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CONSTRAINED EXTENDED KALMAN FILTER FOR NONLINEAR STATE ESTIMATION

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Abstract: Constraints on state variables are commonly encountered in dynamic state estimation in the form of algebraic equality and/or inequality constraints. For weakly nonlinear systems, the extended Kalman filter (EKF) has found numerous uses as a suboptimal state estimator. Unfortunately the structure of the filter does not include constraints on the states. The failure of unconstrained EKF is frequently cited as motivation for moving horizon estimation (MHE) methods for constrained state estimation. However, work on actually imposing the constraints in the existing EKF framework is scarce. This paper presents analytical solutions to the state constrained EKF (CEKF) for a class of linear constraints. It is possible to implement the CEKF efficiently with little additional computation cost and avoid expensive online optimization in MHE. The MHE is a general suboptimal strategy to impose constraints on states, noise processes and inputs, but for a class of state constraints, the proposed CEKF is sufficient. The performance of CEKF is illustrated with a simulation study of a nonlinear batch reactor.

Keywords: State estimation, Extended Kalman filter, Constraints

1. INTRODUCTION

Process measurements typically contain errors due to inherent limitations posed by measurement equipment and stochastic characteristics of the process. State estimation is the task of reducing errors from measurements in an optimal manner. It is an important task since other operations tasks such as, fault detection, supervisory control, scheduling and planning depend on the estimated states. The extended Kalman filter (EKF) is a suboptimal recursive solution to dynamic estimation problems. It is computationally efficient but often fails to achieve desired accuracy and is known to diverge for even simple nonlinear systems. Estimation in nonlinear dynamic systems has received a great deal of attention since the development of the EKF (Jazwinski, 1970; Jang *et al.*, 1986; Liebman *et al.*, 1992).

Dynamic state estimation must also frequently account for constraints on state variables and inputs of the process. For instance, mole fractions are nonnegative and sum of fractions must be equal to one. Until recently, methods of state estimation for both linear and nonlinear dynamic systems either ignored the constraints or incorporated them in an ad hoc manner. The EKF estimates are not constrained in any way, which is often a cause for the filter's instability.

The arrival of optimization based state estimation methods widened the scope of the problem. The lack of constraints in the EKF formulation has been cited by numerous researchers as motiva-

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tion for many variants of the popular moving horizon estimation (MHE) methods (Robertson *et al.*, 1996; Vachhani *et al.*, 2004*b*; Haseltine and Rawlings, 2005). In fact the MHE class of estimators have become the natural choice whenever there are equality and/or inequality constraints and nonlinear system dynamics. However, we find that work on actually imposing the constraints in the existing EKF framework is scarce.

In this paper we show the development of a constrained extended Kalman filter (CEKF) for a class of linear constraints. We first show the necessary results for imposing constraints on a linear estimation problem. The treatment of CEKF is similar to the derivation of the EKF based on the Kalman filter. The importance of this result may be appreciated by comparing it with online optimization based moving horizon formulations. The CEKF analytical solution can be efficiently implemented compared to expensive optimization in MHE. The utility of the MHE as a general purpose suboptimal estimator is not questioned here. For instance, to impose strict inequality constraints or constraints on inputs, the MHE is the preferred formulation. However, for imposing a class of state constraints considered in this paper, the proposed CEKF is shown to be sufficient.

The performance of CEKF is demonstrated with a simulation study of a nonlinear batch reactor, for which earlier researchers have argued against the EKF citing its inability to impose constraints (Vachhani *et al.*, 2004*a*; Haseltine and Rawlings, 2003). We show that the CEKF provides more accurate and robust estimation results with constraints than the EKF. Moreover, performance similar to that of MHE is achieved at a fraction of the computation cost of MHE.

