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Exact Determinations of the Maximal Output Admissible Set for a Class of Nonlinear Systems

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Abstract— **This paper is concerned with obtaining necessary and sufficient conditions for fulfilling specified state and control pointwise-in-time constraints against a certain class of nonlinear dynamics. The results are generalizations of the maximal output admissible set theory to the case of nonlinear systems. Main contribution of the present paper is that explicit algorithmic procedures to determine the maximal output admissible set are proposed. Another contribution is that we discuss on a finite determinability of the maximal output admissible set for nonlinear systems. Some relations between observability of nonlinear systems and finite characterizations of the maximal output admissible set are clarified.**

I. Introduction

This paper considers necessary and sufficient conditions for fulfilling the specified *state and control constraints* against *nonlinear dynamics*. Constraints on actuators and often also on the states are present inherently in all real physical systems. They can lead performance deterioration and even instability if not properly accounted for in design stages [8].

An *invariance* property of a certain subset of the state space plays fundamentally important role in analysis and design of constrained systems [4]. The concept of *maximal output admissible sets*, especially, delivers necessary and sufficient conditions to fulfill the specified constraints [8], [14], [16]. The maximal output admissible set is the largest constraint admissible positively invariant set or, in other word, the set of all initial conditions such that resulting trajectories never exceed the specified constraints.

The maximal output admissible set has been well studied especially for *linear* systems with state and control constraints [8], [14], [16]. The concept of maximal output admissible sets makes it possible to insight into analysis of constrained control systems, and it is also true that the maximal output admissible set is used extensively throughout control system design methods [2], [3], [5], [7], [10], [11], [12], [17], [23], [25]. Among them multimode controller switching strategies and reference management problems have interesting separation or hierarchical features in constrained control system designs. Theses control design techniques allow us to ignore the effects of constraints on design of the primal feedback control system. The effect of constraints are maintained by introducing additional controller switching or reference management mechanisms. The maximal output admissible set is an essential tool to realize these hierarchical control design strategies.

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The problems consider in the present paper are motivated by controlling nonlinear systems with state and control constraints. Although there are variety of methods for designing feedback controllers for nonlinear plants, it is difficult to handle directly the effects of state and control constraints. Thus there is a good reason to ignore, at least in primal feedback control design stages, the effects of state and control constraints on design of closed-loop systems. The maximal output admissible set for nonlinear systems could be an essential tool to realize separation or hierarchical control design approaches for nonlinear systems. Actually it is also true that separation approaches to nonlinear constrained systems are utilized in literatures [1], [6], [20], [21]. However there exist no results to exactly determine the maximal output admissible set for nonlinear systems, therefore certain subsets of the maximal output admissible set, i.e., sublevel sets of Lyapunov functions or simulation based estimations of constrains admissible regions have been employed. Exact characterizations of the maximal output admissible set could be utilized in theses control strategies.

This paper considers generalization of the concept of maximal output admissible sets for nonlinear systems and is concerned with obtaining necessary and sufficient conditions for constraint fulfillments. Nonlinear dynamics considered in this paper are represented as polynomial functions of state variables. We propose a procedure to exactly determine the maximal output admissible set. Specific numerical computations required by the propose procedure can be reduced to certain nonlinear optimization problems, and recent developments of so called polynomial optimization techniques make it possible to obtain globally optimal solutions to the problem [15], [18], [19], [22]. Another contribution of this paper is deriving sufficient conditions which assure finite characterizations of the maximal output admissible set for nonlinear systems. For liner systems with constraints, basically observability assumptions assure finite characterizations of the maximal output admissible set. We discuss nonlinear generalizations for this finite characterization problem and show that observability assumptions with some additional conditions assure finite determinations of the maximal output admissible set. Due to the limitation of space, some numerical examples and proofs are omitted. Detailed version of this paper are available for an interested reader [13] .

Notation: The set of nonnegative integers is denoted by $\mathcal{Z}^+$. For $x \in \mathcal{R}^n$, let x_k denote the kth component of x. For $M \in \mathcal{R}^{m\times n}$, let $M_{(k,:)}$ denote the kth row of M. For $A \subset \mathcal{R}^n$, let $\mathrm{int}A$ and $\mathrm{cl}A$ denote interior and closure of the subset A, respectively.

