

Cubature H_∞ Information Filter

Kumar Pakki, Bharani Chandra, Da-Wei Gu, and Ian Postlethwaite

Abstract—This paper presents a state estimation algorithm referred to as a cubature H_∞ information filter (CH_∞ IF) for nonlinear systems. The proposed algorithm is developed from a cubature Kalman filter, an H_∞ filter and an extended information filter. The CH_∞ IF is a derivative free filter, where the information state vector and information matrix are propagated rather than the state vector and error covariance matrix. Furthermore, the CH_∞ IF is extended for multi-sensor state estimation. The efficacy of the CH_∞ IF is demonstrated by a simulation example of a permanent magnet synchronous motor in the presence of Gaussian and non-Gaussian noises.

I. INTRODUCTION

The Kalman filter has been the preferred state estimation method over the last few decades. It was initially developed for linear systems [1], and then extended to nonlinear systems as the extended Kalman filter (EKF) [2]. The naive EKF is mainly suitable for mild nonlinear systems with Gaussian noises and requires Jacobians for the prediction and measurement update. The calculation or evaluation of Jacobians can be avoided by using the derivative free filters like unscented Kalman filter (UKF) [21], particle filter [22], cubature Kalman filter (CKF)[3], etc. In many real life applications, these derivative free filters show improved performance over the EKF. Unlike the particle filter, the UKF and CKF only have a limited capability to deal with non-Gaussian noises. One of the promising approaches to deal with non-Gaussian noises is the H_∞ filter which requires neither statistical noise properties nor the exact process model [4-10]. For nonlinear systems, an extended H_∞ filter (EH_∞ F) can be used. However, the EH_∞ F is not a derivative free filter as it require Jacobians and hence is not suitable for systems with severe nonlinearities. To improve the performance of EH_∞ F, a cubature H_∞ filter (CH_∞ F) was proposed in [11]. CH_∞ F is a derivative free filter and can handle nonlinear and non-Gaussian systems.

For multi-sensor state estimation, information filters are preferred over Kalman filters. The information filter is an algebraically equivalent form of Kalman filter. Similar to the EKF, the information filter can also be extended for nonlinear state estimation as the extended information filter (EIF). In EIF, the parameters of interest are the information state vector and the information matrix (inverse of covariance

matrix). Information filters are easy in initialisation compared to conventional Kalman filters and the update stage is computationally economic. EIF has indeed several advantages over EKF; for more details see [12, 24]. But, both EKFs and EIFs are only suitable for ‘mild’ nonlinearities where the first-order approximations of the nonlinear functions are suitable and require analytical Jacobians for state estimation. Recently, a cubature information filter (CIF) is proposed as an alternative to EIF [14, 15]. The CIF is a derivative free filter and is suitable for multi-sensor nonlinear state estimation in the presence of Gaussian noises.

In this paper, we extend the CH_∞ F and CIF to form a cubature H_∞ information filter (CH_∞ IF). The CH_∞ IF is not only useful for multi-state estimation but it can also handle nonlinear and non-Gaussian systems.

The rest of the paper is structured as follows. Section II includes the preliminaries of the EIF, H_∞ F and CKF. Section III describes the proposed cubature H_∞ information filter and its extension for multi-sensor state estimation. Section IV is devoted to numerical simulations and Section V concludes the paper.

II. EXTENDED INFORMATION FILTER, H_∞ FILTER AND CUBATURE KALMAN FILTER

This section presents a brief introduction to EIF, H_∞ Filter and CKF. For detailed formulation and derivation of these filtering algorithms, please see for example [12] for EIF, [16] for H_∞ filter and [3] for CKF.

A. Extended information filter

In EIF, the information state vector and the information matrix are propagated rather than state vector and covariance. Similar to EKF, EIF can also be represented by a recursive process of prediction and measurement update. The EIF equations are summarized below.

