

# Nonlinear Control with Approximately Linear Tracking Error

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**Abstract**—Exact feedback linearization is a well-established method in nonlinear control, where the system is transformed into a linear system by a nonlinear change of coordinates in connection with a nonlinear feedback law. For a controllable single-input system, exact linearizability is equivalent to flatness. Unfortunately, the existence conditions are quite restrictive. Even if the existence conditions are met, the explicit computation of the flat output can be very difficult. In this contribution we suggest tracking controllers based on the tangent system, where the flat output is not required explicitly. The controller can even be calculated for controllable non-flat systems.

## I. INTRODUCTION

The general idea of exact feedback linearization is the transformation of a given nonlinear system into a controllable linear system [1]. The controller design is straightforward if a flat output is known. On the other hand, to find a flat output can be very difficult. In particular, the existence conditions are not well suited for a construction of a flat output. We suggest a tracking controller with approximately linear error dynamics, where the actual calculation of the control law can be carried out without the explicit knowledge of the flat output. The approximation is formally carried out in the coordinates of the controller canonical form, where the explicit computation of this form is avoided. The linearization along the reference trajectory leads to a time varying system [2] which contrasts the set-point oriented approaches as in [3], [4]. This contribution extends first results reported in [5].

This paper is structured as follows: In Section II we recall the existence conditions for exact and restricted feedback linearizability. In Section III, tracking controllers with approximately linear error dynamics are derived. These controllers are applied on two example systems in Section IV.

## II. EXACT LINEARIZATION

### A. Existence Conditions

We consider the problem of controller design for a nonlinear input-affine single-input system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \quad (1)$$

$$y = h(\mathbf{x}) \quad (2)$$

with smooth vector fields  $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . The output map  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth scalar field. System (1),(2)

is said to have *relative degree*  $r$  at a point  $\mathbf{x}_0 \in \mathbb{R}^n$  if  $L_{\mathbf{g}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}h(\mathbf{x}) = \dots = L_{\mathbf{g}}L_{\mathbf{f}}^{r-2}h(\mathbf{x}) = 0$  in an open neighborhood of  $\mathbf{x}_0$  and  $L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}h(\mathbf{x}_0) \neq 0$ , see [1]. System (1) is called *exactly (feedback) linearizable* if there exists a scalar field  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  (acting as a virtual output) such that the resulting system has relative degree  $n$ , i.e.,

$$L_{\mathbf{g}}\lambda(\mathbf{x}) = 0, \dots, L_{\mathbf{g}}L_{\mathbf{f}}^{n-2}\lambda(\mathbf{x}) = 0, L_{\mathbf{g}}L_{\mathbf{f}}^{n-1}\lambda(\mathbf{x}) \neq 0 \quad (3)$$

holds for all  $\mathbf{x}$  in a neighborhood of  $\mathbf{x}_0$ . Such an output is a *flat output* [6]. It has been established that a single-input system (1) is exactly transformable into a controllable linear system if and only if the system is (differentially) flat [7].

Using standard operations from differential geometry it can be shown that (3) is equivalent to the condition

$$L_{\mathbf{g}}\lambda(\mathbf{x}) = 0, \dots, L_{\text{ad}_{-\mathbf{f}}^{n-2}\mathbf{g}}\lambda(\mathbf{x}) = 0, L_{\text{ad}_{-\mathbf{f}}^{n-1}\mathbf{g}}\lambda(\mathbf{x}) \neq 0 \quad (4)$$

for all  $\mathbf{x}$  in a neighborhood of  $\mathbf{x}_0$ . The first  $n - 1$  equations occurring in (4) can be written as first order partial differential equation (PDE)

$$d\lambda(\mathbf{x}) \cdot \left( \mathbf{g}(\mathbf{x}), \text{ad}_{-\mathbf{f}}\mathbf{g}(\mathbf{x}), \dots, \text{ad}_{-\mathbf{f}}^{n-2}\mathbf{g}(\mathbf{x}) \right) = \mathbf{0}^T. \quad (5)$$

Therefore, the existence of a flat output  $\lambda$  is directly linked to the existence of a nontrivial solution of (5). The nontrivial solvability of (5) can be formulated using Frobenius' Theorem [1, Theorem 4.2.3]:

*Theorem 1:* System (1) is exactly transformable into a controllable linear system in a neighborhood of  $\mathbf{x}^0 \in \mathbb{R}^n$  if and only if

- 1) the controllability matrix

$$\mathbf{R}(\mathbf{x}^0) := \left( \mathbf{g}(\mathbf{x}^0), \text{ad}_{-\mathbf{f}}\mathbf{g}(\mathbf{x}^0), \dots, \text{ad}_{-\mathbf{f}}^{n-1}\mathbf{g}(\mathbf{x}^0) \right) \quad (6)$$

has rank  $n$ , and

- 2) the distribution  $\text{span}\{\mathbf{g}, \text{ad}_{-\mathbf{f}}\mathbf{g}, \dots, \text{ad}_{-\mathbf{f}}^{n-2}\mathbf{g}\}$  is involutive in a neighborhood of  $\mathbf{x}_0$ .