II. A Class of Nonlinear Systems and Problem Descriptions

A. Constrained Nonlinear Systems

The system dynamics under consideration are

$$x(t+1) = f(x) \tag{1a}$$

$$z_0(t) = h(x) \tag{1b}$$

The vector signals are defined as follows: $x \in \mathcal{R}^n$ denotes the state, $z_0 \in \mathcal{R}^p$ denotes constrained output which are subject to specified pointwise-in-time constraints $z_0(t) \in Z \subset \mathcal{R}^p$ for all $t \in \mathcal{Z}^+$. The vector valued functions $f(\cdot)$ and $h(\cdot)$ are assumed to be polynomial functions of x_k, $k = 1, 2, \ldots, n$. The subset Z is a convex polyhedral set in the following form:

$$Z = \{z_0 \in \mathcal{R}^p | \ M_Z z_0 \leq m_Z\}$$

where $M_Z \in \mathcal{R}^{sz \times p}$ and $m_Z \in \mathcal{R}^{sz}$ describe linear constraints which specify Z.

Remark 1: Constraints considered here are not explicit *output* constraints, and the definition can handle any *state and control* constraints. Descriptions of systems with state and control constraints mentioned above are quiet realistic in practical applications. Certainly they apply to systems involve actuator saturations [2], [3], [5], [7], [8], [10], [11], [12], [14], [16], [17], [25].

B. The Maximal Output Admissible Set

The definition of the maximal output admissible set is stated as follows:

Definition 1: Let $z_0(t; x(0))$ denote the output (1b) for the initial condition $x(0)$. Define the maximal output admissible set by

$$O_\infty = \{x(0) \in \mathcal{R}^n | \ z_0(t; x(0)) \in Z \text{ for all } t \in \mathcal{Z}^+\}$$

Because of the definition, the condition $x(0) \in O_\infty$ is equivalent to constraints fulfillments for nonlinear systems (1). Note here that constraints considered here are not explicit *output* constraints. The definition can handle any *state and control* constraints. From this point of view, the subset O_∞ may also be called the maximal positively invariant set which might be a more classic and widely used term [4]. In this paper, we employ a notion of the maximal output admissible set following the definition in [8].

Explicit procedures to determine O_∞ will be proposed in Section IV. An important property of the maximal output admissible set for *linear* systems is that finite characterization is possible (see Section III). Discussions in Section V include some conditions which assure finite determinations of O_∞ for *nonlinear* system.

III. Review of Results for Linear Systems

This section briefly reviews results for linear systems. The contents of this section focus on the issue of algorithmic determinations of the maximal output admissible set. A key observation in algorithmic determinations is summarized in the following Remark 2. A corresponding observation in nonlinear case is stated in Remark 3.

The system dynamics considered in this section are

$$x(t+1) = Ax(t) \tag{2a}$$

$$z_0(t) = C_0 x(t) \tag{2b}$$

The vector signals are defined as similar to that for (1).

The maximal output admissible set is defined as follows [8], [16]:

Definition 2: Let $z_0(t; x(0))$ denote the output (2b) for the initial condition $x(0)$. Define the maximal output admissible set by

$$O_\infty^{\mathrm{L}} = \{x(0) \in \mathcal{R}^n | \ z_0(t; x(0)) \in Z \text{ for all } t \in \mathcal{Z}^+\}$$

Necessary and sufficient conditions for constraint fulfillments are an initial condition satisfies $x(0) \in O_\infty^{\mathrm{L}}$. The basic idea for algorithmic determinations of O_∞^{L} is to construct the following j-steps constraint admissible sets K_j^{L}, $j = 0, 1, \ldots$

$$K_j^{\mathrm{L}} = \{x(0) \in \mathcal{R}^n | \ z_0(t; x(0)) \in Z \quad t = 0, 1, \ldots, j\}$$

Each K_j^{L} consists of all initial conditions which at least assure j-steps constraint fulfillments. The following recursive descriptions of K_j^{L}, $j = 1, 2, \ldots$ will be useful.

$$K_j^{\mathrm{L}} = \{x \in K_{j-1}^{\mathrm{L}} | \ Ax \in K_{j-1}^{\mathrm{L}}\} \qquad j \geq 1$$
$$K_0^{\mathrm{L}} = \{x \in \mathcal{R}^n | \ C_0 x \in Z\}$$

Although, from the definition of K_j^{L}, we have $O_\infty^{\mathrm{L}} = \bigcap_{j \in \mathcal{Z}^+} K_j^{\mathrm{L}}$, it might not make it possible to construct O_∞^{L} since infinitely many recursive computations may be required. However, a finite characterization of O_∞^{L} is actually possible. We start with the following definition.

