A Computationally Efficient Formulation of Robust Model Predictive Control using Linear Matrix Inequalities

Zhaoyang Wan* and Mayuresh V. Kothare†
Department of Chemical Engineering
Lehigh University
Bethlehem, PA 18015, U.S.A.

Abstract

In this paper, we present an off-line approach for robust constrained MPC synthesis that gives an explicit control law using Linear Matrix Inequalities (LMIs). This off-line approach can address a broad class of model uncertainty descriptions with guaranteed robust stability of the closed-loop system and substantial reduction of the on-line MPC computation.

Keywords

Model predictive control, Linear matrix inequalities, Asymptotically stable invariant ellipsoid

Introduction

The practicality of Model Predictive Control (MPC) is partially limited by its ability to solve optimization problems in real time. Moreover, when MPC incorporates explicit plant-model uncertainty, the additional constraints imposed to guarantee robust stability result in significant growth of the on-line MPC computation. Researchers have begun to study methods for fast computation of an optimal or suboptimal solution to the quadratic programming problem associated with nominal MPC (Bemporad et al., 1999; Zheng, 1999; Van Antwerp and Braatz, 2000). For systems with polytopic model uncertainty and input constraints, receding horizon dual-mode paradigm can be used to reduce the computational complexity in MPC (Lee and Kouvaritakis, 2000). In this paper, we present an off-line approach for robust constrained MPC synthesis for both polytopic uncertain systems and norm bounded uncertain systems.

Background

Models for Uncertain Systems

Consider a linear time varying (LTV) system

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

$$\begin{bmatrix} A(k) & B(k) \end{bmatrix} \in \Omega$$

where $u(k) \in \mathcal{R}^{n_u}$ is the control input, $x(k) \in \mathcal{R}^{n_x}$ is the state of the plant and $y(k) \in \mathcal{R}^{n_y}$ is the plant output.

For polytopic uncertainty, Ω is the polytope $Co\{[A_1 \quad B_1], ..., [A_L \quad B_L]\}$, where Co denotes the convex hull, $[A_i \quad B_i]$ are vertices of the convex hull. Any $[A \quad B]$ within the convex set Ω is a linear combination of the vertices $A = \sum_{j=1}^{L} \alpha_j A_j,\ B = \sum_{j=1}^{L} \alpha_j B_j$ with $\sum_{j=1}^{L} \alpha_j = 1,\ 0 \leq \alpha_j \leq 1$.

For norm bounded uncertainty, the LTV system is expressed as a LTI system with uncertainties or perturbations appearing in a feedback loop:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_p p(k) \\ y(k) &= Cx(k) \\ q(k) &= C_q x(k) + D_{qu} u(k) \\ p(k) &= (\Delta q)(k) \end{aligned} \tag{2}$$

where the operator $\Delta = diag(\Delta_1, ..., \Delta_l)$ with $\Delta_i : \mathcal{R}^{n_i} \longrightarrow \mathcal{R}^{n_i}$, $i = 1, ..., l$. Δ can represent either a memoryless time varying matrix with $\|\Delta_i(k)\|_2 \equiv \bar{\sigma}(\Delta_i(k)) \leq 1,\ k \geq 0$, or a convolution operator (for, e.g., a stable LTI dynamical system), with the operator norm induced by the truncated l_2 -norm less than 1, i.e., $\sum_{j=0}^{k} p_i(j)^T p_i(j) \leq \sum_{j=0}^{k} q_i(j)^T q_i(j),\ i = 1, ..., l,\ \forall k \geq 0$.

