LQG control with Markovian Packet Loss

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Abstract— This paper is concerned with the optimal LQG control of a system through lossy data networks. In particular we will focus on the case where control commands are issued to the system over a communication network where packets may be randomly dropped according to a two-state Markov chain. Under these assumptions, the optimal finite-horizon LQG problem is solved by means of dynamic programming arguments. The infinite horizon LQG control problem is explored and conditions to ensure its convergence are investigated. Finally it is shown how the results presented in this paper can be employed in the case that also the observation packet may be dropped. A numerical simulation shows the relationship between the convergence of the LQG cost and the value of the parameters of the Markov chain.

I. INTRODUCTION

Today, an increasing number of applications demand remote control of plants over unreliable networks. The recent development of sensor web technology enables the development of wireless sensor networks that can be immediately used for estimation and control purposes [1]. In these systems issues such as communication delay, data loss and time synchronization play a critical role. As a matter of fact, communication and control are tightly coupled and so they cannot be addressed independently. For this reason, the study of stability of dynamical systems where components are connected via communication channels has received considerable attention in the past few years, see e.g. [2], [3], [4], [5].

In this paper we will focus on the Linear Quadratic Gaussian (LQG) optimal control problem for the case where control packets can be randomly lost accordingly to a two-states Markov process. With a few notable exceptions, such as [6] and [7], most of the work on control present in the literature focuses on the case where packet losses are governed by independent identically distributed Bernoulli processes. In particular, for what concerns the LQG problem, Schenato et al. [8] consider the case where both sensors and command packets can be lost according to a Bernoulli process, under the assumption that a perfect acknowledgment mechanism is available (the so called TCP-like case [9]).

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In that paper the authors prove that the separation principle holds and the controller and the observer are linear functions of the estimated state. More recently, the authors of [10] generalize the results in the TCP-like framework to the case where multiple memoryless erasure channels are interposed between the sensors, the controller and the actuators. In this case partial observation and control losses may occur. Despite that it is possible to show that the separation principle holds. In stark contrast, when either no acknowledgment or only imperfect acknowledgment occurs, the separation principle does not hold and joint design of estimator and controller becomes a non convex problem, as shown in [11] and [12].

This paper generalizes the aforementioned results to the case where channels have memory. This situation is very common in wireless communication where effects like fading make the assumption of independent losses not suitable. We model such phenomena using a particular Gilbert-Elliott Channel model [13], i.e. a two-state Markov Chain. As already stated above the packet arrival sequence is not i.i.d. anymore.

The remainder of this paper is organized as follows. Section 2 provides the problem formulation in the case only control packets may be lost accordingly to a Gilbert-Elliot Channel model. In Section 3 the design the optimal LQG control law is addressed both in the finite and in the infinite horizon case. Section 4 discusses how the results can be generalized to the case where observation packets can be dropped as well. Section 5 presents a numerical example showing how the Markov chain parameters affect the convergence of the LQG cost in the infinite horizon case. Finally, Section 6 draws some conclusions.

II. PROBLEM FORMULATION

Consider the linear system

$$\begin{aligned}
x_{k+1} &= Ax_k + Bu_k^a + w_k, \\
y_k &= Cx_k + v_k,
\end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k^a \in \mathbb{R}^p$ is the control signal applied by the actuator and $w_k \in \mathbb{R}^n$ is process noise, assumed to be i.i.d. Gaussian with mean 0 and covariance $Q > 0$. $y_k \in \mathbb{R}^m$ is the observation vector and $v_k \in \mathbb{R}^m$ is the measurement noise, also assumed to be i.i.d. Gaussian with mean 0 and covariance $R > 0$. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$ are the system matrixes. We assume that the system is both detectable and stabilizable. The initial state $x_0 \in \mathbb{R}^n$ is assumed to be Gaussian with mean 0 and covariance Σ. We further assume that $x_0, v_0, \ldots, w_0, \ldots$ are independent random vectors.