Definition 3: The maximal output admissible set, O_∞^{L}, is said to be *finitely determined* if there exists an index j such that $K_j^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$.

Suppose O_∞^{L} is finitely determined with an index j. It is clear that $O_\infty^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$. Under the following Assumption 1, finite determinability of O_∞^{L} can be summarized as in Theorem 1 stated below [8].

Assumption 1:

1. Z is bounded and satisfies $0 \in \mathrm{int} Z$.
2. The eigenvalues of A satisfy $|\lambda_j(A)| < 1$.
3. (C_0, A) is an observable pair.

Theorem 1: The maximal output admissible set, O_∞^{L}, is finitely determined.

Algorithmic procedures for computing O_∞^{L} can be summarized as follows:

step 1) Set $K_0^{\mathrm{L}} = \{x \in \mathcal{R}^n | \ N_0 x \leq n_0\}$ and $j = 1$ where

$$N_0 = M_Z C_0 \qquad n_0 = m_Z$$

step 2) Set $K_j^{\mathrm{L}} = \{x \in \mathcal{R}^n | \ N_j x \leq n_j\}$ where

$$N_j = \begin{bmatrix} N_0 A^j \\ N_{j-1} \end{bmatrix} \qquad n_j = \begin{bmatrix} n_0 \\ n_{j-1} \end{bmatrix}$$

if $K_j^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$

$O_\infty^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$ and stop.

else

set $j = j + 1$ and goto step 2).

Remark 2: In step 2), the procedure requires to check whether $K_j^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$ or not. Since K_j^{L} has an explicit representation $K_j^{\mathrm{L}} = \{x \in K_{j-1}^{\mathrm{L}} |\ N_0 A^j x \le n_0\}$, the condition $K_j^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$ indicate that all of new constraints $N_0 A^j x \le n_0$ which are added to represent K_j^{L} are redundant. Therefore the condition $K_j^{\mathrm{L}} = K_{j-1}^{\mathrm{L}}$ is equivalent to the following inequality constraints

$$\max_{x \in K_{j-1}^{\mathrm{L}}} (N_0)_{(k,:)} A^j x < (n_0)_k \quad \text{for all } k = 1, 2, \cdots, s_Z$$

Maximizing the left hand side is a standard linear programming problem, and it can be handled in a numerically efficient way.

IV. Algorithmic Determination of the Maximal Output Admissible Set

This section proposes explicit algorithmic procedures to determine the maximal output admissible set for nonlinear dynamics (1). Main observation in this aspect will be summarized in Remark 3.

We first define j-steps constraint admissible sets

$$K_j = \{x(0) \in \mathcal{R}^n |\ z_0(t; x(0)) \in Z \quad t = 0, 1, \ldots, j\}$$

The subsets K_j, $j = 0, 1, \ldots$ consists of all initial conditions which at least assure j-steps constraint fulfillments. The following recursive descriptions are also valid even for nonlinear case.

$$\begin{aligned} K_j &= \{x \in K_{j-1} |\ f(x) \in K_{j-1}\} \qquad j \ge 1 \\ K_0 &= \{x \in \mathcal{R}^n |\ h(x) \in Z\} \\ &= \{x \in \mathcal{R}^n |\ M_0(x) \le m_0\} \\ M_0(\cdot) &= M_Z h(\cdot) \qquad m_0 = m_Z \end{aligned}$$

Finite determinability of O_∞ is defined as follows:

Definition 4: Suppose $O_\infty \neq \varnothing$[1]. The maximal output admissible set, O_∞, is said to be *finitely determined* if there exists an index j such that $K_j = K_{j-1}$.