On-line Robust Constrained MPC Using LMIs

Consider the following problem, which minimizes the robust or worst case infinite horizon quadratic objective function:

$$\min_{u(k+i|k)=F(k)x(k+i|k)} \quad \max_{[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0} J_\infty(k) \tag{3}$$

subject to

$$|u_r(k+i|k)| \leq u_{r,\max},\ \ i \geq 0,\ r = 1, 2, ..., n_u \tag{4}$$

$$|y_r(k+i|k)| \leq y_{r,\max},\ \ i \geq 1,\ r = 1, 2, ..., n_y \tag{5}$$

where $J_\infty(k) = \sum_{i=0}^{\infty}[x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T R u(k+i|k)]$ with $Q_1 > 0,\ \ R > 0$. In (3), we assume that at each sampling time k, a constant state feedback law $u(k+i|k) = F(k)x(k+i|k)$ is used to minimize the worst case value of $J_\infty(k)$. Following an approach given in (Kothare et al., 1996), it is easy to derive an upper bound on $J_\infty(k)$. First, at sampling time k, define a quadratic function $V(x) = x^T P(k) x,\ P(k) > 0$. For any $[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0$, suppose $V(x)$ satisfies the

*email: zhw2@lehigh.edu
†Corresponding author: phone (610) 758-6654, fax (610) 758-5057, email: mayuresh.kothare@lehigh.edu

following robust stability constraint:

$$\begin{aligned} V(x(k+i+1|k)) - V(x(k+i|k)) \\ \leq -\left[x(k+i|k)^T Q_1 x(k+i|k)\right. \\ \left. +u(k+i|k)^T R u(k+i|k)\right] \end{aligned} \quad (6)$$

Summing (6) from $i = 0$ to $i = \infty$ and requiring $x(\infty|k) = 0$ or $V(x(\infty|k)) = 0$, it follows that

$$\max_{[A(k+i)\ B(k+i)]\in\Omega,\ i\geq 0} J_\infty(k) \leq V(x(k|k)) \leq \gamma \quad (7)$$

(6) and (7) give an upper bound on $J_\infty(k)$. The condition $V(x(k|k)) \leq \gamma$ in (7) can be expressed equivalently as the LMI

$$\begin{bmatrix} 1 & x(k|k)^T \\ x(k|k) & Q \end{bmatrix} \geq 0, \quad (8)$$

where $Q = \gamma P(k)^{-1}$.

The robust stability constraint (6) for the system (1) is satisfied if for each vertex of Ω

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_1^{1/2} & Y^T R^{1/2} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_1^{1/2} Q & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0,\ j = 1, ..., L \quad (9)$$

where, $Q = \gamma P(k)^{-1}$ and $F(k)$ is parameterized by YQ^{-1}. This set of conditions is convex in Ω. So if (9) is satisfied, then for any $[A(k+i)\ B(k+i)] \in \Omega,\ i \geq 0$, (6) is satisfied. For the system (2), the stability constraint (6) is satisfied if

$$\begin{bmatrix} Q & QA^T + Y^T B^T & QC_q^T + Y^T D_{qu}^T & QQ_1^{1/2} & Y^T R^{1/2} \\ AQ + BY & Q - B_p \Lambda B_p^T & 0 & 0 & 0 \\ C_q Q + D_{qu} Y & 0 & \Lambda & 0 & 0 \\ Q_1^{1/2} Q & 0 & 0 & \gamma I & 0 \\ R^{1/2} Y & 0 & 0 & 0 & \gamma I \end{bmatrix} \geq 0 \quad (10)$$

where $\Lambda = diag(\lambda_1 I_{n_1}, ..., \lambda_l I_{n_l}) > 0$. The input constraints (4) are satisfied if there exists a symmetric matrix X such that

$$\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \geq 0 \quad (11)$$

with $X_{rr} \leq u_{r,\max}^2,\ \ r = 1, 2, ..., n_u$. Similarly, the output constraints (5) for the system (1) are satisfied if there exists a symmetric matrix Z such that for each vertex of Ω,

$$\begin{bmatrix} Z & C\left(A_j Q + B_j Y\right) \\ \left(A_j Q + B_j Y\right)^T C^T & Q \end{bmatrix} \geq 0 \quad (12)$$
$$j = 1, ..., L$$

with $Z_{rr} \leq y_{r,\max}^2,\ r = 1, 2, ..., n_y$. The output constraints (5) for the system (2) are satisfied if for each row of C

$$\begin{bmatrix} y_{r,\max}^2 Q & (C_q Q + D_{qu} Y)^T & (AQ + BY)^T C^{<r>T} \\ C_q Q + D_{qu} Y & T_r & 0 \\ C^{<r>}(AQ + BY) & 0 & I - C^{<r>} B_p T_r B_p^T C^{<r>T} \end{bmatrix} \geq 0 \quad (13)$$

where $C^{<r>}$ is the rth row of C, $T_r = diag(t_{r,1} I_{n_1}, ..., t_{r,l} I_{n_l}) > 0,\ r = 1, 2, ..., n_y$