If system (1) is exactly linearizable, i.e., there exists a flat output  $\lambda$ , then the diffeomorphism

$$\mathbf{z} = \mathbf{T}(\mathbf{x}) = \begin{pmatrix} \lambda(\mathbf{x}) \\ L_{\mathbf{f}}\lambda(\mathbf{x}) \\ \vdots \\ L_{\mathbf{f}}^{n-1}\lambda(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} = \mathbf{S}(\mathbf{z}) := \mathbf{T}^{-1}(\mathbf{z}) \quad (7)$$

transforms the system into *controller canonical* form [8], [9]

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= \alpha(\mathbf{z}) + \beta(\mathbf{z})u \end{aligned} \quad (8)$$

with  $\alpha(\mathbf{z}) = L_{\mathbf{f}}^n \lambda(\mathbf{S}(\mathbf{z}))$  and  $\beta(\mathbf{z}) = L_{\mathbf{g}} L_{\mathbf{f}}^{n-1} \lambda(\mathbf{S}(\mathbf{z})) \neq 0$ . The controller canonical form (8) can be written in matrix notation

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}(\alpha(\mathbf{z}) + \beta(\mathbf{z})u), \quad (9)$$

where the pair  $(\mathbf{A}, \mathbf{b})$  is in Brunovsky form:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

System (1) can be transformed into a controllable linear system by the change of coordinates (7) in combination with the feedback

$$u = \frac{1}{\beta(\mathbf{z})} (-\alpha(\mathbf{z}) + v) \quad (10)$$

with the new input  $v$ . The feedback (10) is a state-dependent input-affine change of coordinates in the input space. For the closed-loop system, the feedback (10) affects both vector fields  $\mathbf{f}$  and  $\mathbf{g}$  of (1). If the linearizing feedback affects only the drift vector field  $\mathbf{f}$ , system (1) is called *restricted feedback linearizable* [10], [11]. The input vector field of the closed-loop system is unchanged if and only if  $\beta(\mathbf{z}) \equiv 1$ , or equivalently  $L_{\mathbf{g}} L_{\mathbf{f}}^{n-1} \lambda(\mathbf{x}) \equiv 1$ . Condition (4) of exact feedback linearizability is modified for a restricted feedback linearizable system (1) into

$$L_{\mathbf{g}} \lambda(\mathbf{x}) = 0, \dots, L_{\text{ad}_{-\mathbf{f}}^{n-2}} \mathbf{g} \lambda(\mathbf{x}) = 0, L_{\text{ad}_{-\mathbf{f}}^{n-1}} \mathbf{g} \lambda(\mathbf{x}) = 1 \quad (11)$$

for all  $\mathbf{x}$  in a neighborhood of  $\mathbf{x}_0$ . The  $n$  scalar equations of (11) can be brought together into the system of PDEs

$$d\lambda(\mathbf{x}) \cdot \mathbf{R}(\mathbf{x}) = \mathbf{e}_n^T, \quad (12)$$

where  $\mathbf{e}_n$  denotes the  $n$ th unit vector and  $\mathbf{R}$  is the controllability matrix defined in (6).

Replacing the gradient  $d\lambda$  by a general covector field  $\omega$ , Eq. (12) becomes a system of linear equations, which has the unique solution

$$\omega = \mathbf{e}_n^T \mathbf{R}^{-1}(\mathbf{x}) \quad (13)$$

if and only if the controllability matrix  $\mathbf{R}$  is invertible. Based on (13), the solvability of the PDE (12) w.r.t.  $\lambda$  can be expressed as follows [10, Theorem 2]:

*Theorem 2:* System (1) is restricted feedback linearizable into a controllable system in a neighborhood of  $\mathbf{x}^0 \in \mathbb{R}^n$  if and only if

- 1) the controllability matrix has rank  $\mathbf{R}(\mathbf{x}^0) = n$ , and
- 2) the covector field  $\omega$  resulting from (13) is exact in a neighborhood of  $\mathbf{x}_0$ , i.e., there exists a scalar field  $\lambda$  with  $d\lambda \equiv \omega$ .

Using Poincaré's Lemma, the covector field  $\omega$  is locally exact if and only if the covector field is closed, i.e., its exterior derivative is zero:

$$d\omega(\mathbf{x}) \equiv 0. \quad (14)$$

In terms of the components  $\omega_1, \dots, \omega_n$  of  $\omega$ , condition (14) is equivalent to the symmetry of the second order derivatives

$$\frac{\partial \omega_i}{\partial x_j} \equiv \frac{\partial \omega_j}{\partial x_i} \quad (15)$$

for all  $i, j = 1, \dots, n$ . Additionally allowing an integrating factor (i.e., we relax the restriction  $L_{\mathbf{g}} L_{\mathbf{f}}^{n-1} \lambda(\mathbf{x}) \equiv 1$  to  $L_{\mathbf{g}} L_{\mathbf{f}}^{n-1} \lambda(\mathbf{x}) \neq 0$ ), Frobenius' Theorem immediately leads to the following result (e.g. [10], [12]):

*Theorem 3:* System (1) is exactly linearizable into a controllable system (i.e., it is flat) in a neighborhood of  $\mathbf{x}^0 \in \mathbb{R}^n$  if and only if

- 1) the controllability matrix has rank  $\mathbf{R}(\mathbf{x}^0) = n$ , and
- 2) the covector field  $\omega$  resulting from (13) satisfies

$$d\omega(\mathbf{x}) \wedge \omega(\mathbf{x}) = 0 \quad (16)$$

in a neighborhood of  $\mathbf{x}_0$ .