The nonlinear process and measurement models can be represented as

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) + \mathbf{w}_{k-1} \quad (1)$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k \quad (2)$$

where k is the time index, \mathbf{x}_k is the state vector, \mathbf{u}_k is the control input, \mathbf{z}_k is the measurement, \mathbf{w}_{k-1} and \mathbf{v}_k are the process and measurement noises, respectively. These noises are assumed to be zero mean Gaussian-distributed random variables with covariances of \mathbf{Q}_{k-1} and \mathbf{R}_k .

The predicted information state and information matrix are

$$\hat{\mathbf{y}}_{k|k-1} = \mathbf{Y}_{k|k-1} \hat{\mathbf{x}}_{k|k-1} \quad (3)$$

$$\mathbf{Y}_{k|k-1} = \mathbf{P}_{k|k-1}^{-1} = \left[\nabla \mathbf{f}_x \mathbf{Y}_{k-1|k-1}^{-1} \nabla \mathbf{f}_x^T + \mathbf{Q}_{k-1} \right]^{-1} \quad (4)$$

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where $\mathbf{P}_{k|k-1}$ is the predicted covariance matrix and

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1}). \quad (5)$$

The updated information state and information matrix are

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \mathbf{i}_k \quad (6)$$

$$\mathbf{Y}_{k|k} = \mathbf{Y}_{k|k-1} + \mathbf{I}_k. \quad (7)$$

The information state contribution, \mathbf{i}_k , and its associated information matrix, \mathbf{I}_k , are

$$\mathbf{i}_k = \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} [\nu_k + \nabla \mathbf{h}_x \hat{\mathbf{x}}_{k|k-1}] \quad (8)$$

$$\mathbf{I}_k = \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} \nabla \mathbf{h}_x \quad (9)$$

where the measurement residual, ν_k , is

$$\nu_k = \mathbf{z}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}_k) \quad (10)$$

and $\nabla \mathbf{f}_x$, and $\nabla \mathbf{h}_x$ are the Jacobians of \mathbf{f} and \mathbf{h} evaluated at the best available state.

For the nonlinear information filter, the state vector and covariance matrix can be recovered from information vector and information matrix using MATLAB's *left division* [18]

$$\hat{\mathbf{x}}_{k|k} = \mathbf{Y}_{k|k} \setminus \hat{\mathbf{y}}_{k|k} \quad (11)$$

$$\mathbf{P}_{k|k} = \mathbf{Y}_{k|k} \setminus \mathbf{I}_n \quad (12)$$

where \mathbf{I}_n is the state vector sized identity matrix.

It is easy to initialise the information filter than the Kalman filter. The update stage of information filter is computationally simpler than the Kalman filter. EIF can be shown to be more efficient than the EKF. But some of the drawbacks inherent in the EKF still affect the EIF. These include the nontrivial nature of the derivations of the Jacobian matrices (and computation) and linearisation instability [12].

B. H_∞ Filter

This section presents a brief introduction to an H_∞ filter. For a detailed formulation and derivation see for example [27], [17], [16] and [8].

Consider the discrete process and measurement models given in (1) and (2). The noise terms \mathbf{w}_k and \mathbf{v}_k may be random with possibly unknown statistics, or they may be deterministic. They may also have a nonzero mean.

A solution to the H_∞ filter based on the game theory is given in [17]. In H_∞ filter, instead of directly estimating the state one can estimate a linear combination of states

$$\mathbf{z}_k = \mathbf{L}_k \mathbf{x}_k \quad (13)$$

By replacing \mathbf{L} with the identity matrix, the state vector can be estimated.

The performance measure for the H_∞ filter is

$$\mathbf{J}_\infty = \frac{\sum_{k=0}^{N-1} \|\mathbf{z}_k - \hat{\mathbf{z}}_k\|_{\mathbf{M}_k}^2}{\|\mathbf{x}_0 - \hat{\mathbf{x}}_0\|_{\mathbf{P}_0}^2 + \sum_{k=0}^N (\|\mathbf{w}_k\|_{\mathbf{Q}_k}^2 + \|\mathbf{v}_k\|_{\mathbf{R}_k}^2)} \quad (14)$$

where \mathbf{P}_0 , \mathbf{Q}_k , \mathbf{R}_k , and \mathbf{M}_k are symmetric positive definite weighing matrices chosen by the user based on the problem at hand. The norm notation used in this section is $\|e\|_{S_k}^2 = e^T S_k e$.

