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Robust Explicit/Multi–parametric Model Predictive Control for Box-Constrained Linear Dynamic Systems ⋆

Christos Panos * **Konstantinos Kouramas** *
Efstratios N. Pistikopoulos *

* *Centre for Process Systems Engineering, Department of Chemical Engineering, Imperial College London, SW7 2AZ London, UK (e-mail: christos.panos08, k.kouramas, e.pistikopoulos@imperial.ac.uk).*

Abstract: Robust explicit/multi-parametric controllers are designed for constrained, linear discrete–time systems with box-constrained states and inputs, involving uncertainty in the left–hand side (LHS) of the Model Predictive Control (MPC) optimization model. Based on previous results, this work presents a new algorithm that features: (i) a dynamic programming reformulation of the MPC optimization, (ii) a robust reformulation of the constraints that accounts for uncertainty and (iii) a multi–parametric programming solution step where the controls are obtained as an explicit function of the states.

Keywords: Explicit/multi–parametric MPC, robust control, multi–parametric programming

1. INTRODUCTION

In the last decades significant advances have been achieved in the areas of Robust Model Predictive Control (MPC) (see Zafiriou, 1990; Bemporad and Morari, 1999; Rawlings and Mayne, 2009, and references within) and explicit/multi–parametric MPC (mp–MPC) (Pistikopoulos et al., 2007b,a). Robust MPC has been popular mainly for its ability to explicitly handle for the uncertainties in the control process while explicit/multi–parametric MPC for its ability to obtain the control inputs as explicit functions of the system states. Nevertheless, the area of robust explicit/multi–parametric MPC has received rather limited attention compared to the two former methods (Bemporad et al., 2003; Pistikopoulos et al., 2009). This is obvious from the limited number of publications, with the key publications presented in Table 1. Since, even for the case of linear MPC, the resulting optimization model of the robust explicit MPC formulation is nonlinear (due to the uncertainties appearing in the left and right hand side of the optimization constraints), this imposes significant difficulties for the direct application of existing multi–parametric programming techniques to robust MPC.

Dynamic programming (DP) based methods have been proposed for the solution of the explicit/multi–parametric MPC problem, where the mp–MPC optimization problem is recast as a multi–stage problem and is decomposed into a number of smaller optimization problems (Bemporad et al., 2003; Pistikopoulos et al., 2007a). However, the main issue of applying DP to the mp–MPC problem, especially for the case of problems with quadratic objective functions, is that a nonlinear multi–parametric programming problem has to be solved for each stage of the mp–MPC problem, thus requiring the use of global optimization. In (Pistikopoulos et al., 2009) a new method was proposed that overcomes this problems and only solves a multi–parametric Quadratic Programming (mp–QP) problem for each stage of the mp–MPC, thus overcoming the need for global optimization.

In this work, we propose a new method, based on the work of (Pistikopoulos et al., 2009), for the explicit/multi–parametric MPC of "boxed–constrained" linear discrete–time systems, i.e. when the state and input constraints are described by upper and lower bounds of the state and control variables. More specifically, we focus on the Robust Explicit Model Predictive Control (MPC) problem

$$\begin{aligned} V^*(x) &= \min_{\mathbf{U}} J(\mathbf{U}, x) \\ &= \min_{\mathbf{U}} \sum_{t=0}^{N-1} \{x_t^T Q x_t + u_t^T R u_t\} + x_N^T P x_N \qquad (1) \\ \text{s.t. } & x_{t+1} = A x_t + B u_t \qquad (2) \\ & A = A_0 + \Delta A,\ B = B_0 + \Delta B \qquad (3) \\ & \forall\, \Delta A \in \mathcal{A},\ \Delta B \in \mathcal{B} \qquad (4) \\ & x_t \in \mathcal{X} = \{x \in \mathbb{R}^n \mid x_{min} \le x \le x_{max}\} \qquad (5) \\ & u \in \mathcal{U} = \{u \in \mathbb{R}^m \mid u_{min} \le u \le u_{max}\} \qquad (6) \\ & t = 0, 1, \ldots, N-1 \\ & x_N \in \mathcal{X}_f\ ,\ x = x_0 \qquad (7) \end{aligned}$$

where N is the prediction time horizon, x_0 is the initial state, $\mathbf{U} = [u_0^T \ldots u_{N-1}^T]$ is the sequence of current and future control variables, $Q, P \succeq 0$ and $R \succ 0$ are symmetric matrices, (5) and (6) are the state and input constraints and $\mathcal{X}_f = \{x \in \mathbb{R}^n | Tx \le \tau\}$ is the terminal constraint set. The system (2) is uncertain in that the system matrices A, B are given by (3) where A_0, B_0 are of known constant values but the values of matrices $\Delta A, \Delta B$ are not known but are bounded and given by

⋆ Corresponding author E.N. Pistikopoulos.

$$\Delta A \in \mathcal{A} = \{\Delta A \in \mathbb{R}^{n\times n} | -\varepsilon_a|A_0| \leq \Delta A \leq \varepsilon_a|A_0|\}$$
$$\Delta B \in \mathcal{B} = \{\Delta B \in \mathbb{R}^{n\times n} | -\varepsilon_\beta|B_0| \leq \Delta B \leq \varepsilon_\beta|B_0|\}$$

where $\epsilon_a, \epsilon_\beta \in [0, 1)$. The objective is to obtain the control sequence $\mathbf{U}$, and in extension the control variable u_t, as explicit functions of the state variable x_t, such that the state and input constraints are satisfied for all values of the uncertain matrices $\Delta A \in \mathcal{A}$, $\Delta B \in \mathcal{B}$. Such a solution of the explicit/multi–parametric MPC problem will be called a *robust solution*.