The remainder of this section considers explicit descriptions of K_j and O_∞. Let us denote $f^1(x) = f(x)$ and $f^{j+1}(x) = f^1(f^j(x))$, $j \ge 1$. From the definition of K_1, we have

$$\begin{aligned} K_1 &= \{x \in K_0 |\ f^1(x) \in K_0\} \\ &= \{x \in \mathcal{R}^n |\ M_1(x) \le m_1\} \\ M_1(\cdot) &= \begin{bmatrix} M_0(f^1(\cdot)) \\ M_0(\cdot) \end{bmatrix} \qquad m_1 = \begin{bmatrix} m_0 \\ m_0 \end{bmatrix} \end{aligned}$$

Similarly, definition of K_2 allows us the following possible representation

$$\begin{aligned} K_2 &= \{x \in K_1 |\ f^1(x) \in K_1\} \\ &= \{x \in \mathcal{R}^n | \begin{bmatrix} M_0(f^2(x)) \\ M_0(f^1(x)) \\ M_0(f^1(x)) \\ M_0(x) \end{bmatrix} \le \begin{bmatrix} m_0 \\ m_0 \\ m_0 \\ m_0 \end{bmatrix} \} \end{aligned}$$

[1] O_∞ is not empty in general. See Lemma 2 in Section V.

Clearly, the second row block of the above inequality constraints are redundant to describe K_2. Thus, we have

$$\begin{aligned} K_2 &= \{x \in \mathcal{R}^n |\ M_2(x) \le m_2\} \\ M_2(\cdot) &= \begin{bmatrix} M_0(f^2(\cdot)) \\ M_1(\cdot) \end{bmatrix} \qquad m_2 = \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \end{aligned}$$

It is exactly same for all $j \ge 1$, by removing the redundant constraints, we have the following explicit representation of K_j

$$\begin{aligned} K_j &= \{x \in \mathcal{R}^n |\ M_j(x) \le m_j\} \qquad j \ge 1 \\ M_j(\cdot) &= \begin{bmatrix} M_0(f^j(\cdot)) \\ M_{j-1}(\cdot) \end{bmatrix} \qquad m_j = \begin{bmatrix} m_0 \\ m_{j-1} \end{bmatrix} \end{aligned}$$

Algorithmic procedures for computing O_∞ can be summarized as follows:

step 1) Set $K_0 = \{x \in \mathcal{R}^n |\ M_0(x) \le m_0\}$ and $j = 1$ where

$$M_0(\cdot) = M_Z h(\cdot) \qquad m_0 = m_Z$$

step 2) Set $K_j = \{x \in \mathcal{R}^n |\ M_j(x) \le m_j\}$ where

$$M_j(\cdot) = \begin{bmatrix} M_0(f^j(\cdot)) \\ M_{j-1}(\cdot) \end{bmatrix} \qquad m_j = \begin{bmatrix} m_0 \\ m_{j-1} \end{bmatrix}$$

if $K_j = K_{j-1}$
  $O_\infty = K_{j-1}$ and stop.
else
  set $j = j + 1$ and goto step 2).

Remark 3: In step 2), the procedure requires to check whether $K_j = K_{j-1}$ or not. Since K_j has an explicit representation $K_j = \{x \in K_{j-1} |\ M_0(f^j(x)) \le m_0\}$, the condition $K_j = K_{j-1}$ indicate that all of new constraints $M_0(f^j(x)) \le m_0$ which are added to represent K_j are redundant. Therefore the condition $K_j = K_{j-1}$ is equivalent to the following inequality constraints (see Remark 2).

$$\max_{x \in K_{j-1}} (M_0(f^j(x)))_k < (m_0)_k \quad \text{for all } k = 1, \ldots, s_Z$$

The left hand side of the above inequality is a maximization problem of a polynomial objective function over a feasible region described by semidefinite constraints of multi-variate polynomials [15]. Recent developments of so called polynomial optimization techniques make it possible to solve this class of optimization problems [15], [22], [26], [27]. Some techniques can actually obtain globally optimal solutions for polynomial optimizations [9], [18], [19]. Therefore, checking the condition $K_j = K_{j-1}$ can be handled by utilizing these polynomial optimization techniques. In the following Example 1, we use the results in [9], [24] to show a numerical example of exact computation of O_∞.