### B. Exactly Linear Tracking Control

We want to design a tracking controller for system (1). The system's input

$$u = u^* + \hat{u} \quad (17)$$

is decomposed into an open-loop control part  $u^*$ , which is the reference input, and a closed-loop part  $\hat{u}$ . The reference trajectory will be stabilized by a state feedback  $r$  around the reference input:

$$\hat{u} = u - u^* = -r(\mathbf{x}). \quad (18)$$

This results in a closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})(u^* - r(\mathbf{x})),$$

where the reference input  $u^*$  and the reference trajectory  $\mathbf{x}^*$  are compatible with the dynamics of system (1), i.e., we have the reference system

$$\dot{\mathbf{x}}^* = \mathbf{f}(\mathbf{x}^*) + \mathbf{g}(\mathbf{x}^*)u^*. \quad (19)$$

The controller design is carried out in the coordinates of the controller canonical form (9) for a restricted feedback linearizable system, i.e.,  $\beta(\mathbf{z}) \equiv 1$ . With (17) and (18), the transformed closed loop system reads as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} + \mathbf{b}(\alpha(\mathbf{z}) + u) \\ &= \mathbf{A}\mathbf{z} + \mathbf{b}(\alpha(\mathbf{z}) + u^* - r(\mathbf{S}(\mathbf{z}))). \end{aligned} \quad (20)$$

In transformed coordinates, the reference system (19) becomes

$$\dot{\mathbf{z}}^* = \mathbf{A}\mathbf{z}^* + \mathbf{b}(\alpha(\mathbf{z}^*) + u^*). \quad (21)$$

The tracking error  $\hat{\mathbf{z}} = \mathbf{z} - \mathbf{z}^*$  between (20) and (21) is governed by

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}\hat{\mathbf{z}} + \mathbf{b}(\alpha(\mathbf{z}) - \alpha(\mathbf{z}^*) - r(\mathbf{S}(\mathbf{z}))). \quad (22)$$

Choosing the feedback

$$r(\mathbf{S}(z)) = \alpha(z) - \alpha(z^*) + \mathbf{k}^T \hat{z} \quad (23)$$

with the constant gain vector

$$\mathbf{k} = (p_0, \dots, p_{n-1})^T \quad (24)$$

results in exactly linear tracking error dynamics

$$\dot{\hat{z}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T) \hat{z} \quad (25)$$

with the characteristic polynomial

$$\det(sI - (\mathbf{A} - \mathbf{b}\mathbf{k}^T)) = p_0 + p_1s + \dots + p_{n-1}s^{n-1} + s^n. \quad (26)$$

In the original coordinates, the feedback law (23) becomes

$$r(\mathbf{x}) = \sum_{i=0}^n p_i (L_f^i \lambda(\mathbf{x}) - L_f^i \lambda(\mathbf{x}^*)) \quad (27)$$

with  $p_n := 1$ . As expected, this is a special case of the tracking law given in [1, Section 4.5] for  $r = n$  and  $L_g L_f^{n-1} \lambda(\mathbf{x}) \equiv 1$ .

### III. APPROXIMATE LINEARIZATION

The feedback law (27) requires the explicit knowledge of the flat output  $\lambda$ . We want to derive feedback laws based on  $\omega = d\lambda$ , where an integration of  $d\lambda$  can be omitted.

#### A. Zeroth Order Approximation

We consider a Taylor series approximation of the control law  $r(\mathbf{S}(z))$  along the reference trajectory  $z^*$

$$\begin{aligned} r(\mathbf{S}(z)) &= \left. \frac{\partial r}{\partial \mathbf{S}} \frac{\partial \mathbf{S}}{\partial z} \right|_{z^*} \cdot (z - z^*) + \mathcal{O}(\|\hat{z}\|^2) \\ &= dr(\mathbf{x}^*) \mathbf{S}'(z^*) \hat{z} + \mathcal{O}(\|\hat{z}\|^2). \end{aligned} \quad (28)$$

The constant part of this series expansion can be omitted since it is already included in the input  $u$  via the open-loop control part  $u^*$ , see (18). Inserting (28) in the tracking error dynamics (22) yields

$$\begin{aligned} \dot{\hat{z}} &= \mathbf{A}\hat{z} + \mathbf{b}(\alpha(z) - \alpha(z^*) - r(\mathbf{S}(z))) \\ &= (\mathbf{A} - \mathbf{b} dr(\mathbf{x}^*) \mathbf{S}'(z^*)) \hat{z} \\ &\quad + \mathbf{b}(\alpha(z) - \alpha(z^*)) + \mathcal{O}(\|\hat{z}\|^2). \end{aligned} \quad (29)$$