Table 1. Robust Explicit/Multi–parametric Model Predictive Control – Main developments

Robust mp-MPC - *Parametric Model Uncertainties*	Bemporad et al. (2003); Manthanwar et al. (2005); Kouramas et al. (2009); Pistikopoulos et al. (2009)
Robust mp-MPC - *Additive disturbances*	Sakizlis (2003); Sakizlis et al. (2004); Kerrigan & Maciejowski (2004); De la Peña et al. (2005)
Robust tracking control and disturbance rejection	Sakizlis et al. (2004)
Robustification of explicit control laws	Olaru & Ayerbe (2006)

2. A METHOD FOR ROBUST EXPLICIT/MULTI–PARAMETRIC MPC

The proposed approach for the solution of the robust explicit MPC problem (1) is realized in three main steps (Faísca et al., 2008; Pistikopoulos et al., 2009)

i) dynamic programming step: the MPC optimization (1) is recast in a multi–stage optimization setting
ii) robust reformulation of the constraints: the optimization constraints at each stage of the resulting multi–stage problem are reformulated to account for the worst–case uncertainty, and
iii) multi–parametric programming: the reformulated stage optimization problems are solved as multi–parametric problems

These steps of the proposed procedure are presented in the following sections.

2.1 Dynamic programming – multi–stage optimization

Following the method presented in Faísca et al. (2008) and Pistikopoulos et al. (2009), problem (1) can be expressed as a multi–stage optimization problem and can be decomposed into a set of *stage-wise* problems of smaller dimensions(Bellman, 2003; Bertsekas, 2005; Pistikopoulos et al., 2009)

$$V_t(x_t) = \min_{u_t\in\mathcal{U}} \sum_{i=t}^{N-1} \{x_i^T Q x_i + u_i^T R u_i\} + x_N^T P x_N \quad (8)$$
$$\text{s.t. } x_{i+1} = Ax_i + Bu_i,\ i = t, \ldots, N$$
$$\forall\, \Delta A \in \mathcal{A},\ \Delta B \in \mathcal{B}$$
$$x_{\min} \leq x_t \leq x_{max}, \quad u_{\min} \leq u_t \leq u_{\max}$$
$$x_{\min} \leq x_{t+1} \leq x_{\max}$$
$$x_{t+1} \in \mathcal{X}^{t+1}$$

where the optimization variable for (8) is only the current input variable u_t and only the state and input constraints at times t and $t+1$ are considered. The smaller problem (8) is solved at each stage t, starting from $t = N-1$ and ending at $t = 0$, to derive the control inputs $u_0, \ldots, u_{N-1}$, instead of solving the multi–stage problem as a single large–scale optimization problem. The set $\mathcal{X}^{t+1}$ is the set of all states x_{t+1} for which a solution $u_{t+1} \in \mathcal{U}$ exists for the problem (8) at stage $t+1$ and is known as the feasibility set (Pistikopoulos et al., 2009). If a control input u_t exists that satisfies the constraint $x_{t+1} \in \mathcal{X}^{t+1}$ then a control input u_{t+1} exists that satisfies the constraints of problem (8) for stage $t+1$. The use and method for obtaining the feasibility set will be further discussed in Section 2.5.

In (Faísca et al., 2008) it was shown that (8) can be solved as a multi–parametric Quadratic Programming (mp–QP) problem for the case $\Delta A = \Delta B = 0$. However ,the presence of the uncertain matrices ΔA, ΔB do not allow for the use of the known multi–parametric programming methods to solve (8) and derive the input variable u_t as an explicit function of the state. We will show in Sections 2.4, 2.5 a procedure to reformulate (8) to an mp–QP problem and derive the input variable as an explicit function of the state.

We first consider the following state and input transformations

$$\bar{x} = x - x_{\min}\ ,\ \ \bar{u} = u - u_{\min} \quad (9)$$

By substituting (9) in the optimization problem (8) we obtain the following transformed optimization problem

$$V_t(\bar{x}_t) = \min_{\bar{u}_t\in\bar{\mathcal{U}}} \sum_{i=t}^{N-1} \{(\bar{x}_i + \bar{x}_{\min})^T Q(\bar{x}_i + \bar{x}_{\min}) \quad (10)$$
$$+ (\bar{u}_i + \bar{u}_{\min})^T R(\bar{u}_i + \bar{u}_{\min})\}$$
$$+ (\bar{x}_N + \bar{x}_{\min})^T P(\bar{x}_N + \bar{x}_{\min})$$
$$\text{s.t. } \bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i + \bar{g},\ i = t, \ldots, N \quad (11)$$
$$0 \leq \bar{x}_t \leq x_{\max} - x_{\min} \quad (12)$$
$$0 \leq \bar{x}_{t+1} \leq x_{\max} - x_{\min} \quad (13)$$
$$0 \leq \bar{u}_t \leq u_{\max} - u_{\min} \quad (14)$$
$$\bar{x}_{t+1} \in \bar{\mathcal{X}}^{t+1} \quad (15)$$

for all $\Delta \in \mathcal{A}$, $\Delta B \in \mathcal{B}$ and where $\bar{g} = (A-I)x_{\min} + Bu_{\min}$ and the set $\bar{\mathcal{X}}^{t+1}$ is obtained by substituting (9) into the feasibility set $\mathcal{X}^{t+1}$. Note that the objective function in (10) is a *convex quadratic function* of $\bar{x}+x_{\min}$ and $\bar{u}+u_{\min}$ and hence the minimization in (10) forces $\bar{x}_k \to -x_{min}$ and $\bar{u}_k \to -u_{min}$ and therefore, from relations (9), $x_k \to 0$ and $u_k \to 0$.