2. CONSTRAINED LINEAR ESTIMATION

Consider the following measurement equation, where $y \in \mathbb{R}^m$ is the measurement vector of a linear combination of process state variables $x \in \mathbb{R}^n$, with an additive independent identically distributed (iid) random vector $\nu \in \mathbb{R}^m$, distributed according to $N(0, R)$, representing Gaussian measurement errors,

$$y = Cx + \nu. \tag{1}$$

While it is desired to estimate the process states from the measurements, the states may be subject to additional linear constraints of the form,

$$Fx = b. \tag{2}$$

where $b \in \mathbb{R}^p$, and $\mathrm{rank}(F) = p < n$. State estimation is posed as,

$$\begin{aligned} \min_{\tilde{x}} \ & \nu^T R^{-1} \nu, \\ \text{s.t. } & y = Cx + \nu, \\ & Fx = b, \end{aligned} \tag{3}$$

which minimizes the errors in a least squares sense. The solution is the minimum variance or maximum likelihood estimator given by an extension of the Gauss-Markov theorem. We state the following theorem without proof for brevity (Ungarala and Bakshi, 2001).

Theorem 1: *Given $y = Cx + \nu$, constraints $Fx = b$, C of full column rank, F of full row rank, the least squares estimate $\bar{x}$ and covariance $\bar{P}$ are*

$$\begin{aligned} \bar{x} &= (I - GF)JC^T R^{-1} y + Gb, \\ \bar{P} &= (I - GF)J, \\ J &= \left(C^T R^{-1} C\right)^{-1}, \qquad G = JF^T \left(FJF^T\right)^{-1}. \end{aligned}$$

In many estimation problems, additional information may be available as *a priori* knowledge of the process states gained from historical or multiple data sets. The prior information is cast as the *a priori* state estimate $\tilde{x}$ and the associated error covariance $\tilde{P}$.

The available information is now written as,

$$\begin{bmatrix} y \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} C \\ I \end{bmatrix} x + \begin{bmatrix} \nu \\ \rho \end{bmatrix}, \tag{4}$$

where $\rho \sim N(0, \tilde{P})$ to give the following form

$$\mathbf{y} = \mathbf{C}x + \mathbf{v}, \tag{5}$$

where $\mathbf{C}$ is of full column rank and $\mathbf{v}$ is zero mean noise with a block diagonal covariance matrix $\mathbf{R} = \mathrm{diag}(R, \tilde{P})$. When Theorem 1 is applied to equation (5) it yields the constrained maximum *a posteriori* (MAP) state estimates as viewed from a Bayesian perspective for linear Gaussian systems.

The following extension to Theorem 1 is stated without proof for brevity, which is based on a result from constrained maximum likelihood estimation (Lewis and Odell, 1971).

Theorem 2: *For C of any rank, the least squares estimate is*

$$\begin{aligned} \bar{x} &= HC^T R^{-1} y + (I - HC^T R^{-1} C)F^+ b, \\ \bar{P} &= H, \\ H &= \left((I - F^+ F)C^T R^{-1} C (I - F^+ F)\right)^+ \end{aligned}$$

where $(\cdot)^+$ is generalized inverse.

3. CONSTRAINED EXTENDED KALMAN FILTER

3.1 Equality constraints

Consider the following nonlinear process model and measurement equations,

$$x_k = f(x_{k-1}) + w_{k-1},$$
$$y_k = h(x_k) + \nu_k$$

where f and h are nonlinear vector valued functions and w_k and ν_k are iid Gaussian noise processes distributed according to $N(0, Q)$ and $N(0, R)$ respectively. The knowledge about the initial condition is summarized as the estimate $\tilde{x}_0$ with covariance $\tilde{P}_0$. First we first consider strict equality constraints on the states,

$$Fx_k = b \tag{6}$$

The linearized version of this estimation problem falls under the purview of Theorem 1. The result of Theorem 1 can be split into two separate stages where the states are estimated from the measurements and subsequently corrected with constraints. The prior information is propagated in time using a time-varying linearized process model. Now we can write the constrained extended Kalman filter algorithm in the standard form as follows (see Figure 1):

Algorithm 1: CEKF for equality constraints

(1) Time update

$$\tilde{x}_k = f(\bar{x}_{k-1}),$$
$$\tilde{P}_k = A\bar{P}_{k-1}A^T + Q, \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_{k-1}}.$$

(2) Measurement update (from Theorem 1)

$$K = \tilde{P}_k C^T (C\tilde{P}_k C^T + R)^{-1}, \qquad C = \left.\frac{\partial h}{\partial x}\right|_{\tilde{x}_k},$$
$$\hat{x}_k = \tilde{x}_k + K(y_k - h(\tilde{x}_k)),$$
$$\hat{P}_k = (I - KC)\tilde{P}_k.$$