Example 1: Consider the following nonlinear (quadratic) system [28] with input constraints $-5 \le z_0 = -u \le 5$

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + \Delta(x(t))Hx(t) \\ z_0(t) &= -u(t) = C_0 x(t) \end{aligned}$$

where the matrices A, B and C_0 are given by

$$A = \begin{bmatrix} 0.1 & 2 \\ -0.8 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad C_0 = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}$$

The quadratic term is defined by $\Delta(x) = \mathrm{diag}[\ x^{\mathrm{T}} \quad x^{\mathrm{T}}\ ]$ and $H = [\ H_1^{\mathrm{T}} \quad H_2^{\mathrm{T}}\ ]^{\mathrm{T}}$ where

$$H_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.02 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0.02 & 0.05 \\ 0.005 & 0.01 \end{bmatrix}$$

O_∞ is finitely determined as $O_\infty = K_3 = K_4$. Fig. 1 shows K_0 (dashed lines) and K_1. In Fig. 1, each '$\circ$' indicate the region of K_1 as '$\circ \in K_1$'. Similarly, each '$\circ$' in Fig. 2 indicate the region of $O_\infty = K_3$.

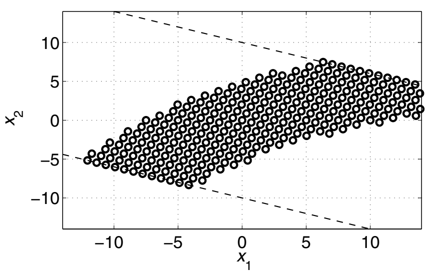

Fig. 1. K_0 (dashed line) and K_1. '$\circ$' indicates a subset K_1 such as '$\circ \in K_1$'.

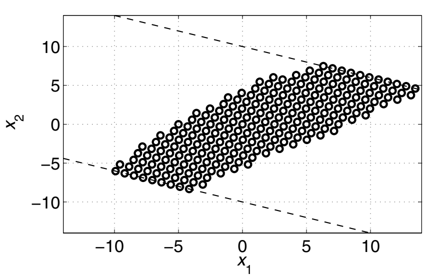

Fig. 2. K_0 (dashed line) and $O_\infty = K_3$. '$\circ$' indicates a subset O_∞ such as '$\circ \in O_\infty$'.

Fig. 3 shows sample state trajectories. Note here that $z_0(\tau) \in Z$ is equivalent to $x(\tau) \in K_0$ for each time instance $\tau \in \mathrm{Z}^+$, and it can be seen that trajectories starting from the inside of O_∞ never exceed the specified constraints. On the other hand, trajectories with $x(0) \notin O_\infty$ necessarily violate the specified constraints.

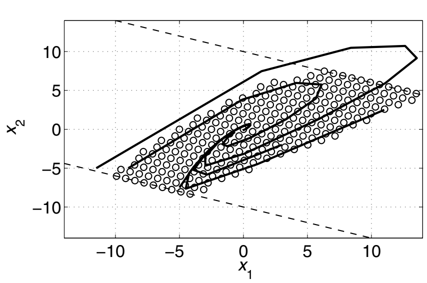

Fig. 3. $O_\infty = K_3$, K_0 and sample state trajectories. Trajectories starting from $x(0) \notin O_\infty$ necessarily violate the specifi ed input constraints.

V. Finite Determinations of the Maximal Output Admissible Sets

This section discusses finite determinability of the maximal output admissible set for nonlinear systems. In Section IV, we propose numerically tractable recursive procedures to determine O_∞. However, a finite termination of the procedure or, in other word, a finite determination of O_∞ is not clear yet. We will show that observability assumptions and some additional conditions can assure finite determinations of O_∞.

The following SectionV-A shows some basic conditions which assure finite determinations of O_∞. Section V-B considers observability and some related properties of nonlinear systems. Finally, Section V-C discusses relation between the condition derived in SectionV-A and the properties related to observability of nonlinear systems. In the remainder of this paper the following assumptions are considered to investigate finite determinability of O_∞.

Assumption 2:

1. Z is bounded and satisfies $0 \in \mathrm{int}Z$.
2. $f(0) = 0$ and $h(0) = 0$, i.e., $x = 0$ is an equilibrium point of (1).

Due to the limitation of space, proofs of all lemmas in this section are omitted. Those are available in [13].