In (29), the choice of  $dr(\mathbf{x}^*)$  via  $r(\mathbf{x}^*)$  offers some degrees of freedom in the controller design. Since we want to carry out the design in terms of  $dr$  instead of  $r$ , we replace the gradient by a general covector field

$$\boldsymbol{\rho}(\mathbf{x}^*) = dr(\mathbf{x}^*). \quad (30)$$

Choosing

$$\boldsymbol{\rho}(\mathbf{x}^*) = \mathbf{k}^T [\mathbf{S}'(z^*)]^{-1} \quad (31)$$

results in tracking error dynamics

$$\dot{\hat{z}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T) \hat{z} + \mathbf{b}(\alpha(z) - \alpha(z^*)) + \mathcal{O}(\|\hat{z}\|^2). \quad (32)$$

Error dynamics of the structure (32) occur in high-gain design [13]. Since the linear part  $d\alpha(z^*)$  in the difference  $\alpha(z) - \alpha(z^*)$  is not compensated, we finally obtain a zeroth order approximation.

Now, we want to express the feedback (31) in the coordinates of the original system (1). With  $dL_f \lambda(\mathbf{x}) = L_f d\lambda(\mathbf{x}) = L_f \omega(\mathbf{x})$  (i.e., the exterior derivative commutes with the Lie derivative) and (7) we can express the Jacobian matrix of the coordinate transformation by

$$[\mathbf{S}'(z^*)]^{-1} = \mathbf{T}'(\mathbf{x}^*) = \begin{pmatrix} \omega(\mathbf{x}) \\ L_f \omega(\mathbf{x}) \\ \vdots \\ L_f^{n-1} \omega(\mathbf{x}) \end{pmatrix}. \quad (33)$$

Together with (24) and (31) we finally obtain the controller gain

$$\boldsymbol{\rho}(\mathbf{x}^*) = p_0 \omega(\mathbf{x}^*) + p_1 L_f \omega(\mathbf{x}^*) + \dots + p_{n-1} L_f^{n-1} \omega(\mathbf{x}^*), \quad (34)$$

where the linear part of (32) has the characteristic polynomial (26). The controller gain (34) is feed back into the system via (18) by

$$\hat{u} = -\boldsymbol{\rho}(\mathbf{x}^*) \hat{\mathbf{x}} \quad (35)$$

due to (28) and (30).

*Remark 1:* We could also achieve a zeroth order approximation (32) with  $r(\mathbf{x}) = \mathbf{k}^T \hat{z}$ . However, this control law would require the direct knowledge of the flat output  $\lambda$  since  $z_i = L_f^{i-1} \lambda(\mathbf{x})$  for  $i = 1, \dots, n$ , see (7).

#### B. First Order Approximation

In this section we will derive a state feedback resulting in approximately linear tracking error dynamics. Consider the series expansion (28) of the state feedback and the error dynamics (29). Now, we will also carry out a series expansion of the systems nonlinearity along the reference trajectory:

$$\alpha(z) = \alpha(z^*) + d\alpha(z^*) \hat{z} + \mathcal{O}(\|\hat{z}\|^2). \quad (36)$$

From (29) and (36) we obtain the tracking error dynamics governed by

$$\begin{aligned} \dot{\hat{z}} &= \mathbf{A}\hat{z} + \mathbf{b}(\alpha(z) - \alpha(z^*) - r(\mathbf{S}(z))) \\ &= (\mathbf{A} + \mathbf{b}(d\alpha(z^*) - dr(\mathbf{x}^*) \mathbf{S}'(z^*))) \hat{z} + \mathcal{O}(\|\hat{z}\|^2). \end{aligned} \quad (37)$$

In order to obtain approximately linear error dynamics

$$\dot{\hat{z}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T) \hat{z} + \mathcal{O}(\|\hat{z}\|^2), \quad (38)$$

where all other first order terms along the reference trajectory are compensated and the linear part is designed with the characteristic polynomial (26), we have to choose the covector field (30) as

$$\boldsymbol{\rho}(\mathbf{x}^*) = \mathbf{k}^T [\mathbf{S}'(z^*)]^{-1} + d\alpha(z^*) [\mathbf{S}'(z^*)]^{-1}. \quad (39)$$

The first summand already occurred in (34). In order to express the second summand in  $\mathbf{x}^*$ -coordinates we consider the canonical form (8), where the systems nonlinearity  $\alpha$  is given by

$$\begin{aligned} \alpha(z^*) &= L_f^n \lambda(\mathbf{x}^*) \\ &= \langle dL_f^{n-1} \lambda, \mathbf{f} \rangle(\mathbf{x}^*) \\ &= \langle L_f^{n-1} d\lambda, \mathbf{f} \rangle(\mathbf{x}^*) \\ &= \langle L_f^{n-1} \omega, \mathbf{f} \rangle(\mathbf{x}^*). \end{aligned}$$

Hence, the associated gradient reads

$$d\alpha(\mathbf{z}^*) = \frac{\partial \langle L_f^{n-1} \omega, \mathbf{f} \rangle(\mathbf{x}^*)}{\partial \mathbf{x}^*} \frac{\partial \mathbf{x}^*}{\partial \mathbf{z}^*} = \frac{\partial \langle L_f^{n-1} \omega, \mathbf{f} \rangle(\mathbf{x}^*)}{\partial \mathbf{x}^*} \mathbf{S}'(\mathbf{z}^*).$$