(3) Constraints update (from Theorem 1)

$$G = \hat{P}_k F^T (C\hat{P}_k F^T)^{-1},$$
$$\bar{x}_k = \hat{x}_k + G(b - F\hat{x}_k),$$
$$\bar{P}_k = (I - GF)\hat{P}_k.$$

3.2 Inequality constraints

We consider inequality constraints on the states to be of the form,

$$Fx \leq b. \tag{7}$$

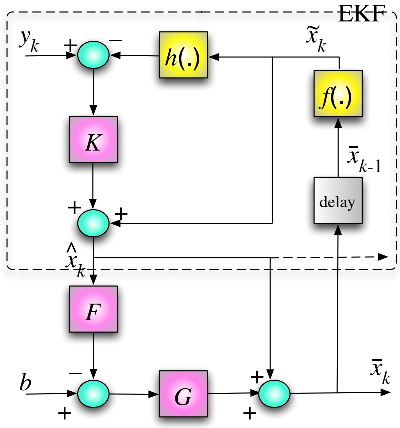

Fig. 1. Constrained extended Kalman filter

Of the p inequality constraints, a subset may be active at the optimal solution, which act as equality constraints,

$$\Lambda F x = \Lambda b, \tag{8}$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ and $\lambda_i = 1$ or $\lambda_i = 0$ if the constraint is active or inactive respectively. If the subset of active constraints are known *a priori*, the analytical solution for the equality constrained problem shown in Theorem 1 and the CEKF in the previous section are clearly applicable. However, if the active constraints are not known an iterative solution strategy is needed.

The inequality constraints are rewritten using non-negative slack variables as follows,

$$Fx + s = b, \qquad s \geq 0. \tag{9}$$

Now the available information is,

$$\begin{bmatrix} y \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} C & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} + \begin{bmatrix} \nu \\ \rho \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} F & I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b \qquad s \geq 0, \tag{11}$$

which are in the form of,

$$\mathbf{y} = \bar{\mathbf{C}}\mathbf{x} + \mathbf{v}, \tag{12}$$
$$\mathbf{F}\mathbf{x} = b, \qquad \mathbf{x}(i) \geq 0;\ i = n+1, \ldots, n+p \tag{13}$$

The matrix $\mathbf{F}$ is of full row rank but $\bar{\mathbf{C}}$ is not of full column rank, hence the result of Theorem 2 is applicable, provided, the non-negativity conditions are ignored to estimate both states and the slack variables.

If the estimate $\hat{\mathbf{x}}$ satisfies the non-negativity conditions in Eq. 13 then $\hat{\mathbf{x}}$ is clearly the inequality constrained estimate. However, in general the non-negative conditions will not be satisfied since they were not imposed in Theorem 2. The following iterative algorithm is used to compute inequality constrained MAP estimate.

Algorithm 2: CEKF for inequality constraints

BEGIN

(1) Calculate $[\bar{x}_k,\ \bar{s}_k]^T$ (Theorem 2) and the set: $V(\bar{s}_k) = \{i : s_k(i) \leq 0\}$.
  IF $V(\bar{s}_k) = \emptyset$
    STOP ($\bar{x}_k$ is constrained estimate).
  ELSE
    GO TO step 2.

(2) FOR each $i \in V(\bar{s}_k)$,
  (a) SET $\bar{s}_k(i) = 0$, reduce states by one
  (b) GO TO step 1.

(3) FOR each $i \in V(\bar{s}_k)$,
  FOR each $j \in V(\bar{s}_k),\ i \neq j$,
  (a) SET $\bar{s}_k(i) = \bar{s}_k(j) = 0$, reduce states by two
  (b) GO TO step 1.

(4) REPEAT step 3 for triples, 4-tuples, ..., p-tuples of $V(\bar{s}_k)$.

END

The time update equations remain the same using the inequality constrained estimate.