A. Basic Conditions for Finite Determinations

This section describes some basic conditions which assure finite determinations of O_∞. Observations from an engineering point of view on the conditions considered here are stated in Remark 4.

We first state the following Lemma 1 and 2 which summarize some important properties of K_j and O_∞.

Lemma 1: The subsets K_j, $j = 0, 1, 2, \ldots$ and O_∞ are closed.

Lemma 2: Suppose the equilibrium point $x = 0$ of (1) is stable. The subsets K_j, $j = 0, 1, \ldots$ and O_∞ contain $x = 0$ in its interior.

The following Lemma 3 and 4 clarify basic mechanisms which assure finite determinations of O_∞. In the remainder of this paper we consider the following assumption:

Assumption 3: The equilibrium point $x = 0$ of (1) is asymptotically stable.

Lemma 3: O_∞ is finitely determined if there exists a $T \in \mathcal{Z}^+$ such that K_{T-1} is bounded.

We need to introduce some notations to state Lemma 4. For $t \in \mathcal{Z}^+ \setminus \{0\}$ and $x(0) \in \mathcal{R}^n$, let us define the mapping H_{t-1} as follows:

$$\begin{bmatrix} z_0(0) \\ z_0(1) \\ \vdots \\ z_0(t-1) \end{bmatrix} = \begin{bmatrix} h(x(0)) \\ h(f^1(x(0))) \\ \vdots \\ h(f^{t-1}(x(0))) \end{bmatrix} = H_{t-1}(x(0))$$

The mapping $H_{t-1} : \mathcal{R}^n \to \mathcal{R}^{pt}$ is continuously differentiable on $\mathcal{R}^n$. Note here that, by using H_{j-1}, the subset K_{j-1} has the following representation:

$$K_{j-1} = \{x \in \mathcal{R}^n |\ H_{j-1}(x) \in Z^j\}$$

Lemma 4: O_∞ is finitely determined if there exists a $T \in \mathcal{Z}^+$ such that $H_{T-1} : K_{T-1} \to H_{T-1}(K_{T-1})$ is a homeomorphism.

Remark 4: The mapping H_{T-1} is injective indicate that the initial condition $x(0)$ and the resulting output $z_0(\cdot; x(0))$ have one-to-one correspondence. This is a necessary requirement from a basic concept of observability such that knowledge of the output suffices to uniquely determine the initial state (see the following Definition 5, 6 and 8). On the other hand, continuity of the inverse mapping F_{T-1} of H_{T-1} gets rid of small fluctuations of the output observations result in large differences of the initial state. Arbitrary small observation noises may cause extremely large changes in estimated initial state for system without this continuity properties, thus mechanisms so called observers may not work well. From an engineering pint of view, systems to be called observable should hold this continuity properties. Actually standard results of nonlinear system theory require this continuity property at least locally (see Definition 6 and Lemma 5). Lemma 4 states that natural requirements from an engineering point of view assure finite determinations of O_∞.

B. Observability Assumptions and Related Properties

We will consider some properties of nonlinear systems derived under observability assumptions. The results stated here might be simple applications of standard results in nonlinear system theories [29]. The relations to the conditions derived in Section V-A will be discussed in the next Section V-C.

First, let us introduce a notation G_{t-1}, $t \in \mathcal{Z}^+ \setminus \{0\}$.

$$G_{t-1} = \{h_j(f^k(\cdot)) : \mathcal{R}^n \to \mathcal{R} | \\ j = 1, 2, \ldots, p \quad k = 0, 1, \ldots, t-1\}$$

G_{t-1} is a set of all functions which are components of H_{t-1}. Each element of G_{t-1} is a continuously differentiable function on $\mathcal{R}^n$. We use $\mathrm{d}\alpha$ to denote derivative of $\alpha \in G_{t-1}$.

The following four definitions are standard in nonlinear system theories.

Definition 5: Two states x_0 and x_1 are said to be *distinguishable* if $z_0(\cdot; x_0) \neq z_0(\cdot; x_1)$ where $z_0(\cdot; x_j)$, $j = 0, 1$ denotes the output of the system (1) corresponding to the initial condition $x(0) = x_i$.

Definition 6: The system (1) is said to be *locally observable at* $x_0 \in \mathcal{R}^n$ if there exists an open neighborhood U of x_0 such that every $x \in U$ other than x_0 is distinguishable from x_0.