The H_∞ filter is designed to minimize the state estimation error so that \mathbf{J}_∞ is bounded by a prescribed threshold under the worst case \mathbf{w}_k , \mathbf{v}_k , and \mathbf{x}_0

$$\sup \mathbf{J}_\infty < \gamma^2 \quad (15)$$

where ‘sup’ stands for supremum, $\gamma > 0$ is the error attenuation parameter.

Based on (15), the designer should find $\hat{\mathbf{x}}_k$ so that $\mathbf{J}_\infty < \gamma^2$ holds for any disturbances in \mathbf{w}_k , \mathbf{v}_k , and \mathbf{x}_0 . The best the designer can do is to minimize \mathbf{J}_∞ under worst case disturbances, then the H_∞ filter can be interpreted as the following ‘minimax’ problem

$$\min_{\hat{\mathbf{x}}_k} \max_{\mathbf{w}_k, \mathbf{v}_k, \mathbf{x}_0} \mathbf{J}_\infty \quad (16)$$

For detailed analysis and solution to the H_∞ filtering problem see [17] and [16]. In this paper, we have used the $EH_\infty F$ algorithm given in [19]. The predicted state vector and auxiliary matrix are

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1}) \quad (17)$$

$$\mathbf{P}_{k|k-1} = \nabla \mathbf{f}_x \mathbf{P}_{k-1|k-1} \nabla \mathbf{f}_x^T + \mathbf{Q}_k \quad (18)$$

and the inverse of the updated auxiliary matrix can be obtained as

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} \nabla \mathbf{h}_x - \gamma^{-2} \mathbf{I}_n \quad (19)$$

where \mathbf{I}_n denotes the identity matrix of dimension $n \times n$.

The updated state is

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_\infty [\mathbf{z}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{u}_k)] \quad (20)$$

where

$$\mathbf{K}_\infty = \mathbf{P}_{k|k-1} \nabla \mathbf{h}_x^T [\nabla \mathbf{h}_x \mathbf{P}_{k|k-1} \nabla \mathbf{h}_x^T + \mathbf{R}_k]^{-1} \quad (21)$$

The Jacobians of \mathbf{f} and \mathbf{h} , $\nabla \mathbf{f}_x$ and $\nabla \mathbf{h}_x$, are evaluated at $\hat{\mathbf{x}}_{k-1|k-1}$ and $\hat{\mathbf{x}}_{k|k-1}$, respectively.

C. Cubature Kalman filter

The CKF is an appealing option for nonlinear state estimation when compared with EKF or UKF [3]. CKF use the cubature rule to approximate different n-dimensional Gaussian weighted integrals. See [3] for more details on the problem formulation and derivation of CKF.

Consider the discrete process and measurement models with Gaussian noises given in (1) and (2). The cubature points required for the prediction step are

$$\mathcal{X}_{i,k-1|k-1} = \sqrt{\mathbf{P}_{k-1|k-1}} \xi_i + \hat{\mathbf{x}}_{k-1|k-1} \quad (22)$$

where $i = 1, 2, \dots, 2n$, n is the size of the state vector and ξ_i is the i -th element of the following set

$$\sqrt{n} \left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \right]. \quad (23)$$

The propagated cubature points through the process model are

$$\mathcal{X}_{i,k|k-1}^* = \mathbf{f}(\mathcal{X}_{i,k-1|k-1}, \mathbf{u}_{k-1}). \quad (24)$$

The predicted mean and error covariance matrix are

$$\hat{\mathbf{x}}_{k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathcal{X}_{i,k|k-1}^* \quad (25)$$

$$\mathbf{P}_{k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathcal{X}_{i,k|k-1}^* \mathcal{X}_{i,k|k-1}^{*T} - \hat{\mathbf{x}}_{k|k-1} \hat{\mathbf{x}}_{k|k-1}^T + \mathbf{Q}_{k-1} \quad (26)$$