The differentiation of the inner product yields

$$\begin{aligned} d\alpha(\mathbf{z}^*) [\mathbf{S}'(\mathbf{z}^*)]^{-1} &= \frac{\partial \langle L_f^{n-1} \omega, \mathbf{f} \rangle(\mathbf{x}^*)}{\partial \mathbf{x}^*} \\ &= \mathbf{f}^T(\mathbf{x}^*) \frac{\partial \langle L_f^{n-1} \omega(\mathbf{x}^*) \rangle^T}{\partial \mathbf{x}^*} + L_f^{n-1} \omega(\mathbf{x}^*) \mathbf{f}'(\mathbf{x}^*) \\ &= \mathbf{f}^T(\mathbf{x}^*) \left( \frac{\partial \langle L_f^{n-1} \omega(\mathbf{x}^*) \rangle^T}{\partial \mathbf{x}^*} \right)^T + L_f^{n-1} \omega(\mathbf{x}^*) \mathbf{f}'(\mathbf{x}^*) \\ &= L_f^n \omega(\mathbf{x}^*), \end{aligned} \quad (40)$$

because we have the symmetry of second derivatives since the covector field  $\omega$  is assumed to be exact, see (15). With (31), (34) and (40) we can finally express the controller gain (39) in the original coordinates:

$$\begin{aligned} \boldsymbol{\rho}(\mathbf{x}^*) &= p_0 \omega(\mathbf{x}^*) + p_1 L_f \omega(\mathbf{x}^*) + \dots \\ &\quad + p_{n-1} L_f^{n-1} \omega(\mathbf{x}^*) + L_f^n \omega(\mathbf{x}^*). \end{aligned} \quad (41)$$

This equation can be interpreted as a generalization of Ackermann's formula for nonlinear systems [14]. Eq. (41) is in some sense dual to the gain vector of the extended Luenberger observer derived by M. Zeitz and his coworkers [15], [16]. In contrast to Ackermann's formula for linear time-varying systems [2], our approach can be extended to higher order approximations, where we have to proceed similar as in case of nonlinear observer design [17], [18].

#### IV. EXAMPLES

##### A. Tracking Control of an Elastic Manipulator Arm

To illustrate the described approach we apply it to an elastic manipulator arm, cf. Fig. 1 and e.g. [19]. Its model is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{c}{J_1}(x_2 - x_1) \\ -\frac{mgl}{J_2} \sin(x_2) + \frac{c}{J_2}(x_1 - x_2) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{J_1} \\ 0 \end{pmatrix} u, \quad (42)$$

where  $J_1, J_2$  denote moments of inertia (w.r.t. the rotation axis),  $m$  and  $l$  denote the mass and the distance to the axis of the load and  $g$  represents the gravitational acceleration.

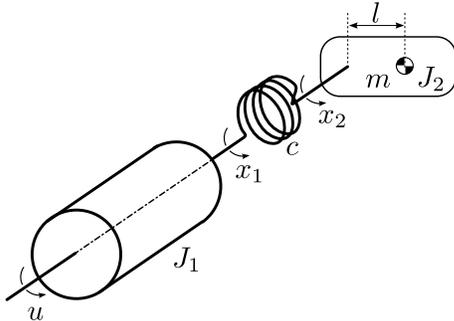


Fig. 1. Elastic manipulator arm.

The controllability matrix (6) for this system reads,

$$\mathbf{R}(\mathbf{x}) = \begin{pmatrix} 0 & \frac{1}{J_1} & 0 & -\frac{c}{J_1^2} \\ 0 & 0 & 0 & \frac{c}{J_1 J_2} \\ \frac{1}{J_1} & 0 & -\frac{c}{J_1^2} & 0 \\ 0 & 0 & \frac{c}{J_1 J_2} & 0 \end{pmatrix} \quad (43)$$

and by formula (13) we obtain

$$\omega = (0, \frac{J_1 J_2}{c}, 0, 0).$$

Obviously, the conditions of Theorem 3 are fulfilled, which means that  $\omega = d\lambda$  with the flat output  $\lambda(\mathbf{x}) = \frac{J_1 J_2}{c} x_2$ , where the scaling factor ensures  $\beta(\mathbf{x}) = 1$ .

Hence, for this system it is possible to compare the approximations of zeroth (34) and first order (41) to the exact formulation (27) of the derived control law. Additionally, we design an LTI-controller with constant state feedback based on the Jacobian-linearized model of (42) in  $\mathbf{x} = (0, 0, 0, 0)^T$ .

As reference trajectory we consider a periodic signal for the flat output:

$$x_2^*(t) = \pi \sin(2\pi t).$$

Note that, another possibility for stabilizing a periodic solution would be to explicitly generate a stable limit cycle, e.g. cf. [20].