Theorem 1 *(On-line robust constrained MPC) (Kothare et al., 1996). For the system (1) or (2), at sampling time k, let $x(k) = x(k|k)$ be the state. Then the state feedback matrix $F(k)$ in the control law $u(k+i|k) = F(k)x(k+i|k),\ i \geq 0$, which minimizes the upper bound γ on the worst case robust MPC objective function $J_\infty(k)$, is given by $F(k) = YQ^{-1}$ where $Q > 0$ and Y are obtained from the solution (if it exists) of one of the following linear objective minimization problems:*
(a) for system (1), $\min_{\gamma,Q,X,Y,Z} \gamma$ subject to (8), (9), (11) and (12).
(b) for system (2), $\min_{\gamma,Q,X,Y,\Lambda,T_1,...,T_{n_y}} \gamma$ subject to (8), (10), (11) and (13).

Furthermore, the time varying state feedback matrix $F(k)$ robustly asymptotically stabilizes the closed-loop system.

Off-line Robust Constrained MPC

In this section, we present an off-line approach based on the concept of *the asymptotically stable invariant ellipsoid*. Without loss of generality, we use the algorithm for polytopic uncertain systems (Theorem 1 (a)) to illustrate the following lemmas, corollaries and theorem. Similar results can be obtained for Theorem 1 (b).

Asymptotically Stable Invariant Ellipsoid

Definition 1 *(Asymptotically stable invariant ellipsoid). Given a discrete dynamical system $x(k+1) = f(x(k))$, a subset $\mathcal{E} = \{x \in \mathcal{R}^{n_x} | x^T Q^{-1} x \leq 1\}$ of the state space $\mathcal{R}^{n_x}$ is said to be an asymptotically stable invariant ellipsoid, if it has the property that, whenever $x(k_1) \in \mathcal{E}$ at $k_1 \geq 0$, then $x(k) \in \mathcal{E}$ for all times $k \geq k_1$ and $x(k) \longrightarrow 0$ as $k \longrightarrow \infty$.*

Lemma 1 *Consider a closed-loop system composed of a plant (1) and a static state feedback controller $u = YQ^{-1}x$, where Y and Q^{-1} are obtained by applying the robust constrained MPC defined in Theorem 1 (a) to a system state x_0. Then, the subset $\mathcal{E} = \{x \in \mathcal{R}^{n_x} | x^T Q^{-1} x \leq 1\}$ of the state space $\mathcal{R}^{n_x}$ is an asymptotically stable invariant ellipsoid.*

Proof. When the robust constrained MPC defined in Theorem 1 (a) is applied to a state x_0 of a plant (1), the only LMI in Theorem 1 (a) that depends on the system state is (8) which is automatically satisfied for all states within the ellipsoid $\mathcal{E}$. So the minimizer γ, Q, X, Y and Z at the state x_0 is also feasible (though not necessarily optimal) for any other state in $\mathcal{E}$, which means we can apply the state feedback law $u = YQ^{-1}x$ to all the states in $\mathcal{E}$ with the satisfation of (9), (11) and (12).

Consider the closed-loop system composed of the plant (1) and the static state feedback controller $u = YQ^{-1}x$, where Y and Q^{-1} are obtained by applying the robust constrained MPC defined in Theorem 1 (a) to a system state x_0. Then, for any state $\tilde{x}(k) \in \mathcal{E}$, the satisfaction of (9) ensures that in real time $\tilde{x}(k+i+1)^T Q^{-1} \tilde{x}(k+i+1) < \tilde{x}(k+i)^T Q^{-1} \tilde{x}(k+i) \leq 1$, $i \geq 0$. Thus, $\tilde{x}(k+i) \in \mathcal{E}$, $i \geq 0$ and $\tilde{x}(k+i) \longrightarrow 0$ as $i \longrightarrow \infty$, which establish that $\mathcal{E}$ is an asymptotically stable invariant ellipsoid.