Along with the obtained measurements and predicted states and covariance matrix, the measurement update can be obtained. The predicted measurement and the associated covariances required for the measurement update are

$$\hat{\mathbf{z}}_{k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{z}_{i,k|k-1} \quad (27)$$

$$\mathbf{P}_{zz,k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{z}_{i,k|k-1} \mathbf{z}_{i,k|k-1}^T - \hat{\mathbf{z}}_{k|k-1} \hat{\mathbf{z}}_{k|k-1}^T + \mathbf{R}_k \quad (28)$$

$$\mathbf{P}_{xz,k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathcal{X}_{i,k|k-1} \mathbf{z}_{i,k|k-1}^T - \hat{\mathbf{x}}_{k|k-1} \hat{\mathbf{z}}_{k|k-1}^T \quad (29)$$

where

$$\mathbf{z}_{i,k|k-1} = \mathbf{h}(\mathcal{X}_{i,k|k-1}, \mathbf{u}_k) \quad (30)$$

$$\mathcal{X}_{i,k|k-1} = \sqrt{\mathbf{P}_{k|k-1}} \xi_i + \hat{\mathbf{x}}_{k|k-1}. \quad (31)$$

The updated state and covariance are

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}) \quad (32)$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{P}_{zz,k|k-1} \mathbf{K}_k^T \quad (33)$$

with the Kalman gain is

$$\mathbf{K}_k = \mathbf{P}_{xz,k|k-1} \mathbf{P}_{zz,k|k-1}^{-1} \quad (34)$$

III. THE CUBATURE H_∞ INFORMATION FILTER

This section describes the EH_∞ IF and CH_∞ IF. CH_∞ IF uses CKF and H_∞ filter in an EIF framework.

Consider the discrete process and measurement models given in (1) and (2). The noise terms \mathbf{w}_k and \mathbf{v}_k may be random with possibly unknown statistics, or they may be deterministic. They may also have a nonzero mean. The update step of EH_∞ IF is required to derive the CH_∞ IF and is given below.

Let the updated information state vector and information matrix of EH_∞ IF are $\hat{\mathbf{y}}_{k|k}$ and $\mathbf{Y}_{k|k}$. The updated information matrix of EH_∞ IF is

$$\begin{aligned} \mathbf{P}_{k|k}^{-1} &= \mathbf{P}_{k|k-1}^{-1} + \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} \nabla \mathbf{h}_x - \gamma^{-2} \mathbf{I}_n \\ \mathbf{Y}_{k|k} &= \mathbf{Y}_{k|k-1} + \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} \nabla \mathbf{h}_x - \gamma^{-2} \mathbf{I}_n \\ &= \mathbf{Y}_{k|k-1} + \mathbf{I}_k \end{aligned} \quad (35)$$

where

$$\mathbf{I}_k = \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} \nabla \mathbf{h}_x - \gamma^{-2} \mathbf{I}_n \quad (36)$$

The updated information state vector is the same as in Section II. A.

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \mathbf{i}_k \quad (37)$$

where

$$\mathbf{i}_k = \nabla \mathbf{h}_x^T \mathbf{R}_k^{-1} [\mathbf{v}_k + \nabla \mathbf{h}_x \hat{\mathbf{x}}_{k|k-1}] \quad (38)$$

The evaluation of Jacobians are required for (35) and (37). By using the below linear error propagation property [32, 33, 34] of the error covariance and cross covariance along with CKF equations, these Jacobians can be avoided.

$$\mathbf{P}_{zz,k|k-1} \approx \nabla \mathbf{h}_x \mathbf{P}_{k|k-1} \nabla \mathbf{h}_x^T \quad (39)$$

$$\mathbf{P}_{xz,k|k-1} \approx \mathbf{P}_{k|k-1} \nabla \mathbf{h}_x^T \quad (40)$$

From (40), transpose of the measurement Jacobian can be written as

$$\nabla \mathbf{h}_x^T = \mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{xz,k|k-1} \quad (41)$$