The simulation is carried out using the following parameter values:  $J_1 = 0.015 \text{ kg m}^2$ ,  $J_2 = 0.04 \text{ kg m}^2$ ,  $c = 25 \frac{\text{Nm}}{\text{rad}}$ ,  $m = 1 \text{ kg}$ ,  $l = 0.1 \text{ m}$  and  $g = 9.81 \frac{\text{m}}{\text{s}^2}$ . We choose a phase shift of half a period to obtain an initial value error:  $\mathbf{x}(0) = \mathbf{x}^*(0.5 \text{ s})$ . For all four controllers the coefficients  $p_i$  are chosen such that the poles of the closed loop are placed at  $-30$ .

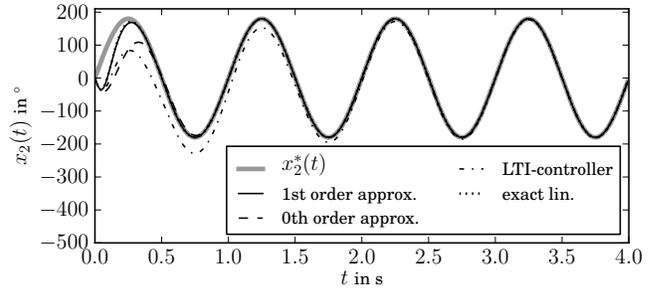


Fig. 2. Convergence of  $x_2(t)$  for different controllers.

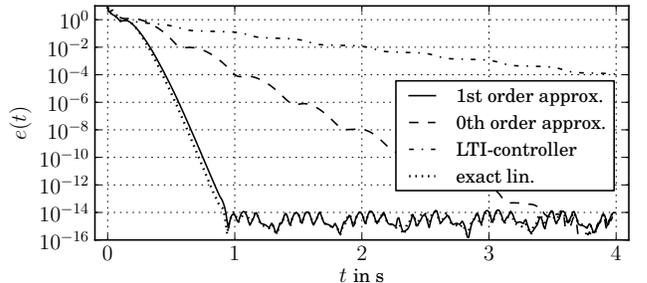


Fig. 3. Logarithmic error over time.

Simulation results are shown in Fig. 2 and 3. In the second figure the time dependent error

$$e(t) := \sum_{i=1}^n \left( \frac{x_i^*(t) - x_i(t)}{\max_{\tau} (|x_i^*(\tau)|)} \right)^2, \quad (44)$$

is depicted, in which every quantity is normalized to the maximum of the absolute value of the respective component of the reference signal. Thus, the error is invariant w.r.t. scaling of the state components.

The simulation results indicate that all four controllers lead to convergence to the reference trajectory. However, the speed of convergence gets worse with less exact approximations. Further, observe that the difference between the exact implementation and the first order approximation is very small.

### B. Tracking Control of a Gantry Crane

Now, we consider a model of another simple underactuated mechanical system: a pendulum attached to a cart, cf. Fig. 4. This system can be seen as a idealized model of a gantry crane with fixed cable length.

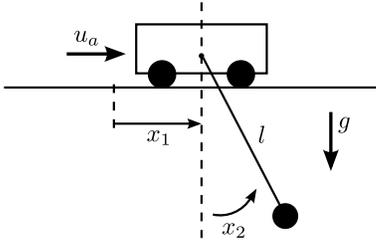


Fig. 4. Gantry Crane (cart with pendulum).

If the cart acceleration  $u_a$  is considered as input, the state-space model is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ 0 \\ -\frac{g}{l} \sin x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{l} \cos x_2 \end{pmatrix} u_a, \quad (45)$$

where the state vector  $x = (x_1, x_2, x_3, x_4)^T$  consists of the cart position, the pendulum angle and the respective velocities. The constant  $l$  denotes the length of the pendulum and  $g$  is the gravitational acceleration.

The  $4 \times 4$ -controllability matrix (6) for that system is given by

$$\mathbf{R}(x) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{\cos(x_2)}{l} \\ 1 & 0 \\ -\frac{\cos(x_2)}{l} & -\frac{x_4 \sin(x_2)}{l} \\ 0 & 0 \\ -\frac{2x_4 \sin(x_2)}{l} & \frac{-4g \sin^2(x_2) + g + 3lx_4^2 \cos(x_2)}{l^2} \\ 0 & 0 \\ \frac{-2g \sin^2(x_2) + g + lx_4^2 \cos(x_2)}{l^2} & \frac{x_4(8g \cos(x_2) + lx_4^2) \sin(x_2)}{l^2} \end{pmatrix}. \quad (46)$$

The covector field  $\omega$  (i.e., the last row of  $\mathbf{R}(x)^{-1}$ ) can be calculated easily using a computer algebra system (CAS) such as *Maxima* or *Sympy*.

To check whether system (45) is exactly linearizable, either the distribution spanned by the first three columns of  $\mathbf{R}(x)$  must be tested for involutivity or, alternatively, condition (16) has to be evaluated. However, as it is well known, this system is not exactly linearizable.

The considered control problem is to perform an equilibrium transition of the gantry crane in finite time  $T$ , i.e., to horizontally move the cart from one point to another such that all velocities vanish at  $t = 0$  and  $t = T$ .