Remark 1 *From Lemma 1, we know that for system (1), an asymptotically stable invariant ellipsoid can be constructed by applying Theorem 1 (a) to an arbitrary feasible state x_0. The minimization at x_0 gives a state feedback law $u = YQ^{-1}x$. Once a state enters into the ellipsoid, the static state feedback controller $u = YQ^{-1}x$ can keep it within the ellipsoid and converge it to the origin. The following corollaries state the dependence of the ellipsoidal weighting matrix Q^{-1} and the state feedback matrix YQ^{-1} on the state x_0.*

Corollary 1 *For a nominal and unconstrained system, the ellipsoid weighting matrix Q^{-1} and the feedback matrix YQ^{-1} obtained by applying Theorem 1 (a) without (11) and (12) to an arbitrary state x_0 are $\frac{1}{x_0^T P_{are}\, x_0} P_{are}$ and $-(R+B^T P_{are} B)^{-1} B^T P_{are} A$ respectively, where P_{are} is the solution of the Algebraic Riccati Equation $P_{are} - A^T P_{are} A + A^T P_{are} B (R + B^T P B)^{-1} B^T P_{are} A - Q_1 = 0$.*

Corollary 2 *For an uncertain and unconstrained system (1), the ellipsoid weighting matrix Q^{-1} and the feedback matrix YQ^{-1} obtained by applying Theorem 1 (a) without (11) and (12) to an arbitrary state x_0 are $\frac{1}{x_0^T P\, x_0} P$ and F, where P and F are constant along an arbitrary one-dimensional subspace $\mathcal{S} = \{x \in R^{n_x} | \alpha x_0,\ \alpha \in R\}$ of the state space $\mathcal{R}^{n_x}$.*

Proof. Along the one-dimensional subspace $\mathcal{S}$, let $\gamma_{\rm opt}$, $Q_{\rm opt}$ and $Y_{\rm opt}$ be the minimizers at x_0. It is easy to verify that $\alpha^2 \gamma_{\rm opt}$, $\alpha^2 Q_{\rm opt}$ and $\alpha^2 Y_{\rm opt}$ is the minimizers at αx_0. Therefore, at αx_0, $F = \alpha^2 Y_{\rm opt} \frac{1}{\alpha^2} Q_{\rm opt}^{-1} = F_{\rm opt}$ and $P = \alpha^2 \gamma_{\rm opt} \frac{1}{\alpha^2} Q_{\rm opt}^{-1} = P_{\rm opt}$.

Remark 2 *From Corollary (1) and (2), we can see that when we apply Theorem 1 (a) without (11) and (12) to an arbitrary state x_0, for a nominal system, the state feedback matrix is independent of the state, while for an uncertain system (1), the state feedback matrix is independent of the state along a one dimensional subspace, but may change according to different orientations of the one dimensional subspace.*

Off-line Robust Constrained MPC

For a constrained system, we know that the nearer the state is to the origin, the less restrictions on the choice of the feedback matrix. To provide the state space with a sense of distance, we define the norm of any vector within the asymptotically stable invariant ellipsoid $\mathcal{E} = \{x \in \mathcal{R}^{n_x} | x^T Q^{-1} x \leq 1\}$ as $\|x\|_{Q^{-1}} \triangleq \sqrt{x^T Q^{-1} x}$. The distance between the state and the origin is the norm of the state. On-line MPC in Theorem 1 has the advantage of determining the control law based on the norm of the state (see the derivation of (11) and (12) in (Kothare et al., 1996)). Off-line we can still achieve this. When we apply Theorem 1 to a state far from the origin, the resulting asymptotically stable invariant ellipsoid has a more restrictive feedback matrix. It is not necessary to keep this feedback matrix constant while the state is converging to the origin. We can construct inside the ellipsoid another asymptotically stable invariant ellipsoid based on a state nearer to the origin, which can have a less restrictive feedback matrix. By adding asymptotically stable invariant ellipsoids one inside another, we have more freedom to adopt suitable feedback matrices based on the distance between the state and the origin.