Using (41) in (38) and (36), we get

$$\mathbf{I}_k = \mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{xz,k|k-1} \mathbf{R}_k^{-1} \mathbf{P}_{xz,k|k-1}^T \mathbf{P}_{k|k-1}^{-1} - \gamma^{-2} \mathbf{I}_n \quad (42)$$

$$\mathbf{i}_k = \mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{xz,k|k-1} \mathbf{R}_k^{-1} [\mathbf{v}_k + \mathbf{P}_{xz,k|k-1}^T \mathbf{P}_{k|k-1}^{-1} \hat{\mathbf{x}}_{k|k-1}] \quad (43)$$

The derivative free error covariance and cross error covariance, $\mathbf{P}_{k|k-1}$ and $\mathbf{P}_{k|k-1}$, can be obtained from (28) and (29). The CH_∞ IF is summarised in Algorithm 1. In a similar way, the unscented H_∞ information filter (UH_∞ IF) can also be derived.

A. CH_∞ IF in Multi-Sensor State Estimation

One of the main advantages of the information filter is its ability to deal with multi-sensor data fusion [12, 28]. The information from different sensors can be easily fused by simply adding the information contributions to the information matrix and information vector [12, 28]. In multi-sensor state estimation, the available observations consist of measurements taken from different sensors. The prediction step for multi-sensor state estimation is similar to that of the Kalman or information filter. In the measurement update step, the data from different sensors are fused for an efficient and reliable estimation [29].

Let the different sensors used for state estimation be given by

$$\mathbf{z}_{j,k} = \mathbf{h}_{j,k}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_{j,k}; \quad j = 1, 2, \dots, D \quad (44)$$

where 'D' is the number of sensors.

The CH_∞ IF algorithm can be easily extended for multi-sensor data fusion in which the basic update step of CH_∞ IF is similar to CIF [15]. The updated information vector and information matrix for multi-sensor CH_∞ IF are

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \sum_{j=1}^D \mathbf{i}_{j,k} \quad (45)$$

$$\mathbf{Y}_{k|k} = \mathbf{Y}_{k|k-1} + \sum_{j=1}^D \mathbf{I}_{j,k}. \quad (46)$$

The information contributions of multi-sensor CH_∞ IF are

$$\mathbf{I}_{j,k} = \mathbf{M}_{j,k|k-1}^T \mathbf{R}_{j,k}^{-1} \mathbf{M}_{j,k|k-1} - \gamma^{-2} \mathbf{I}_n \quad (47)$$

$$\mathbf{i}_{j,k} = \mathbf{M}_{j,k|k-1}^T \mathbf{R}_{j,k}^{-1} [\mathbf{v}_{j,k} + \mathbf{M}_{j,k|k-1} \hat{\mathbf{x}}_{k|k-1}] \quad (48)$$

where

$$\mathbf{M}_{j,k|k-1}^T = \mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{j,xz,k|k-1}. \quad (49)$$

Algorithm 1 Cubature H_∞ Information Filter

Prediction

- 1: Evaluate the information matrix and the information state vector

$$\mathbf{Y}_{k|k-1} = \mathbf{P}_{k|k-1}^{-1}$$

$$\hat{\mathbf{y}}_{k|k-1} = \mathbf{Y}_{k|k-1} \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{X}_{i,k-1|k-1}^*$$

where,

$$\mathbf{P}_{k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{X}_{i,k-1|k-1}^* \mathbf{X}_{i,k-1|k-1}^{*T} - \hat{\mathbf{x}}_{k|k-1} \hat{\mathbf{x}}_{k|k-1}^T + \mathbf{Q}_{k-1}$$

$$\mathbf{X}_{i,k-1|k-1} = \sqrt{\mathbf{P}_{k-1|k-1}} \boldsymbol{\xi}_i + \hat{\mathbf{x}}_{k-1|k-1}$$

$$\mathbf{X}_{i,k-1}^* = \mathbf{f}(\mathbf{X}_{i,k-1|k-1}, \mathbf{u}_{k-1}).$$

Measurement Update

- 1: Evaluate the information state contribution and its associated information matrix

$$\mathbf{I}_k = \mathbf{Y}_{k|k-1} \mathbf{P}_{xz,k|k-1} \mathbf{R}_k^{-1} \mathbf{P}_{xz,k|k-1}^T \mathbf{Y}_{k|k-1}^T - \gamma^{-2} \mathbf{I}_n$$