The reference trajectory  $x^*(\cdot)$  for the desired state transition is calculated by numerically solving a boundary value problem with free parameters [21]. The chosen boundary conditions are<sup>1</sup>  $x^*(0) = (0, 0, 0, 0)^T$  and  $x^*(T) = (1, 0, 0, 0)^T$ , with  $T = 4$  s.

To implement the approximate feedback law  $r(x)$  the successive Lie derivatives  $L_f \omega, \dots, L_f^n \omega$  of the covector field  $\omega$  have to be calculated. This is again, facilitated by use of a CAS. For evaluation the resulting  $n \cdot (n + 1) = 20$  scalar expressions<sup>2</sup>, they are converted to C-code and compiled to machine-code. This code can then be easily included to a simulation environment such as Matlab or Scipy in form of a shared library.

For the simulation the parameter values  $l = 5$  m and  $g = 9.81 \frac{\text{m}}{\text{s}^2}$  and the initial value  $x(0) = (0, 5 \frac{\pi}{180}, 0, 0)^T \neq x^*(0)$  are used.

As system (45) is not exactly linearizable only three controllers can be compared: approximations of zeroth (34) and first order (41) as well as a standard LTI-controller with constant feedback gain. To model actuator limitations the system input is calculated by  $u_a = \text{sat}_{\pm\kappa}(u)$  with  $\kappa = 200 \text{ m s}^{-2}$ , where  $u$  is the output of the respective controller. For all three controllers poles are placed at  $-10$  which results in the time depended gains  $\rho_1(t), \dots, \rho_4(t)$  along the reference trajectory shown in Fig. 5.

As in the previous example all three controllers lead to convergence to the reference, cf. Fig. 6 and 7. In Fig. 7 the error according to (44) is shown. Note that in the present situation zeroth and first order approximation lead to more similar results than in the first example.

## V. SUMMARY AND OUTLOOK

We derived tracking controllers for flat single-input systems, where the controller gain can be computed even for controllable non-flat systems. The feasibility was demonstrated on two example systems. The controllers derived in the paper achieve a zeroth or first order approximation of linear tracking error dynamics. However, our approach can be extended for a higher order approximation of nonlinear systems. The problem of very long symbolic expressions for the controller gain can be circumvented by calculating the Lie derivatives using algorithmic differentiation [22].

<sup>1</sup>For better readability in vector notation we omit explicitly writing SI-units.

<sup>2</sup>In the case of zeroth order approximation, the  $n$  components of  $L_f^n \omega$  are not needed.

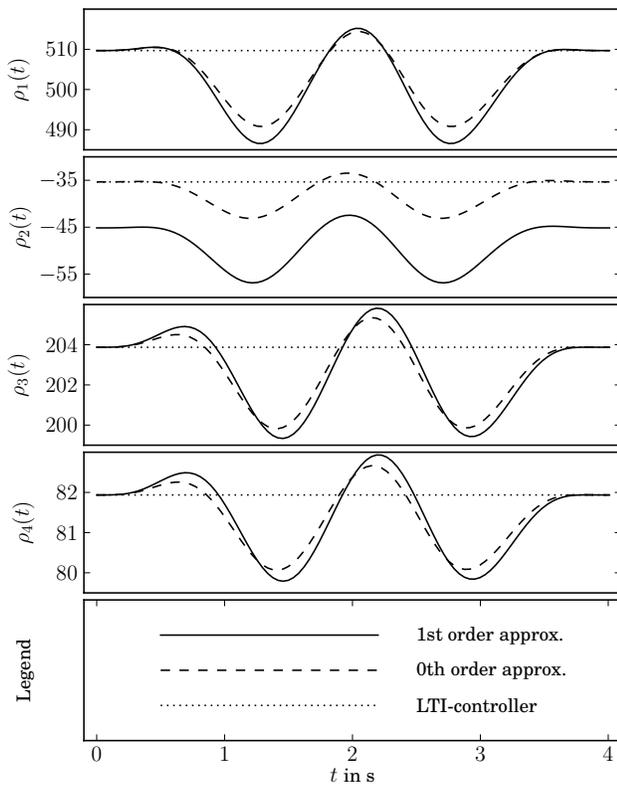


Fig. 5. Time dependent gains of  $\rho(x^*(t))$ .

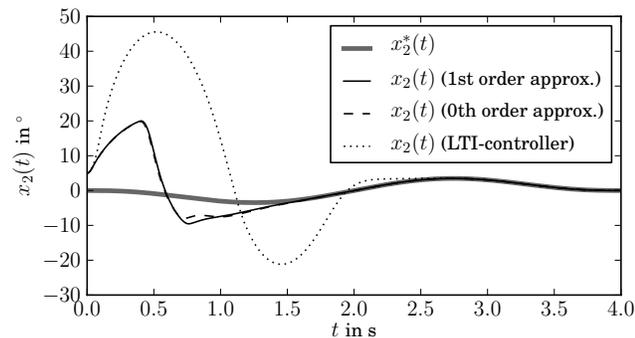


Fig. 6. Pendulum angle  $x_2(t)$ .