4. SIMULATION EXAMPLE

Consider the gas-phase irreversible reaction in a well mixed, constant volume, isothermal batch reactor (Haseltine and Rawlings, 2003),

$$2\mathrm{A} \overset{k=0.16}{\longrightarrow} \mathrm{B}. \tag{14}$$

The following nonlinear ODEs describe the dynamics of the partial pressures,

$$\frac{d}{dt}\begin{bmatrix} p_\mathrm{A} \\ p_\mathrm{B} \end{bmatrix} = \begin{bmatrix} -2kp_\mathrm{A}^2 \\ kp_\mathrm{A}^2 \end{bmatrix}. \tag{15}$$

The measurements are generated by sampling at $\Delta t = 0.1$ min according to the the reactor pressure measurement equation,

$$P_k = [1\ 1]\begin{bmatrix} p_\mathrm{A} \\ p_\mathrm{B} \end{bmatrix}_k + \nu_k, \tag{16}$$

with $\nu_k \sim N(0, 0.1^2)$. The initial conditions are $p_{\mathrm{A}_0} = 3$ and $p_{\mathrm{B}_0} = 1$. The following discrete-time process model is available to the state estimators,

$$[p_\mathrm{i}]_k = [p_\mathrm{i}]_{k-1} + \int\limits_{(k-1)\Delta t}^{k\Delta t} [\dot{p}_\mathrm{i}]\ d\tau + w_k, \tag{17}$$

which is integrated using `ode45` in MatLab with $w_k \sim N(0, 0.001^2 I_2)$. The estimators are initiated with $\tilde{p}_{\mathrm{i}_0} = [3,\ 1]^T$ and $\tilde{P}_0 = I_2$. To be physically meaningful, the pressures cannot be negative, i.e., the states must obey the following inequality constraint

$$[p_\mathrm{i}]_k > 0. \tag{18}$$

Note that the strict inequality constraint can be used only in the MHE framework. Since the prior information is known well, the performance of the EKF is generally acceptable in most simulation runs and the estimates do not violate the constraints. However, often the EKF convergence is slow as shown for two sample paths in Figure 2.

A CEKF can be implemented by considering a new state space model of mole fractions as the state variables and relating them to total the pressure. The aim is to estimate mole fractions and total pressure and then to compute partial pressures. The dynamic model for the mole fractions is,

$$\frac{d}{dt}\begin{bmatrix} x_\mathrm{A} \\ x_\mathrm{B} \end{bmatrix} = \begin{bmatrix} -\alpha k x_\mathrm{A}^2 \\ \alpha k x_\mathrm{A}^2 \end{bmatrix} \tag{19}$$

and the mole fractions are related to pressure as,

$$P_k = \frac{\alpha}{x_{\mathrm{A}_k} + 2x_{\mathrm{B}_k}} + \nu_k \tag{20}$$

where the constant $\alpha = p_{\mathrm{A}_0} + 2p_{\mathrm{B}_0} = 5$. The states must now obey the following constraints,

$$x_{\mathrm{A}_k} + x_{\mathrm{B}_k} = 1, \qquad \begin{bmatrix} x_{\mathrm{A}_k} \\ x_{\mathrm{B}_k} \end{bmatrix}_k \geq 0, \tag{21}$$

which are useful for the implementation of the CEKF. The discrete-time mole fractions model experiences noise from $w_k \sim N(0, 0.001^2 I_2)$. The prior information is adapted from the partial pressures prior into $\tilde{x}_{\mathrm{i}_0} = [0.75,\ 0.25]^T$ and $\tilde{P}_0 = 0.02^2 I_2$. For the same data sets for which EKF performed poorly in Figure 2, the CEKF is shown to yield superior estimates in Figure 3.

Now consider a case where the prior information is poorly known, $\tilde{p}_{\mathrm{i}_0} = [0.1,\ 4.5]^T$ and $\tilde{P}_0 = 6^2 I_2$. Previously it was shown that without imposing the non-negativity constraints on the states, the extended Kalman filter fails to estimate the partial pressures from total pressure measurements (Haseltine and Rawlings, 2003). Figure 4 displays two examples of typical EKF failure with negative pressures and wrong steady states.

An MHE using strict inequality constraints on partial pressures readily corrects the estimates, however the improvement is achieved with high computational cost. Results of CEKF on the same data sets is shown in Figure 5. The performance is summarized in Table 1 with averages for 100 runs. For good initial guess, the MHE does not necessarily improve upon EKF because the constraints are not violated. It is possible to improve MHE estimates by using large horizons, which smooth the estimates. For bad initial guess, the MHE is clearly superior. However, greater accuracy is achieved by CEKF with little computational load over the EKF.