Definition 7: Let $X \subset \mathcal{R}^n$ be an open set. The system (1) is said to be *locally observable in* X if it is locally observable at each $x_0 \in X$.

Definition 8: Let $X \subset \mathcal{R}^n$. The system (1) is said to be *observable in* X if each $x_0 \in X$ is distinguishable from every $x \in X$ other than x_0.

We introduce the following Condition 1 which are standard requirements for nonlinear systems to be observable [29].

Condition 1: Suppose $x \in \mathcal{R}^n$ and $T \in \mathcal{Z}^+$ are given. There exist n functions $\alpha_k(\cdot) \in G_{T-1}$, $k = 1, 2, \ldots, n$ such that n row vectors $\mathrm{d}\alpha_k(x)$ evaluated at x are linearly independent.

Lemma 5: Suppose $x_0 \in \mathcal{R}^n$ is given. The system (1) is *locally observable at* x_0 if there exits a $T \in \mathcal{Z}^+$ that satisfies Condition 1.

Corollary 1: Let $X \subset \mathcal{R}^n$ be an open set. The system (1) is *locally observable in* X, if there exists a $T \in \mathcal{Z}^+$ that satisfies Condition 1 for each $x \in X$.

Remark 5: As it is required in Corollary 1, even if the conditions mentioned in Lemma 5 are satisfied for every $x \in X$, there may exist indistinguishable initial conditions and thus the mapping from the initial conditions to the resulting outputs may not be injective. Therefore, a *global* observability in the sense of Definition 8 is not assured. A global observability in an open set X is important to derive the following Lemma 6.

Lemma 6: Let $X \subset \mathcal{R}^n$ be an open set. Suppose the system (1) is *observable in* X, i.e., there exists a $T \in \mathcal{Z}^+$ such that $H_{T-1} : \mathcal{R}^n \to \mathcal{R}^{pT}$ is an one-to-one mapping from X onto $H_{T-1}(X)$. Then the inverse mapping $F_{T-1} : H_{T-1}(K_{T-1}) \to K_{T-1}$ of H_{T-1} is defined. Suppose that T satisfies Condition 1 for each $x \in X$. Then the mapping F_{T-1} is continuously differentiable.

C. Observability and Finite determinations of O_∞

Sufficient conditions that assure finite determinability of O_∞ is derived by combining the results in Sections V-A and V-B. The following Corollary 2 is immediate from Lemma 6.

Corollary 2: Suppose there exists a $T \in \mathcal{Z}^+$ such that the system (1) is observable on K_{T-1} and Condition 1 is satisfied for each $x \in \mathrm{int}K_{T-1}$. Then the inverse mapping $F_{T-1} : H_{T-1}(\mathrm{int}K_{T-1}) \to \mathrm{int}K_{T-1}$ of $H_{T-1} : \mathrm{int}K_{T-1} \to H_{T-1}(\mathrm{int}K_{T-1})$ is continuously differentiable on $H_{T-1}(\mathrm{int}K_{T-1})$.

Observing Lemma 4, continuity of F_{T-1} on $H_{T-1}(K_{T-1})$ rather than on $H_{T-1}(\mathrm{int}K_{T-1})$ concludes finite determinations of O_∞. We have the following sufficient conditions.

Theorem 2: Suppose there exists a $T \in \mathcal{Z}^+$ that satisfies conditions required in Corollary 2. Suppose that $K_{T-1} = \mathrm{cl}(\mathrm{int}K_{T-1})$, and there exists $\eta > 0$ such that

$$\inf_{\substack{v=1 \\ v \in \mathcal{R}^n}} \left|\frac{\partial H_{T-1}(x)}{\partial x} v\right| \geq \eta |v|$$

is held for all $x \in \mathrm{int}K_{T-1}$. Suppose further, for any $\gamma > 0$, there exists $r > 0$ such that

$$\left|F_{T-1}(\bar{z}) - F_{T-1}(z) - \frac{\partial F_{T-1}(z)}{\partial z}(\bar{z} - z)\right| \leq \gamma |\bar{z} - z|$$

is satisfied for any $\bar{z}, z \in H_{T-1}(\mathrm{int}K_{T-1})$ with $|\bar{z} - z| \leq r$. Then O_∞ is finitely determined.

