimes A)$ is detectable where C^a is defined by

$$C^a \triangleq \begin{pmatrix} e^T_{k_1+1} \otimes C \\ \vdots \\ e^T_{k_m+1} \otimes C \end{pmatrix}, \tag{42}$$

where $m = \#\mathcal{I}^{\mathrm{c}}$ and integers $k_1, k_2, \cdots, k_m$ are all elements of $\mathcal{I}^{\mathrm{c}}$, then there exists an LTI system $(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}}(z) \in \mathbf{RH}_\infty$ such that

$$\begin{aligned} &(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}} R^{-j}_{\mathrm{id}} \mathrm{rrs}\ \tilde{M} - I \\ &= \left\{ \begin{array}{ll} 0 & (\text{if } j \in \mathcal{I}) \\ U^{\mathrm{r}}_j[\tilde{M}, H^{\mathrm{r}}, L^{\mathrm{r}}](z) \in \mathbf{RH}_\infty & (\text{if } j \in \mathcal{I}^{\mathrm{c}}) \end{array} \right. , \end{aligned} \tag{43}$$

where $U^{\mathrm{r}}_j[G, H^{\mathrm{r}}, L^{\mathrm{r}}](z)$ is defined by

$$U^{\mathrm{r}}_j[G, H^{\mathrm{r}}, L^{\mathrm{r}}] \triangleq (U^{\mathrm{c}}_j[G^T, H^{\mathrm{r}T}, L^{\mathrm{r}T}])^T(z), \tag{44}$$

where H^{r} and L^{r} are free parameters of $(\mathrm{rrs}\ \tilde{M})^{\#}_{\mathcal{I}}(z)$ and H^{r} must satisfy $R^T \otimes A + H^{\mathrm{r}} C^a \in \mathcal{S}$.

Proof: The proof of Corollary 1 and Corollary 2 readily follows from Lemma 3 and Theorem 2. ■

VI. ROBUSTNESS OF THE CONTROLLER

A robustness analysis on the proposed multirate controller is discussed in this section. Firstly, we consider additive uncertainty case. Suppose that the actual plant P_Δ consists of a nominal plant $P(z) \in \mathbf{RL}$ and a norm-bounded perturbation $\Delta \in \mathcal{B}_\gamma$:

$$P_\Delta \triangleq P(z) + \Delta, \quad \Delta(t) \in \mathcal{B}_\gamma, \tag{45}$$

$$\mathcal{B}_\gamma \triangleq \{\Delta(t) \mid \forall\ t, \quad \bar{\sigma}(\Delta(t)) \le \gamma\}. \tag{46}$$

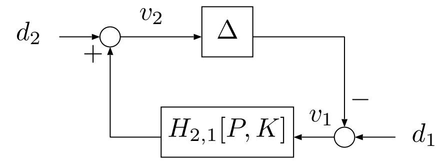

Fig. 5. Robust feedback loop block diagram

Then, the closed-loop with P_Δ can be expressed as Fig. 5. Therefore, from small gain theorem (see [7] for example) the closed-loop system in Fig. 5 is internally stable, (i.e. $H[P_\Delta, K]$ is stable) for any $\Delta \in \mathcal{B}_\gamma$ if the following conditions are satisfied:

1) The system $H[P(z), K]$ is stable.
2) The following equation is satisfied:

$$\|H_{2,1}[P(z), K]\|_{\mathrm{ind}} < \gamma^{-1}. \tag{47}$$

And when the controller $K[Q]$ is designed based on P1 problem, the second condition is equivalent to

$$\|\Pi_{\mathcal{I}} H_{2,1}[P(z), K]\|_{\mathrm{ind}} < \gamma^{-1} \tag{48}$$

because of the similarity of the structure of $K[Q]$ and that of $H_{2,1}[P(z), K]$. Hence, to keep the robustness of the multirate controller it is desirable to minimize the value:

$$\begin{aligned} \|\Pi_{\mathcal{I}} H_{2,1}[P(z), K[Q]]\|_{\mathrm{ind}} &= \left\|\Pi_{\mathcal{I}}(Y - MQ)\tilde{M}\right\|_{\mathrm{ind}} \\ &= \left\|\sum_{i \in \mathcal{I}} \mathrm{Pr}^i (I - \mathrm{ccs}\ MR^{N-1-i}_{\mathrm{id}} (\mathrm{ccs}M)^{\#}_{\mathcal{I}_0}) Y\tilde{M}\right\|_{\mathrm{ind}} \\ &= \left\|\sum_{i \in \mathcal{I}} \mathrm{Pr}^i U^{\mathrm{c}}_{N-1-i}[M, F^{\mathrm{c}}, L^{\mathrm{c}}](z) Y(z)\tilde{M}(z)\right\|_{\mathrm{ind}} \\ &\le \sum_{i \in \mathcal{I}} \left\|U^{\mathrm{c}}_{N-1-i}[M, F^{\mathrm{c}}, L^{\mathrm{c}}](z)\right\|_\infty \left\|Y(z)\tilde{M}(z)\right\|_\infty . \end{aligned}$$

Therefore, it is reasonable to use the free parameter F^{c} and L^{c} to minimize the $\mathcal{H}^\infty$ norm of $U^{\mathrm{c}}_{N-1-i}[M, F^{\mathrm{c}}, L^{\mathrm{c}}](z)$.