$$\mathbf{i}_k = \mathbf{Y}_{k|k-1} \mathbf{P}_{xz,k|k-1} \mathbf{R}_k^{-1} [\mathbf{v}_k + \mathbf{P}_{xz,k|k-1}^T \mathbf{Y}_{k|k-1}^T \hat{\mathbf{x}}_{k|k-1}]$$

where

$$\mathbf{P}_{xz,k|k-1} = \frac{1}{2n} \sum_{i=1}^{2n} \mathbf{X}_{i,k-1|k-1} \mathbf{Z}_{i,k-1}^T - \hat{\mathbf{x}}_{k|k-1} \hat{\mathbf{z}}_{k|k-1}^T.$$

- 2: The estimated information vector and information matrix of CH_∞ IF are given as:

$$\mathbf{Y}_{k|k} = \mathbf{Y}_{k|k-1} + \mathbf{I}_k$$

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \mathbf{i}_k.$$

Measurement Update for multi-sensor State estimation

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \sum_{j=1}^D \mathbf{i}_{j,k}$$

$$\mathbf{Y}_{k|k} = \mathbf{Y}_{k|k-1} + \sum_{j=1}^D \mathbf{I}_{j,k}$$

where

$$\mathbf{I}_{j,k} = \mathbf{M}_{j,k|k-1}^T \mathbf{R}_{j,k}^{-1} \mathbf{M}_{j,k|k-1} - \gamma^{-2} \mathbf{I}_n$$

$$\mathbf{i}_{j,k} = \mathbf{M}_{j,k|k-1}^T \mathbf{R}_{j,k}^{-1} [\mathbf{v}_{j,k} + \mathbf{M}_{j,k|k-1} \hat{\mathbf{x}}_{k|k-1}]$$

$$\mathbf{M}_{j,k|k-1}^T = \mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{j,xz,k|k-1}.$$

Recovery of Estimated State

$$\hat{\mathbf{x}}_{k|k} = \mathbf{Y}_{k|k}^{-1} \hat{\mathbf{y}}_{k|k}$$

IV. STATE ESTIMATION FOR A PERMANENT MAGNET SYNCHRONOUS MOTOR

In this section, we will consider the state estimation of a two phase permanent magnet synchronous motor (PMSM) in the presence of Gaussian and non-Gaussian noises. The

nonlinear model of PMSM is [16, 15]

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + T_s \left(-\frac{R}{L} x_{1,k} + \frac{\omega \lambda}{L} \sin x_{4,k} + \frac{1}{L} u_{a,k} \right) \\ x_{2,k} + T_s \left(-\frac{R}{L} x_{2,k} - \frac{\omega \lambda}{L} \cos x_{4,k} + \frac{1}{L} u_{b,k} \right) \\ x_{3,k} + T_s \left(-\frac{3L}{2J} x_{1,k} \sin x_{4,k} + \frac{3L}{2J} x_{2,k} \cos x_{4,k} - \frac{F x_{3,k}}{J} \right) \\ x_{4,k} + T_s x_{3,k} \end{bmatrix}$$

the outputs and inputs are

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}, \quad \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} = \begin{bmatrix} \sin(0.002\pi k) \\ \cos(0.002\pi k) \end{bmatrix}.$$

The first two states, x_1 and x_2 , are currents, x_3 is speed and x_4 is rotor angular position. The objective is to estimate the rotor angular position and speed of PMSM using the x_1 and x_2 . The remaining parameters are: $R = 1.9\Omega$, $\lambda = 0.1$, $L = 0.003\text{H}$, $J = 0.00018$, $F = 0.001$ and $T_s = 0.001$ s.

A. State estimation in presence of Gaussian noises

The Gaussian noises for simulations are generated using the Matlab command 'randn'. For the process noise, the standard deviations of the four states are 3.33, 3.33, 0.5 and 0.001, respectively. For the measurement noise, the standard deviations of both the outputs are 0.005.