Remark 1. Note that if the solution of (10) is $\bar{u}_t$ then the solution of (8) is $u_t = \bar{u}_t + u_{min}$.

Furthermore, an mp–QP formulation of (10) can be derived by (i) considering u_t as the optimization variable, (ii) considering $\bar{\theta}_t = \begin{bmatrix} x_t^T\ u_{t+1}^T \ldots u_{N-1}^T \end{bmatrix}$ as the vector of parameters, (iii) combining the feasibility constraint $\bar{\mathcal{X}}$ and the state constraints $0 \leq \bar{x}_t \leq x_{\max} - x_{\min}$ into the inequality constraints

$$\mathcal{G}^{t+1} x_{t+1} \leq b^{t+1} \quad (16)$$

(iv) incorporating the linear system model (11) into the constraints, and (v) incorporating the nominal system dynamics ((11) with $\Delta A = \Delta B = 0$) in the objective function (objective penalizes the nominal system performance). Following the above steps we obtain the following multi–parametric programming problem

$$V_t(\bar{x}_t) = \min_{\bar{u}_t \in \bar{\mathcal{U}}} \left\{ \frac{1}{2}\bar{u}_t^T H \bar{u}_t + \bar{\theta}_t^T F \bar{u}_t + L_u^T \bar{u} \right\} + \bar{\theta}_t^T Y \bar{\theta}_t + L_{\bar{\theta}}^T \bar{\theta} + c \quad (17)$$

$$\text{s.t. } \mathcal{G}^{t+1} A_0 \bar{x}_t + \mathcal{G}^{t+1} \Delta A \bar{x}_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \mathcal{G}^{t+1} \Delta B \bar{u}_t + \mathcal{G}^{t+1} \bar{g} \leq b^t \quad (18)$$

$$0 \leq \bar{x}_t \leq x_{\max} - x_{\min}$$

$$0 \leq \bar{u}_t \leq u_{\max} - u_{\min}$$

where $\Delta A \in \mathcal{A}$ and $\Delta B \in \mathcal{B}$. The matrices H, F, L_u, Y, L_θ and c are of appropriate dimensions and are obtained after substituting the nominal system model in the objective function of (10). Note that since the objective function of (10) is a convex quadratic function of $\bar{u}_i$ and $\bar{x}_i$, $i = t, \ldots, N-1$, the objective function of (17) is also a convex quadratic function of u_i, $i = t, \ldots, N-1$ and x_0. Note also that the matrix coefficients in (18) are uncertain, hence problem (17) is a robust mp–QP problem (Kouramas et al., 2009). In order for a control input $\bar{u}_t$ to be a robust solution of (17), the constraint (18) has to be satisfied for all values of the uncertainty.

2.2 Robust Reformulation

In order to ensure that the constraints of (17) are satisfied at all stages t and for all possible values of the uncertain matrices ΔA, ΔB, we apply the following *robust reformulation* (Ben-Tal and Nemirovski, 2000) of (18)

$$\mathcal{G}^{t+1} A_0 \bar{x}_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1} \quad (19)$$

$$\mathcal{G}^{t+1} A_0 \bar{x}_t + \epsilon_a |\mathcal{G}^{t+1}||A_0||\bar{x}_t| + \mathcal{G}^{t+1} B_0 \bar{u}_t + \epsilon_b |\mathcal{G}^{t+1}||B_0||u_t| + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1} + \delta \max\{1, |b^{t+1}|\} \quad (20)$$

where inequality (19) ensures that the problem is feasible for the nominal system while inequality (20) represents the realization of the first constraint in (17) for the worst–case value of the uncertainty. The variable δ is a measure of the tolerated infeasibility i.e. how much the constraint can be relaxed to ensure a feasible solution. Obviously, no infeasibility is allowed when $\delta = 0$.

The inequality (20) is nonlinear with respect to $\bar{x}$ and $\bar{u}$ and hence replacing it in (17) will result in a multi–parametric nonlinear programming problem. However, since from (12), (14) we have that $\bar{x}_t \geq 0$, $\bar{u}_t \geq 0$, we can replace the absolute values $|\bar{x}_t|$ and $|\bar{u}_t|$ in (20) by $\bar{x}_t$, $\bar{u}_t$ and re–write the inequality (20) as a linear inequality of $\bar{x}$, $\bar{u}$

$$\mathcal{G}^{t+1} A_0 \bar{x}_t + \epsilon_a |\mathcal{G}^{t+1}||A_0|\bar{x}_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \epsilon_b |\mathcal{G}^{t+1}||B_0|\bar{u}_t + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1} + \delta \max\{1, |b^{t+1}|\} \quad (21)$$