Lemma 2 *(Existence) Consider the minimization defined in Theorem 1 (a). If there exists a minimizer γ, Q, X, Y and Z at x, then, at an arbitrary $\tilde{x}$ satisfying $\|\tilde{x}\|^2_{Q^{-1}} < 1$ there exists a minimizer $\tilde{\gamma}, \tilde{Q}, \tilde{X}, \tilde{Y}$ and $\tilde{Z}$ for the minimization defined in Theorem 1 (a) with an additional constraint $Q > \tilde{Q}$.*

Proof. Consider the minimization defined in Theorem 1 (a) at x. The minimizer defines an ellipsoid $\mathcal{E} = \{x \in \mathcal{R}^{n_x} | x^T Q^{-1} x \leq 1\}$. An arbitrary $\tilde{x}$ satisfying $\|\tilde{x}\|^2_{Q^{-1}} < 1$ means that the state $\tilde{x}$ is inside $\mathcal{E}$. So $\exists \alpha > 1$, $\|\alpha \tilde{x}\|^2_{Q^{-1}} = 1$ and $\frac{1}{\alpha^2}\gamma$, $\frac{1}{\alpha^2}Q$, $\frac{1}{\alpha^2}X$, $\frac{1}{\alpha^2}Y$, $\frac{1}{\alpha^2}Z$ is a feasible solution for the minimization defined in Theorem 1 (a) with the additional constraint $Q > \frac{1}{\alpha^2}Q$ satisfied.

Algorithm 1 *(Off-line robust constrained MPC) Off-line, given an initial feasible state x_1, generate a sequence of minimizers γ_i, Q_i, X_i, Y_i and Z_i $(i = 1, \ldots N)$ as follows.*

1. compute the minimizer γ_i, Q_i, X_i, Y_i and Z_i at x_i by using Theorem 1 with an additional constraint $Q_{i-1} > Q_i$ (ignored at $i = 1$), store Q_i^{-1} and $F_i (= Y_i Q_i^{-1})$ in a look-up table;

2. if $i < N$, choose a state x_{i+1} satisfying $\|x_{i+1}\|^2_{Q_i^{-1}} < 1$. Go to 1.

On-line, given a dynamical system (1) and an initial state $x(0)$ satisfying $\|x(0)\|^2_{Q_1^{-1}} \leq 1$, let the state be $x(k)$

at time k. Perform a bisection search over Q_i^{-1} in the lookup table until a Q_i^{-1} is found satisfying $\|x(k)\|^2_{Q_i^{-1}} \le 1$ and $\|x(k)\|^2_{Q_{i+1}^{-1}} > 1$ ($i = 1, ..., N-1$), or $\|x(k)\|^2_{Q_i^{-1}} \le 1$ ($i = N$). Apply the control law $u(k) = F_i x(k)$.

Theorem 2 *Given a dynamical system (1) and an initial state $x(0)$ satisfying $\|x(0)\|^2_{Q_1^{-1}} \le 1$, the off-line robust constrained MPC algorithm 1 robustly asymptotically stabilizes the closed-loop system.*

Proof. For the minimization at x_i, $i = 2, ..., N$, the additional constraint $Q_{i-1} > Q_i$ is equivalent to ${Q_{i-1}}^{-1} < Q_i^{-1}$. This implies that the constructed asymptotically stable invariant ellipsoid $\mathcal{E}_i = \{x \in \mathcal{R}^{n_x} | x^T Q_i^{-1} x \le 1\}$ is inside $\mathcal{E}_{i-1}$, i.e., $\mathcal{E}_i \subset \mathcal{E}_{i-1}$. So for a fixed x, $\|x\|^2_{Q_i^{-1}}$ is monotonic with respect to the index i, which ensures the on-line bisection search over the lookup table finds a unique Q_i^{-1}.