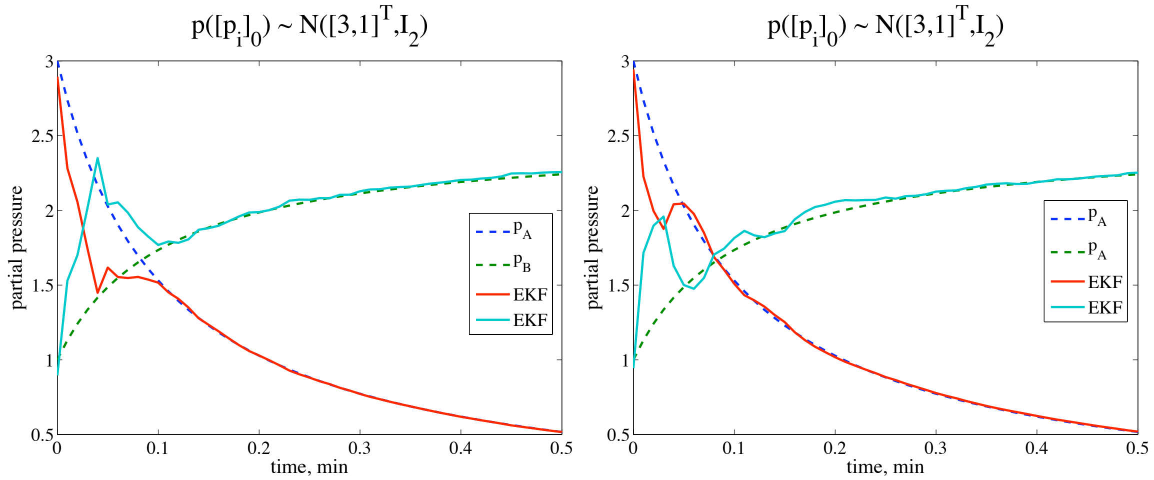

Fig. 2. Estimates of partial pressures using EKF (good prior) for two random sample paths.

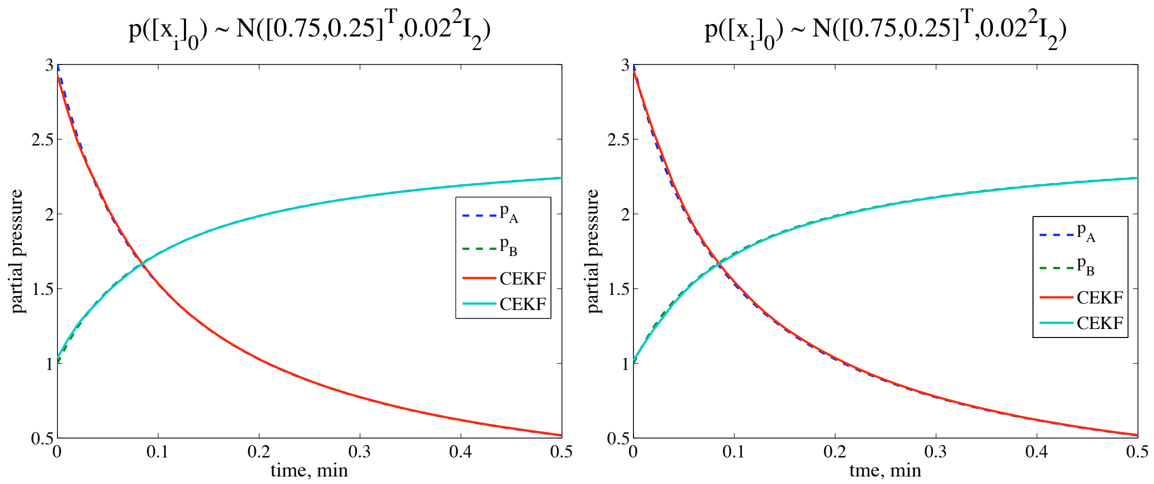

Fig. 3. Partial pressures from mole fractions estimates using CEKF (good prior)

Table 1. Average MSE of partial pressure estimates and CPU time to process 101 measurements.