(21) is then substituted in (17) to obtain the following mp-QP formulation of (17)

$$V_t(\bar{x}_t) = \min_{\bar{u}_t \in \bar{\mathcal{U}}} \left\{ \frac{1}{2}\bar{u}_t^T H \bar{u}_t + \bar{\theta}_t^T F \bar{u}_t + L_u^T \bar{u} \right\} + \bar{\theta}_t^T Y \bar{\theta}_t + L_{\bar{\theta}}^T \bar{\theta} + c \quad (22)$$

$$\text{s.t. } \mathcal{G}^{t+1} A_0 \bar{x}_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1}$$

$$\mathcal{G}^{t+1} A_0 \bar{x}_t + \epsilon_a |\mathcal{G}^{t+1}||A_0|\bar{x}_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \epsilon_b |\mathcal{G}^{t+1}||B_0|\bar{u}_t + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1} + \delta \max\{1, |b^{t+1}|\}$$

$$0 \leq \bar{x}_t \leq x_{\max} - x_{\min}\ , \ 0 \leq \bar{u}_t \leq u_{\max} - u_{\min}$$

where $\bar{u}_t$ is the optimization variable and $\bar{\theta}_t$ is the vector of parameters. Problem (17) is a mp–QP reformulation of the stage optimization problem (8).

If $\bar{u}_t$ is a solution for (22) then $\bar{u}_t$ satisfies (19)–(20) and hence it satisfies the constraint (18) for all values of ΔA, ΔB. This implies that $\bar{u}_t$ is a robust solution for (17) and hence for (10). In addition, since (10) is obtained by applying the linear transformation (9) on (8), then $u_t = \bar{u}_t + u_{\min}$ is also a robust solution of (8). We can now state the following Lemma

Lemma 2. If $\bar{u}_t$ is a feasible solution for (22) then it is also a robust solution for (10) and $u_t = \bar{u}_t + u_{\min}$ is a robust solution for (8).

Remark 3. In (Pistikopoulos et al., 2009) it was shown that the nonlinear inequality (20) can be relaxed to the set of linear inequalities $\mathcal{G}^{t+1} A_0 \bar{x}_t + \epsilon_a |\mathcal{G}^{t+1}||A_0| z_t + \mathcal{G}^{t+1} B_0 \bar{u}_t + \epsilon_b |\mathcal{G}^{t+1}||B_0| \omega_t + \mathcal{G}^{t+1} \bar{g} \leq b^{t+1} + \delta \max\{1, |b^{t+1}|\}$, $-z_t \leq \bar{x}_t \leq z_t$, $-\omega_t \leq \bar{u}_t \leq \omega_t$, $z_t, \omega_t \geq 0$ by introducing two new optimization variables z_t, ω_t and four extra inequalities, thus increasing the number of constraints in (22). However, as we showed above, this is not anymore necessary since $\bar{x}, \bar{u}$ are positive and (20) can be replaced only by (21) without increasing the number of constraints in (22).

2.3 Multi–parametric procedure

Since (22) is an mp–QP problem, the solution to (22) is given by the following explicit form (Pistikopoulos et al., 2007b,a)

$$\bar{u}_t = K_t^i \bar{\theta}_t + c_t^i, \textbf{ if } \bar{\theta}_t \in \mathcal{CR}_t^i,\ \mathcal{CR}_t^i = \{\bar{\theta}_t \mid H_t^i \bar{\theta}_t \leq h_t^i\} \quad (23)$$

$$\bar{u}_t = f_t^*(\bar{\theta}_t) = f_t^*(\bar{x}_t, \bar{u}_{t+1}, \ldots, \bar{u}_{N-1}) \quad (24)$$

where $i = 1, \ldots, s_t$, K_t^i, c_t^i are matrices of appropriate dimensions and $\mathcal{CR}_t^i \subset \mathbb{R}^n$ are the corresponding critical regions. The expression (23) describes the relation between the solution $\bar{u}_t$ at the current stage and the solutions $\bar{u}_{t+1}$, $\ldots$, $\bar{u}_{N-1}$ at the previous stages. However, our objective is to obtain the input $\bar{u}_t$ as an explicit function of the incumbent state $\bar{x}_t$. Hence, in the following we present a procedure for deriving i) $\bar{u}_t$ as an explicit function of the state $\bar{x}_t$ and ii) the feasibility set $\bar{\mathcal{X}}^t$.

Reduction of the mp–QP solution: We will first demonstrate the procedure for deriving an expression $u_t = f_t^*(x_t)$ from (23) for the stages $t = N-1$ and $t = N-2$. For the stages $N-1$ and $N-2$ the control variables are $\bar{u}_{N-1}$ and $\bar{u}_{N-2}$ while the parameters are $\bar{\theta}_{N-1} = \bar{x}_{N-1}$ and $\bar{\theta}_{N-2} = [\bar{x}_{N-1}\ \bar{u}_{N-1}]^T$. The expression (23) for $\bar{u}_{N-1}$ and $\bar{u}_{N-2}$ are then given by

$$\bar{u}_{N-1} = K_{N-1}^i \bar{x}_{N-1} + c_{N-1}^i, \textbf{ if } \bar{x}_{N-1} \in \mathcal{CR}_{N-1}^i \quad (25)$$