Given a dynamical system (1) and an initial state $x(0)$ satisfying $\|x(0)\|^2_{Q_1^{-1}} \le 1$, the closed–loop system becomes

$$x(k+1) = \begin{cases} (A(k) + B(k)F_i)\,x(k) & \begin{array}{c} \|x(k)\|^2_{Q_i^{-1}} \le 1 \\ \cap\ \|x(k)\|^2_{Q_{i+1}^{-1}} > 1 \\ (i = 1, ..., N-1) \end{array} \\ (A(k) + B(k)F_N)\,x(k) & \|x(k)\|^2_{Q_N^{-1}} \le 1 \end{cases}$$

When $x(k)$ satisfies $\|x(k)\|^2_{Q_i^{-1}} \le 1$ and $\|x(k)\|^2_{Q_{i+1}^{-1}} > 1$, $i = 1, ..., N-1$, the control law $u(k) = F_i x(k)$ corresponding to the ellipsoid $\mathcal{E}_i$ is guaranteed to keep the state within $\mathcal{E}_i$ and converge it into the ellipsoid $\mathcal{E}_{i+1}$, and so on. Lastly, the smallest ellipsoid $\mathcal{E}_N$ is guaranteed to keep the state within $\mathcal{E}_N$ and converge it to the origin.

Remark 3 *Algorithm 1 is a general approach to construct a Lyapunov function for uncertain and constrained systems. The Lyapunov function is*

$$V(x) = \begin{cases} x^T Q_i^{-1}\, x & \begin{array}{c} \|x(k)\|^2_{Q_i^{-1}} \le 1\ \cap\ \|x(k)\|^2_{Q_{i+1}^{-1}} > 1 \\ (i = 1, ..., N-1) \end{array} \\ x^T Q_N^{-1} x & \|x(k)\|^2_{Q_N^{-1}} \le 1 \end{cases}$$

Note that this Lyapunov function is not continuous on the boundary of ellipsoids. Due to the special characteristics of the robust asymptotically stable invariant ellipsoid, it is enough to have $V(x)$ be monotonically decreasing within the smallest ellipsoid and within each ring region between two adjacent ellipsoids to stabilize the closed-loop system.

From algorithm 1, we can see that the choice of the state x_{i+1} satisfying $\|x_{i+1}\|^2_{Q_i^{-1}} < 1$ is arbitrary. From the point of view of easy implementation, we provide the following suggestions. We can choose an arbitrary one dimensional subspace $\mathcal{S} = \{x \in R^{n_x} | \alpha x_0,\ \alpha \in R\}$, and discretize it and construct a set of discrete points, $\mathcal{S}^d = \{x \in R^{n_x} | \alpha_i x^{\max},\ 1 \ge \alpha_1 > ... > \alpha_N > 0\}$. Because the asymptotically stable invariant ellipsoid constructed for each discrete point actually passes through that point, $\|\alpha_{i+1} x^{\max}\|^2_{Q_i^{-1}} < \|\alpha_i x^{\max}\|^2_{Q_i^{-1}} = 1$ is satisfied. And in order to obtain a look-up table that can cover a very large portion of the state space with a limited number of discrete points, we suggest a discretization of the one dimensional subspace using a logarithmic scale.

Complexity Analysis

The on-line computation mainly comes from the bisection search in a lookup table. Because a sequence of N stored Q_i^{-1} (N generally less than 20) requires $\log_2 N$ searches and the matrix-vector multiplication in one search has quadratic growth $O(n_s^2)$ in the number of flops with n_s the number of state variables, the total number of flops required to calculate an input move is $O(n_s^2 \log_2 N)$. On the other hand, the fastest interior point algorithms are cubic growth in the number of flops as a function of problem size, which is $O((\frac{n_s^2}{2} + n_s n_c)^3)$ for the robust algorithm in Theorem 1 (a) with n_c the number of manipulated variables. So we can conclude that this off-line approach can substantially reduce the on-line computation.