Proof: We have

$$H_{T-1}(K_{T-1}) \subset \mathrm{cl}(H_{T-1}(\mathrm{int}K_{T-1})) \tag{3}$$

since $H_{T-1}(K_{T-1}) = H_{T-1}(\mathrm{cl}(\mathrm{int}K_{T-1}))$.

Let $\{z_n\}$ be a sequence in $H_{T-1}(\mathrm{int}K_{T-1})$ converges to z_0, i.e., $z_0 \in \mathrm{cl}(H_{T-1}(\mathrm{int}K_{T-1}))$. Define a corresponding sequence in $\mathrm{int}K_{T-1}$ by

$$\{x_n = F_{T-1}(z_n)\} \tag{4}$$

Now suppose that the sequence $\{x_n\}$ is convergent, then we have $z_0 = H_{T-1}(x_0) \in H_{T-1}(K_{T-1})$ since $x_n \to x_0 \in K_{T-1}$. This implies

$$\mathrm{cl}(H_{T-1}(\mathrm{int}K_{T-1})) \subset H_{T-1}(K_{T-1})$$

Combining (3) and the above inclusion, we have

$$H_{K-1}(K) = \mathrm{cl}(H_{T-1}(\mathrm{int}K_{T-1}))$$

Thus, the subset $H_{T-1}(K_{T-1})$ is closed. This also concludes continuity of F_{T-1} on $H_{T-1}(K_{T-1})$ since $F_{T-1}^{-1}(K_{T-1}) = H_{T-1}(K_{T-1})$ and both of K_{T-1} and $H_{T-1}(K_{T-1})$ are closed. Therefore, Lemma 4 assures that O_∞ is finitely determined.

Now we investigate the convergence of the sequence $\{x_n\}$ defined by (4). For $x \in \mathrm{int}K_{T-1}$, let $\frac{\partial H_{T-1}(x)}{\partial x}v = \ell$. Since $\frac{\partial F_{T-1}(z)}{\partial z}\frac{\partial H_{T-1}(x)}{\partial x} = I_n$, we have $|\frac{\partial H_{T-1}(x)}{\partial x}v| \geq \eta|v| = \eta|\frac{\partial F_{T-1}(z)}{\partial z}\ell|$, thus

$$|\frac{\partial F_{T-1}(z)}{\partial z}\ell| \leq M|\ell|$$

where $z = H_{T-1}(x)$ and $M = 1/\eta$. For any $\varepsilon > 0$, let $\delta \leq \min(\frac{\varepsilon}{M+\gamma}, r)$. Since $\{z_n\}$ is convergent, there exists N such that $|z_n - z_m| \leq \delta$ is held for any $n, m \geq N$. Because F_{T-1} is differentiable on $H_{T-1}(\mathrm{inf}K_{T-1})$, we have

$$\begin{aligned} x_n - x_m &= F_{T-1}(z_n) - F_{T-1}(z_m) \\ &= \frac{\partial F_{T-1}(z_m)}{\partial z}(z_n - z_m) + o(|z_n - z_m|) \end{aligned}$$

where $o(|z_n - z_m|)$ is any function that satisfies $o(|z_n - z_m|) \to 0$ when $z_m - z_n \to 0$, and we further have $|o(|z_n - z_m|)| \leq \gamma|z_m - z_n|$ since $\delta \leq r$. Finally we have

$$\begin{aligned} |x_n - x_m| &\leq |\frac{\partial F_{T-1}(z_m)}{\partial z}(z_n - z_m)| + |o(|z_n - z_m|)| \\ &\leq M|z_n - z_m| + \gamma|z_n - z_m| \leq \varepsilon \end{aligned}$$

thus the sequence $\{x_n\}$ defined by (4) is convergent. ■

VI. Conclusions

This paper considers necessary and sufficient conditions for fulfilling the specified state and control constraints against the class of nonlinear dynamics. The results are generalizations of the maximal output admissible set theory to the case of certain class of nonlinear systems. We propose a recursive procedure to determine the maximal output admissible set for nonlinear systems. We also discuss on issues of finite determinations of the maximal output admissible set. Some relations between observability of nonlinear systems and finite characterizations of the maximal output admissible set are clarified.

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