$$\bar{u}_{N-2} = K_{N-2}^j \bar{x}_{N-2} + L_{N-2}^j \bar{u}_{N-1} + c_{N-2}^j\ , \textbf{ if } \bar{x}_{N-2}, \bar{u}_{N-1} \in \mathcal{CR}_{N-2}^j \quad (26)$$

where $i = 1, \ldots, s_{N-1}$ and $j = 1, \ldots, q_{N-2}$. Note that $\bar{u}_{N-1}$ is an explicit PWA function of the state x_{N-1} while $\bar{u}_{N-2}$ is a function of $\bar{x}_{N-2}$ and $\bar{u}_{N-1}$. In order to obtain $\bar{u}_{N-2}$ only as an explicit function of $\bar{x}_{N-2}$, we apply the following steps to eliminate u_{N-1} from (26) (Faísca et al., 2008; Pistikopoulos et al., 2009): i) first, the system model $\bar{x}_{N-1} = A_0 \bar{x}_{N-2} + B_0 \bar{u}_{N-2} + \bar{g}$ is incorporated in (25) in order to express $\bar{u}_{N-1}$ as a function of $\bar{x}_{N-2}$ and $\bar{u}_{N-2}$ and ii) (25) and (26) are combined for all i, j to obtain a set of piecewise affine (PWA) expressions with respect to $\bar{x}_{N-2}$, $\bar{u}_{N-2}$ and $\bar{u}_{N-1}$

$$\bar{u}_{N-1} = K^i_{N-1}A_0\bar{x}_{N-2} + K^i_{N-1}B_0\bar{u}_{N-2} + K^i_{N-1}\bar{g} + c^i_{N-1} \quad (27)$$

$$\bar{u}_{N-2} = K^j_{N-2}\bar{x}_{N-2} + L^j_{N-2}\bar{u}_{N-1} + c^j_{N-2} \quad (28)$$

$$\bar{x}_{N-2}, \bar{u}_{N-2} \in \mathcal{CR}^i_{N-1}\ ,\ \ \bar{x}_{N-2}, \bar{u}_{N-1} \in \mathcal{CR}^j_{N-2} \quad (29)$$

Then, by: i) directly substituting (27) in (28) and (29) and solving for $\bar{u}_{N-2}$ or ii) using elimination methods such as orthogonal projection or Fourier-Motzkin elimination to eliminate $\bar{u}_{N-1}$ from (27), (28) and (29), the control input $\bar{u}_{N-2}$ can be obtained as an explicit function of $\bar{x}_{N-2}$

$$\bar{u}_{N-2} = f^*_{N-2}(\bar{x}_{N-2})$$

$$\bar{u}_{N-2} = K^j_{N-2}\bar{x}_{N-2} + c^j_{N-2},\ \textbf{if}\ \ \bar{x}_{N-2} \in \mathcal{CR}^j_{N-2} \quad (30)$$

where $j = 1, \ldots, s_{N-2}$ and $\mathcal{CR}^j_{N-2}$ is the critical region in which (30) is valid. Note that expressions (27)–(29) are obtained for all possible combinations i, j of the critical regions of (25) and (26), and correspond to feasible values of $\bar{x}_{N-2}$, $\bar{u}_{N-2}$ and $\bar{u}_{N-1}$ for problem (22). It is possible that some combination of i, j is not realizable, which implies that no feasible solutions exists.

The same procedure is applied for all stages t. Let $\bar{u}_{t+1} = f^*_{t+1}(\bar{x}_{t+1})$, ..., $\bar{u}_{N-1} = f^*_{N-1}(\bar{x}_{N-1})$ be the solutions of the (22) for stages $t+1$, ..., $N-1$ and $\bar{u}_t = f^*_t(\bar{\theta}_t)$ the solution for the stage t given by (24). Then, i) by replacing x_k with $\bar{x}_k = A_0^{k-t}\bar{x}_t + \sum_{i=0}^{k-1-t} A_0^j B_0 \bar{u}_{k-1-j}$ in the control variables $\bar{u}_k = f^*_k(\bar{x}_k)$ for all $k = t+1, \ldots, N-1$ and ii) by combining the critical regions and control expressions of all control variables we obtain the following set of PWA expressions on x_t, u_t, ... , u_{N-1}

$$\bar{u}_{N-1} = f^*_{N-1}(\bar{x}_t, \bar{u}_t, \ldots, \bar{u}_{N-2}),\ \ldots, \quad (31)$$

$$\bar{u}_{t+2} = f^*_{t+2}(\bar{x}_t, \bar{u}_t, \bar{u}_{t+1}),\ \bar{u}_{t+1} = f^*_{t+1}(\bar{x}_t, \bar{u}_t) \quad (32)$$

$$\bar{u}_t = f^*_t(\bar{x}_t, \bar{u}_{t+1}, \ldots, \bar{u}_{N-1}) \quad (33)$$

Then the variables $\bar{u}_{t+1}$, ..., $\bar{u}_{N-1}$ are eliminated either by i) substituting (31)-(32) in (33) and solving for $\bar{u}_t$ or ii) applying elimination techniques on (31)–(33), to obtain $\bar{u}_t$ as an explicit function of the state $\bar{x}_t$, $\bar{u}_t = f^*_t(\bar{x}_t)$ where

$$\bar{u}_t = K^i_t\bar{x}_t + c^i_t,\ \textbf{if}\ \ \bar{x}_t \in \mathcal{CR}^i_t\ ,\ \ i = 1, \ldots, s_t \quad (34)$$

and $\mathcal{CR}^i_t$ is the critical region where the control (34) is valid.