Example

Consider the linearized model derived for a single, non-isothermal CSTR (Marlin, 1995), which is discretized using a sampling time of 0.15 min and given in terms of perturbation variables as follows:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.85 - 0.1\alpha(k) & -0.001\alpha(k) \\ \alpha(k)\beta(k) & 0.05 + 0.01\alpha(k)\beta(k) \end{bmatrix} x(k) \\ &\quad + \begin{bmatrix} 0.15 & 0 \\ 0 & -0.9 \end{bmatrix} u(k) \\ y(k) &= x(k) \end{aligned}$$

where x is a vector of the reactor concentration and temperature, and u is a vector of the feed concentration and the coolant flow, $1 \le \alpha(k) = k_0/10^9 \le 10$ and $1 \le \beta(k) = -\Delta H_{rxn}/10^7 \le 10$. The polytopic uncertain set has four vertices. The robust performance objective function is defined as (3) subject to $|u_1(k+i|k)| \le 0.5$ kmole/m^3 and $|u_2(k+i|k)| \le 1$m^3/min, $i \ge 0$, with $Q_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 0.2$. We choose the x_1 axis as the one dimensional subspace, and discretize it in ten points. Figure 1 shows the ellipsoids defined by Q_i^{-1} for all ten discrete points.

Given an initially perturbed state $x(0) = \left[\begin{smallmatrix} 0.1 \\ 2 \end{smallmatrix}\right]$, Figure 2 shows the closed-loop responses of the system cor-

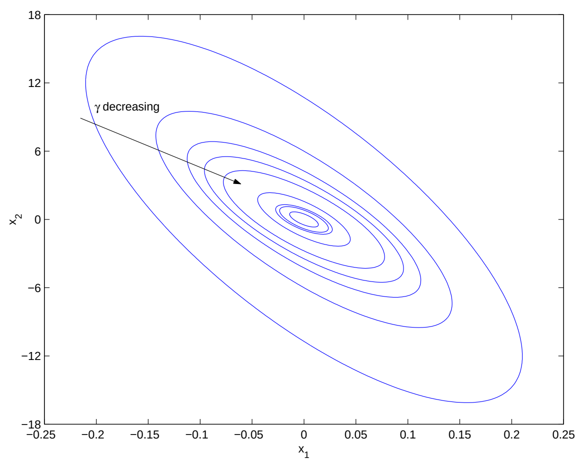

Figure 1: The ellipsoids defined by Q_i^{-1} for all ten discrete points.

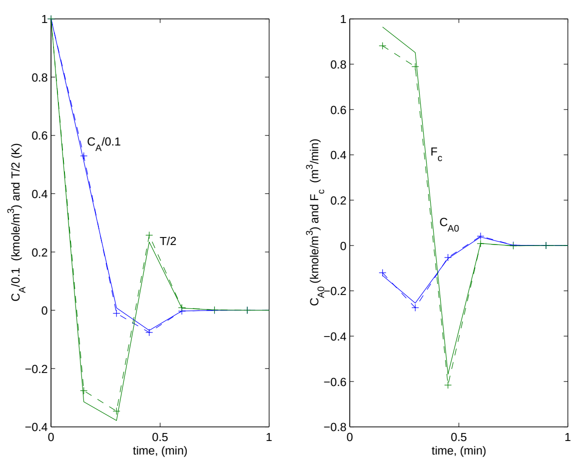

Figure 2: Closed-loop responses: solid lines, on-line MPC algorithm in Theorem 1; dashed lines with (+), off-line MPC in Algorithm 1.

responding to $\alpha(k) \equiv 1.1$ and $\beta(k) \equiv 1.1$. The off-line approach gives nearly the same performance as the on-line robust constrained MPC algorithm (Kothare et al., 1996). The average time for the off-line MPC to get a feedback gain is 6.6×10^{-4} sec, while the average time required for the on-line MPC is 0.6 sec. The calculation of this off-line approach is 900 times faster than that required for on-line MPC.

Conclusions

In this paper, based on the concept of *asymptotically stable invariant ellipsoid* and by using LMIs, we developed an off-line robust constrained MPC algorithm with guaranteed robust stability of the closed-loop system and substantial reduction of on-line MPC computation.

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