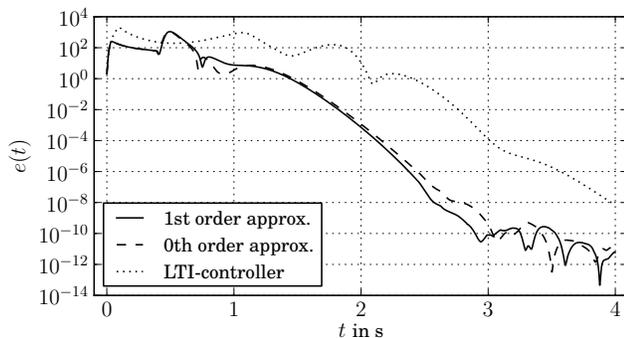


Fig. 7. Logarithmic error over time.

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#### REFERENCES

- [1] A. Isidori, *Nonlinear Control Systems: An Introduction*, 3rd ed. London: Springer-Verlag, 1995.
- [2] O. Föllinger, "Entwurf zeitvarianter Systeme durch Polvorgabe," *Regelungstechnik*, vol. 26, no. 6, pp. 189–196, 1978.
- [3] C. Reboulet and C. Champetier, "A new method for linearizing nonlinear systems: the pseudolinearization," *Int. J. Control*, vol. 40, pp. 631–638, 1984.
- [4] W. T. Baumann and W. J. Rugh, "Feedback control of nonlinear systems by extended linearization," *IEEE Trans. on Automatic Control*, vol. 31, no. 1, pp. 40–46, Jan. 1986.
- [5] K. Röbenack and F. Paschke, "Approximately linear tracking control of nonlinear systems," *Proc. in Applied Mathematics and Mechanics*, 2012, accepted for publication.
- [6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: Introductory theory and examples," *Int. J. Control*, vol. 61, pp. 1327–1361, 1995.
- [7] B. Charlet, J. Lévine, and R. Marino, "On dynamic feedback linearization," *Systems & Control Letters*, vol. 13, no. 2, pp. 143–151, 1989.
- [8] M. Zeitz, "Controllability canonical (phase-variable) form for nonlinear time-variable systems," *Int. J. Control*, vol. 37, no. 6, pp. 1449–1457, 1983.
- [9] —, "Canonical forms for nonlinear systems," in *Geometric Theory of Nonlinear Control Systems*, B. Jakubczyk, W. Respondek, and K. Tchoń, Eds. Wrocław Technical University Press, 1985, pp. 255–278.
- [10] W. Dayawansa, W. M. Boothby, and D. L. Elliott, "Global state and feedback equivalence of nonlinear systems," *Systems & Control Letters*, vol. 6, pp. 229–234, 1985.
- [11] W. Respondek, "Geometry of static and dynamic feedback," Lectures given at the Summer School on Mathematical Control Theory, Trieste, Sept. 2001.
- [12] M. Franke and K. Röbenack, "Some remarks concerning differential flatness and tangent systems," *Proc. in Applied Mathematics and Mechanics*, vol. 12(1), pp. 729–730, 2012.
- [13] K. Röbenack and A. F. Lynch, "High-gain nonlinear observer design using the observer canonical form," *IET Control Theory & Applications*, vol. 1, no. 6, pp. 1574–1579, 2007.
- [14] J. E. Ackermann, "On the synthesis of linear control systems with specified characteristics," *Automatica*, vol. 13, no. 1, pp. 89–94, Jan. 1977.
- [15] D. Bestle and M. Zeitz, "Canonical form observer design for non-linear time-variable systems," *Int. J. Control*, vol. 38, no. 2, pp. 419–431, 1983.
- [16] M. Zeitz, "The extended Luenberger observer for nonlinear systems," *Systems & Control Letters*, vol. 9, pp. 149–156, 1987.
- [17] K. Röbenack and A. F. Lynch, "An efficient method for observer design with approximately linear error dynamics," *Int. J. Control*, vol. 77, no. 7, pp. 607–612, May 2004.
- [18] K. Röbenack, "Direct approximation of observer error linearization for nonlinear forced systems," *IMA Journal of Mathematical Control and Information*, vol. 24, no. 4, pp. 551–566, 2007.
- [19] S. Nicosia, P. Tomei, and A. Tornambé, "A nonlinear observer for elastic robots," *IEEE J. of Robotics and Automation*, vol. 4, no. 1, pp. 45–52, Feb. 1988.
- [20] C. Knoll and K. Röbenack, "Generation of stable limit cycles with prescribed frequency and amplitude via polynomial feedback," in *Proc. of the 9th International Multi-Conference on Systems, Signals and Devices*, 2012.
- [21] K. Graichen, M. Treuer, and M. Zeitz, "Fast side-stepping of the triple inverted pendulum via constrained nonlinear feedforward control design," in *Proc. of the 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05*, December 12–15 2005, pp. 1096–1101.
- [22] K. Röbenack, J. Winkler, and S. Wang, "LIEDRIVERS — a toolbox for the efficient computation of Lie derivatives based on the object-oriented algorithmic differentiation package ADOL-C," in *Proc. of the 4th International Workshop on Equation-Based Object-Oriented Modeling Languages and Tools*, Zurich, 2011, pp. 57–66.