Calculation of the feasibility constraint set: Once the explicit solution (34) has been obtained, the *feasibility constraint* set $\bar{\mathcal{X}}^t$ for stage t can then be obtained from the following expression

$$\bar{\mathcal{X}}^t = \bigcup_{i=1}^{s_t} \mathcal{CR}^i_t \quad (35)$$

Since (22) is a convex multi–parametric quadratic programming problem, the set of all critical regions in the combined PWA expressions (31)–(33) is a convex polyhedral set. Moreover, the set of all critical regions $\mathcal{CR}^i$ is also a convex polyhedral set, since it is obtained by performing linear operations on (31)–(33), and hence can be described by a set of linear inequalities

$$\bar{\mathcal{X}}^t = \{\bar{x} \in \mathbb{R}^n \mid H^t\bar{x} \leq h^t\} \quad (36)$$

Remark 4. The use of the feasibility constraints $\bar{x}_{t+1} \in \bar{\mathcal{X}}^{t+1}$ is very important since it ensures that the future state x_{t+1} lies in the set of states for stage $t+1$, for which a feasible and robust control $\bar{u}_{t+1} = f^*_{t+1}(\bar{x}_{t+1})$ exists. Therefore, by applying $\bar{u}_t$ to the system (11), it will guarantee that a feasible control $\bar{u}_{t+1} = f^*_{t+1}(\bar{x}_{t+1})$ can be obtained at the next time $t+1$.

The control input u_t can then be obtained by substituting (9) in (34)

$$u_t = \mu_t(x_t) = \bar{u}_t + f^*_t(x_t - x_{\min}) \quad (37)$$

Note from (37) that u_t is also an explicit function of the state x_t. Specifically, $\mu_t(x_t)$ is a PWA function of x_t since $f^*_t(x_t - x_{\min})$ is a PWA function of x_t. Finally, the feasibility set $\mathcal{X}^{t+1}$ can be obtained by substituting $\bar{x} = x - x_{\min}$ in (36).

Remark 5. Note that the proposed algorithm, although it is based on DP methods, does not follow the conventional DP approaches (Bemporad et al., 2003). If conventional DP methods were used, then the solutions $\bar{u}_{t+1} = f^*_{t+1}(\bar{x}_{t+1})$, ..., $\bar{u}_{N-1} = f^*_{N-1}(\bar{x}_{N-1})$ from the previous stages should be first incorporated in the formulation of problem (22) which would then become a non-linear multi–parametric programming problem, thus requiring a global optimization procedure for its solution (Faísca et al., 2008; Pistikopoulos et al., 2009). On the other hand, the proposed procedure takes into account the convexity of (22) with respect to the control variables $\bar{u}_i$, $i = t, \ldots, N-1$ and state x_t, to solve an mp–QP problem (22) at each stage. The explicit solution (34) is then derived by performing a set of linear algebraic manipulations ((31)–(33)).

2.4 Algorithm for robust explicit/multi–parametric MPC for linear system with "box"–constraints

Based on the proposed procedure that was described in Sections 2.3–2.5, we can now propose a DP–based algorithm for robust explicit/multi–parametric programming which is shown in Table 2. In Step 1 of the proposed algorithm, problem (22) is solved for stage $t = N-1$ and $u_{N-1} = f_{N-1}(x_{N-1})$ is obtained as a function of x_{N-1}. The algorithm then proceeds iteratively by applying Steps 2i. – 2iv. for each stage t. In Step 2i the mp–QP problem (22) is solved to obtain $\bar{u}_t = f^*_t(\theta_t)$. In Step 2ii the reduction procedure described in section 2.5 is applied and the control variable $\bar{u}_t$ is obtained as an explicit function of the state $\bar{u}_t = f^*_t(x_t)$. In step 2iii. the control input $u_t = \mu_t(x_t)$ is obtained from expression (37) and finally in step 2iv. the feasibility set $\bar{\mathcal{X}}_t$ is obtained from (35). The algorithm then proceeds to the next stage $t-1$ and terminates when stage $t = 0$. After repeating all the steps of the proposed algorithm we obtain a sequence of control laws $\mathbf{U} = \{u_t, \ldots, u_{N-1}\} = \{\mu_0(x_0), \ldots, \mu_{N-1}(x_{N-1})\}$.

Each of the control inputs $u_t = \mu_i(x_t)$, $t = 0, \ldots, N-1$ is a robust solution of (8), hence the constraints of (8) are satisfied for all values of the uncertain matrices $\Delta A, \Delta B$ (see Lemma 2). This also implies that the control inputs $u_t = \mu_i(x_t)$, $t = 0, \ldots, N-1$ satisfy the state and inputs constraints (5)–(6) for all values of $\Delta A, \Delta B$ which are also the constraints for (1). Therefore, the control sequence $\mathbf{U}$ satisfies the constraints of the explicit/multi–parametric MPC problem (1) for all values of $\Delta A, \Delta B$.

Lemma 6. The control sequence $\mathbf{U} = \{u_t, \ldots, u_{N-1}\}$, where $u_t = \mu_i(x_t)$, $t = 0, \ldots, N-1$, is a robust solution of the explicit/multi–parametric MPC problem (1).