The initial conditions for all the plant states are 0, the initial information vector is selected from $\mathcal{N}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T, \mathbf{I}_4\right)$. The tuning parameter γ for CH_∞ IF is considered as 1. 500 Monte-Carlo runs were performed to analyse the performance of the estimates. The accumulated root mean square error (RMSE) for EH_∞ IF, UH_∞ IF and CH_∞ IF are shown in Figure 1. The UH_∞ IF tuning parameters are $\alpha = 0.001$, $\beta = 2$ and $\kappa = 3 - n$ [30]. The CH_∞ IF performance is superior than EH_∞ IF and UH_∞ IF. The convergence rate of CH_∞ IF is faster than EH_∞ IF and UH_∞ IF. The average accumulated RMSEs are 0.8491, 0.6312 and 0.2747 for EH_∞ IF, UH_∞ IF and CH_∞ IF, respectively.

B. State estimation in presence of non-Gaussian noises

In some of the control applications, the process and measurement noise can be approximated by a Rayleigh probability distribution function [31]. Rayleigh noise can be generated using the Matlab command 'raylrnd'. To show the efficacy of the proposed method in the presence of non-Gaussian noise, the simulations in Section IV-A are repeated with Rayleigh noise. In the presence of Rayleigh noise, the accumulated RMSEs using EH_∞ IF, UH_∞ IF and CH_∞ IF are shown in Figure 2. In this case also, the performance of CH_∞ IF is superior than EH_∞ IF and UH_∞ IF. The average accumulated RMSEs are 6.9391, 2.3498 and 2.3284 for EH_∞ IF, UH_∞ IF and CH_∞ IF, respectively.

The proposed cubature H_∞ information filter can be further tested for non-Gaussian noises using high frequency sinusoid disturbances.

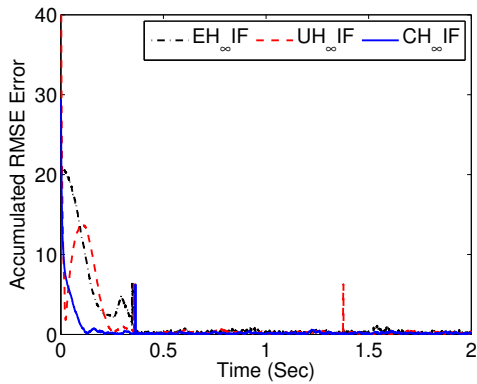


Fig. 1. Accumulated RMSEs of PMSM in the presence of Gaussian noises. Solid, dashed and dashed-dotted lines represents $CH_{\infty}IF$, $UH_{\infty}IF$ and $EH_{\infty}IF$, respectively.

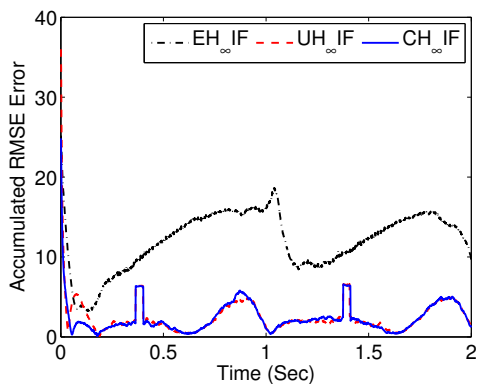


Fig. 2. Accumulated RMSEs of PMSM in the presence of Rayleigh noises. Solid, dashed and dashed-dotted lines represents $CH_{\infty}IF$, $UH_{\infty}IF$ and $EH_{\infty}IF$, respectively.

V. CONCLUSIONS

In this paper, we presented a cubature H_{∞} information filter. The proposed filter is derived from cubature Kalman filter, extended H_{∞} filter and from an extended information filter. Some of the desirable features of the proposed filter includes derivative free nonlinear state estimation, computationally easier measurement update, capability to handle non-Gaussian noises, easy extension for multi-sensor state estimation.

The efficacy of the proposed algorithm is verified on a simulation example of PMSM. The superior performance of cubature H_{∞} information filter over extended H_{∞} and unscented H_{∞} information filters was demonstrated in the presence of Gaussian and non-Gaussian noises.

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