Similarly, when the controller $K[Q]$ is designed based on P2 problem, (47) is equivalent to

$$\|H_{2,1}[P(z), K]\Pi_{\mathcal{I}}\|_{\mathrm{ind}} < \gamma^{-1}, \tag{49}$$

and the following inequality holds:

$$\begin{aligned} \|H_{2,1}[P(z), K]\Pi_{\mathcal{I}}\|_{\mathrm{ind}} &= \left\|M(\tilde{Y} - Q\tilde{M})\Pi_{\mathcal{I}}\right\|_{\mathrm{ind}} \\ &\le \sum_{i \in \mathcal{I}} \left\|U^{\mathrm{r}}_i[\tilde{M}, H^{\mathrm{r}}, L^{\mathrm{r}}](z)\right\|_\infty \left\|M(z)\tilde{Y}(z)\right\|_\infty . \end{aligned}$$

Consequently, minimization of $\left\|U^{\mathrm{r}}_i[\tilde{M}, H^{\mathrm{r}}, L^{\mathrm{r}}](z)\right\|_\infty$ is related to the robust stability margin of the closed-loop.

Multiplicative uncertainty case can be also dealt with as well as additive uncertainty case. If the plant P_Δ is expressed by $P_\Delta = P(z)(I + \Delta)$ with an uncertainty $\Delta \in \mathcal{B}_\gamma$ and a nominal plant $P(z) \in \mathbf{RL}$, the robustness condition from the small gain theorem is the following inequality:

$$\|H_{2,2}[P(z), K] - I\|_{\mathrm{ind}} = \left\|(Y - MQ)\tilde{N}\right\|_{\mathrm{ind}} < \gamma^{-1}. \tag{50}$$

On the other hand, if the plant P_Δ has an uncertainty $\Delta \in \mathcal{B}_\gamma$ as $P_\Delta = (I+\Delta)P(z)$, then the robustness condition for any $\Delta \in \mathcal{B}_\gamma$ results in the inequality:

$$\|H_{1,1}[P(z), K] - I\|_{\mathrm{ind}} = \left\|N(\tilde{Y} - Q\tilde{M})\right\|_{\mathrm{ind}} < \gamma^{-1}. \quad (51)$$

VII. NUMERICAL EXAMPLE

In this section, we show a numerical example for discrete-time MIMO plant with limited controlled input data rate case. The operating period N is selected as $N = 5$. And the set of integers $\mathcal{I}$ is defined by

$$\mathcal{I} \stackrel{\triangle}{=} \{0,1\}, \quad \mathcal{I}^{\mathrm{c}} = \{2,3,4\}, \quad \Phi_5(\mathcal{I}^{\mathrm{c}}) = \{0,1,2\}.$$

As a result, the controlled input data rate is reduced to the ratio $2/5$. The plant $P(z)$ dealt with in this section is the following unstable MIMO system:

$$P(z) = \begin{pmatrix} \frac{z-1.6}{(z-1.3)(z-0.15)} & \frac{-(z-0.2)}{(z+0.1)(z+1.4)} \\ \frac{z-3}{(z-0.1)^2} & \frac{z-2}{(z-0.1)(z-1.55)} \end{pmatrix}. \quad (52)$$

The simulation result is shown in Fig. 6 and Fig. 7. The meaning of the signals $v_1(t)$ and $v_2(t)$ are represented in Fig. 3. These graphs represent the time response of the closed-loop $H[P, K]$ with a unit impulse disturbance on d_2 in Fig. 3.

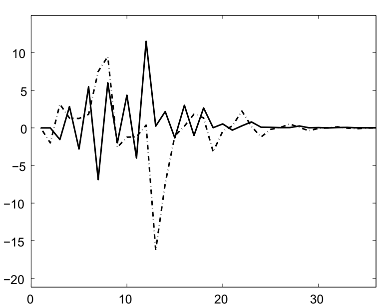

Fig. 6. The measurement signal v_1 vs. time. The solid line and the dashed line represent the first element of v_1 and the second element of v_1, respectively.

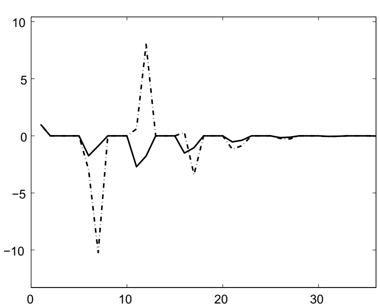

Fig. 7. The input signal v_2 vs. time. The solid line and the dashed line represent the first element of v_2 and the second element of v_2, respectively.

Note that v_2 in Fig. 7 is always zero at phase 2, 3 and 4 in each period. This means that the the condition $\Pi_{\mathcal{I}} u = u$ for the control signal (i.e. the output of the controller) u is satisfied. Therefore, stabilization of the unstable plant with limited controlled input data rate formulated as P1 problem is achieved.

VIII. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

A design method of multirate stabilizing LPTV controllers for LTI plants has been proposed. The required condition to achieve internal stability with limited communication rate resulted in a kind of stabilizability or detectability condition. And the explicit parameterization of the multirate stabilizing controllers was given via Youla parameterization. The Q-parameter of a resulting LPTV controller has a state space representation with two free parameters.

B. Future Works

Optimization methods of $\left\|U_j^{\mathrm{c}}(z)\right\|_\infty$ and $\left\|U_j^{\mathrm{r}}(z)\right\|_\infty$ are necessary but not developed yet. To clarify the trade-off between communication rate and achievable performance by using the parameterization in this paper is an interesting problem to try. Together with the problems, it is also important to obtain a parameterization of multirate controllers which have both input and output communication limitation.

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