Key features of the proposed robust explicit/multi– parametric MPC method: The key features of the proposed robust explicit MPC method are summarized as follows: *(i) the proposed method obtains a sequence of* robust explicit control laws $u_t = \mu_i(x_t)$, $t = 0, \ldots, N-1$ *which guarantee that the state and input constraints are always satisfied for all values of the uncertainty (Lemma 6), (ii) a convex mp–QP problem is solved for each stage of the proposed DP procedure, without the need to solve a Global Optimization problem (Remark 5), and (iii) the number of optimization variables and constraints for each stage optimization problem* (22) *is not increased compared to the previous methods (Faísca et al., 2008; Pistikopoulos et al., 2009) (Remark 3).*

Note that the proposed algorithm was developed for the case of linear discrete–time system with the state and input constraints described by (5)–(6). Note also that at each stage t of the proposed algorithm, the number of parameters $\bar{\theta}_t = \{\bar{x}_t, \bar{u}_{t+1}, \ldots, \bar{u}_{N-1}\}$ involved in (22) increases as t decreases from $N-1$ to 0. Also the number of expressions (31)–(33) increases as t decreases hence increasing the complexity of the algebraic manipulations required to eliminate $\bar{u}_{t+1}, \ldots, \bar{u}_{N-1}$ from (31)–(33) and derive the control law (34). In addition if a relatively larger number of critical regions is obtained at each stage for (34) then this might increase the complexity of the calculations for obtaining the feasibility set from (35).

On–line Implementation: The on–line implementation of the explicit/ multi–parametric MPC can be realized with two approaches. In the first approach, the whole control sequence $\mathbf{U} = \{u_0, u_1, \ldots, u_{N-1}\}$ is applied to the system, with each control input $u_t = \mu_t(x_t)$ applied at the corresponding time instant t. With the second approach, only the first control $u_0 = \mu_0(x_0)$ of the control sequence $\mathbf{U}$ is applied at each time t, by considering the current state x as the initial state x_0.

Table 2. Algorithm for Robust Explicit Multi–Parametric MPC

Step 1. Set $k = N-1$: solve the mp-QP problem (22) with x_{N-1} being the parameters and obtain $u_{N-1} = f^*_{N-1}(x_{N-1})$ (Eq. (25)) and $\bar{\mathcal{X}}^{N-1}$ (Eq. (36)).

Step 2. Set k to the current stage:

i. solve the k^{th} stage–wise mp–QP problem (22) with $\bar{x}_k, \bar{u}_k, \ldots, \bar{u}_{N-1}$ being the parameters and obtain $\bar{u}_k = f^*_k(\bar{x}_k, \bar{u}_k, \ldots, \bar{u}_{N-1})$,

ii. obtain $\bar{u}_k = f^*_k(\bar{x}_k)$ by eliminating $\bar{u}_{k+1}, \ldots, \bar{u}_{N-1}$ from (31)–(33),

iii. obtain $u_t = \mu_t(x_t)$ from (37)

iv. calculate the feasibility set $\mathcal{X}^k$ from Eq. (35)

Step 3. Set $k = k-1$: if $k = 0$ stop, else go to Step 2.

2.5 Example

We illustrate the proposed algorithm for the following example,

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x_k \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\varepsilon_a = \varepsilon_\beta = 0.2\,, \quad -1 \le u_k \le 1$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1\,, \quad P = \begin{bmatrix} 2.6005 & 2.081 \\ 2.081 & 3.3306 \end{bmatrix}, \quad N = 3$$

We also assume that $\delta = 0$. By applying the proposed algorithm the control variables u_0, u_1, u_2 for each stage are obtain, where each control variables is an explicit PWA function of its corresponding state $u_t = \mu_t(x_t)$, $t = 0, 1, 2$. Each of the control laws u_0, u_1, u_2 consists of 344, 105 and 6 critical regions, which are shown in figures 3, 2, 1 respectively. Moreover, Tables 3, 4 and 5 show the PWA expressions of the control variables u_0, u_1, u_2 and their corresponding critical regions. We then implement the first control input $u_0 = \mu_0(x_0)$ in the system. The simulation of these implementation is shown in Figure 3 where we can notice that the trajectories of the system satisfy the state and input constraints at all times.

Table 3. Explicit solution $u_2 = \mu^*_2(x_2)$ for Stage 2.

Control law	Critical Region
$u_2 = 1$	$-x_2^1 - x_2^2 \le 10$
	$0.3845x_2^1 + x_2^2 \le -0.7078$
	$-x_2^2 \le 10,\ -x_2^1 \le 10,\ x_2^1 \le 10$
$u_2 = -0.5432x_2^1 - 1.4127x_2^2$	$-0.3845x_2^1 - x_2^2 \le 0.7078$
	$-0.3845x_2^1 - x_2^2 \le -0.0070$
	$x_2^1 \le 10,\ -x_2^1 \le 10$
$u_2 = -1$	$-0.3845x_2^1 - x_2^2 \le -1.0774$
	$x_2^1 + 0.8181x_2^2 \le 9.0909$
	$0.8181x_2^1 + x_2^2 \le 9.0909$
	$x_2^1 + 0.9090x_2^2 \le 9.09$
	$0.9090x_2^1 + x_2^2 \le 9.09$
	$x_2^1 + x_2^2 \le 9.0909$
	$-10 \le x_2^1 \le 10$