	MSE$\times 10^5$ (good prior)	MSE$\times 10^3$ (bad prior)	CPU sec (2.5GHz G5)
EKF	367	5277	0.33
MHE (N=3)	405	143	293
CEKF	4	14	0.35

5. CONCLUSIONS

An analytical solution for a constrained extended Kalman filter is presented in this paper. For a class of linear equality and inequality constraints, the moving horizon estimation methods are shown to be unnecessarily burdensome because greater accuracy is obtained by CEKF at a small fraction of the cost of MHE.

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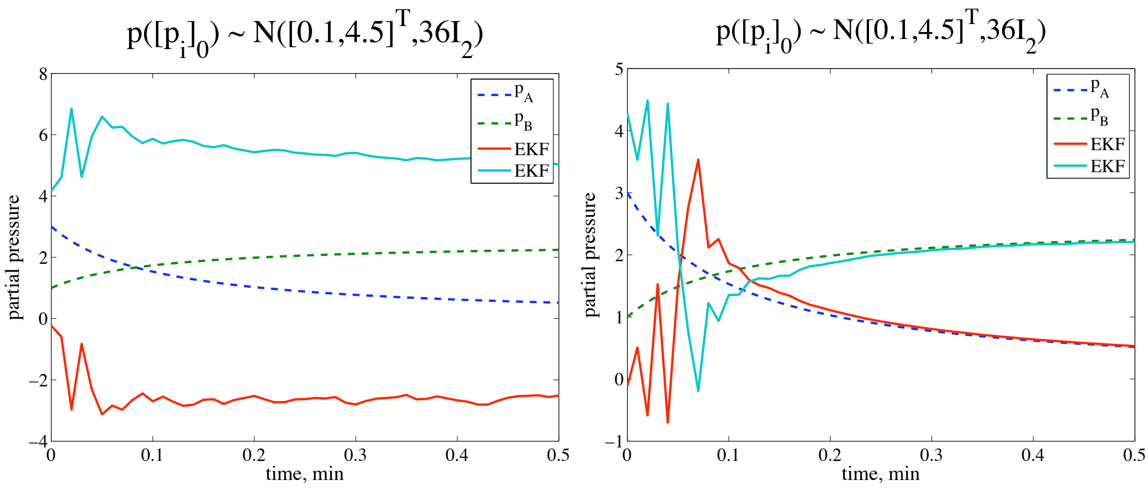

Fig. 4. Estimates of partial pressures using EKF (bad prior) for two random sample paths.

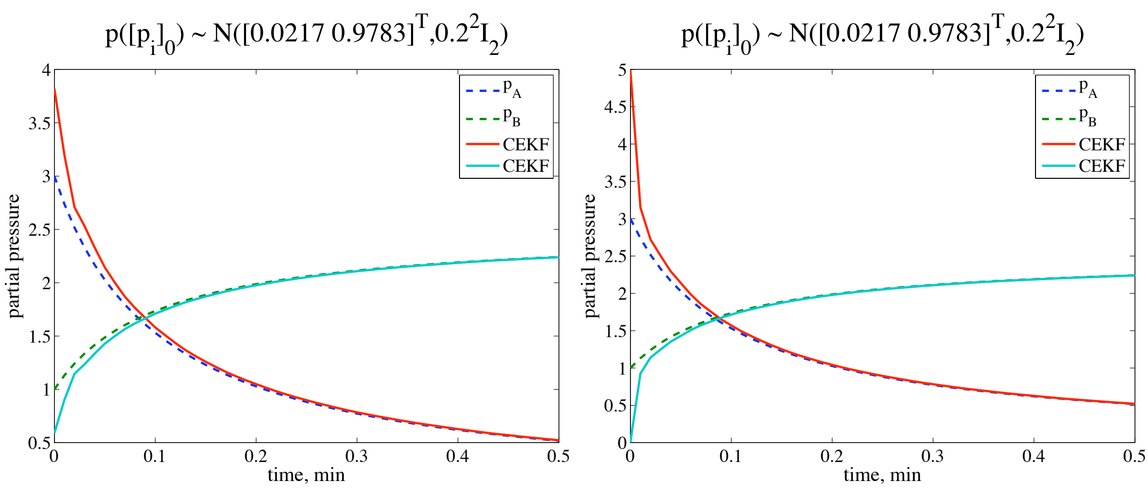

Fig. 5. Partial pressures from mole fractions estimates using CEKF (bad prior)

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