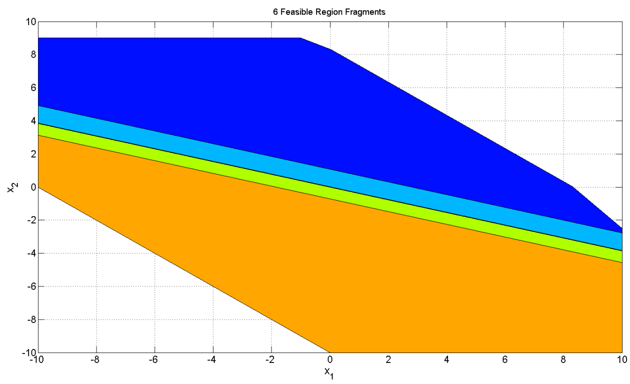

Fig. 1. Critical regions of the explicit robust MPC for stage 2 ($u_2 = \mu^*_2(x_2)$)

3. CONCLUSIONS

A new algorithm for robust explicit/multi–parametric MPC was presented for the case of linear systems with state and input constraints described by upper and lower bounds on the state and input variables. The algorithm features three key steps, based on DP, robust optimization and multi–parametric programming methods and allows for the derivation of robust explicit control solutions to the robust explicit MPC problem.

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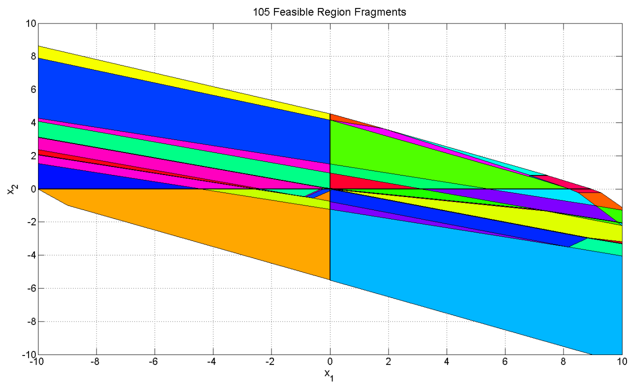

Fig. 2. Critical regions of the explicit robust MPC for stage 1 ($u_1 = \mu_1^*(x_1)$)

Table 4. Explicit solution $u_1 = \mu_1^*(x_1)$ for Stage 1.

Control law	Critical Region
$u_1 = 1$	$0.2777x_1^1 + x_1^2 \leq -1.2335$ $-x_1^1 - x_1^2 \leq 10$ $-0.5x_1^1 - x_1^2 \leq 5.5$ $x_1^1 \leq -0.01,\ x_1^2 \leq -0.01$
$u_1 = -4411x_1^1 - 1.3145x_1^2$	$-08079x_1^1 + x_1^2 \leq -0.1010$ $-0.3356x_1^1 - x_1^2 \leq 0.7607$ $x_1^1 \leq -0.01$
$u_1 = -0.3903x_1^1 - 1.2028x_1^2$ -0.0002	$-0.3245x_1^1 - x_1^2 \leq 0.0085$ $x_1^1 \leq 0.005,\ x_1^2 \leq 0.005$

Table 5. Explicit solution $u_0 = \mu_0^*(x_0)$ for Stage 0.

Control law	Critical Region
$u_0 = 1$	$0.2173x_0^1 + x_0^2 \leq -1.748$ $-0.5x_0^1 - x_0^2 \leq 5.5$ $-0.3333x_0^1 - x_0^2 \leq 4.3333$ $x_0^1 \leq -0.01,\ x_0^2 \leq -101$
$u_0 = -0.3005x_0^1 - 1.2098x_0^2$ -0.8443	$-0.3357x_0^1 + x_0^2 \leq -5.73$ $-0.2484x_0^1 - x_0^2 \leq 0.6896$ $x_0^1 \leq 10$
$u_1 = -0.4712x_0^1 - 1.3486x_0^2$	$-x_0^1 - 0.9041x_0^2 \leq -0.0789$ $x_0^1 + x_0^2 \leq -0.01$ $-0.3493x_0^1 - x_0^2 \leq 0.7414$

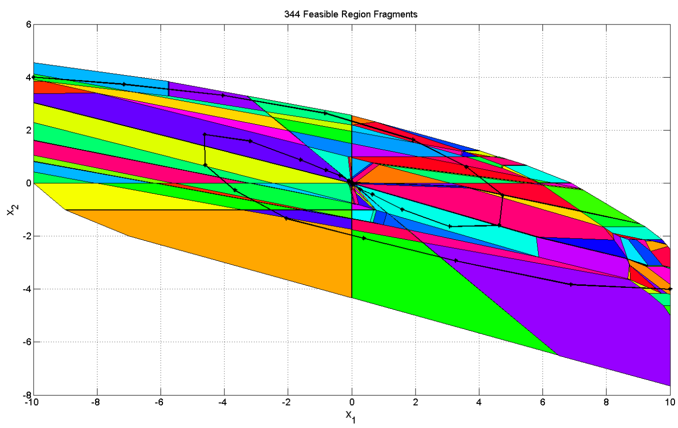

Fig. 3. Critical regions of the explicit robust MPC for stage 0 ($u_0 = \mu_0^*(x_0)$)