Let $u_k \in \mathbb{R}^n$ be the control signal sent by the controller. We assume that such signal is sent through an unreliable network. The random variable ν_k characterizes the correct packet communication, namely we will have $\nu_k = 1$ if the actuator receives u_k and $\nu_k = 0$ otherwise. Assuming that $u_k^a = 0$ if u_k is dropped and $u_k^a = u_k$ if u_k is received, u_k^a will be

$$u_k^a = \nu_k u_k. \tag{2}$$

In this paper we assume that packet losses are modeled through the particular Gilbert-Elliott channel model depicted in Fig.1. This model consists of a two-state Markov process, one of them representing an good behavior of the channel (the packet is received, i.e. $\nu_k = 1$) and the other the bad one (the packet is not received, i.e. $\nu_k = 0$).
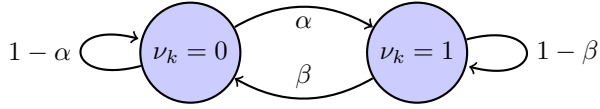

Fig. 1. The *Gilbert-Elliott* model of the communication channel.

We denote with α the probability of passing from the bad state to the good one and with β the one of passing from the good to the bad one. As a consequence the probabilities that a packet is received or not, depending on the previous packet, are

$$\begin{bmatrix} P(\nu_{k+1}=0|\nu_k=0) & P(\nu_{k+1}=1|\nu_k=0) \\ P(\nu_{k+1}=0|\nu_k=1) & P(\nu_{k+1}=1|\nu_k=1) \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}. \tag{3}$$

Furthermore, the latter Markov process is assumed to be:

- **Irreducible**: the variables are such that $0 < \alpha \le 1$, $0 < \beta \le 1$;
- **Stationary**: in absence of any past information, the probability that, at a certain instant k, $\nu_k = 0$ is always the same and in particular $P(\nu_0 = 0) = \ldots = P(\nu_k = 0) = \beta/(\alpha + \beta), \forall k > 0$.

We assume that the information set $\mathcal{F}_k$ available to the controller at each time instant k is

$$\mathcal{F}_k \triangleq (y_0, \ldots, y_k, \nu_0, \ldots, \nu_{k-1}, u_0, \ldots, u_{k-1}, x_0). \tag{4}$$

Please note that this information set contains the sequence $\{\nu_t\}_{t=0}^k$, which means that a reliable acknowledgment mechanism is implemented (the so called TCP-like case [9]). As pointed out in [10], this assumption is reasonable in many practical cases.

The goal of this paper is to determine the optimal control input sequence $\{u_k\}$ as a function of the available information set, i.e. $u_k = f_k(\mathcal{F}_k)$, such that the following cost function is minimized:

$$J_T \triangleq \inf_{\{u_k\}:u_k=f_k(\mathcal{F}_k)} \mathbb{E}\left[\sum_{k=0}^T x_k' W_k x_k + (u_k^a)' U_k u_k^a\right], \tag{5}$$

where W_k, U_k are strictly positive definite matrices. Since $u_k^a = \nu_k u_k$, the above equation can be simplified as

$$J_T = \inf_{\{u_k\}:u_k=f_k(\mathcal{F}_k)} \mathbb{E}\left[\sum_{k=0}^T x_k' W_k x_k + \nu_k u_k' U_k u_k\right]. \tag{6}$$

III. MAIN RESULTS

A. Estimator Design

Using arguments similar to standard Kalman filtering, we can prove that the following estimation equations hold when the control packets are dropped according to a Markov chain:

$$\hat{x}_k = \mathbb{E}[x_k|\mathcal{F}_k] = \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}), \tag{7}$$
$$P_k = \mathbb{E}[(x_k-\hat{x}_k)(x_k-\hat{x}_k)'|\mathcal{F}_k] = P_{k|k-1} - K_k C P_{k|k-1}, \tag{8}$$

where

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1} + \nu_{k-1}Bu_{k-1}, \tag{9}$$
$$P_{k|k-1} = AP_{k-1}A' + Q, \tag{10}$$
$$K_k = P_{k|k-1}C'(CP_{k|k-1}C' + R)^{-1} \tag{11}$$
$$\hat{x}_0 = 0, \ P_0 = \Sigma. \tag{12}$$

Moreover, it is worth to recall the following lemma, whose proof is reported in [14]

Lemma 1: For any matrix S, the following equality holds:

$$\mathbb{E}(x_k'Sx_k|\mathcal{F}_k) = \hat{x}_k'S\hat{x}_k + \text{tr}(SP_k). \tag{13}$$

where $\text{tr}(SP_k)$ denotes the trace of the matrix SP_k.

B. Finite Horizon

In order to derive the optimal control law and the corresponding value for the objective function we follow a dynamic programming approach based on a cost-to-go iterative procedure. To this end, let us define the following optimal value function

$$V_k(x_k) \triangleq \inf_{u_k,\ldots,u_T} \mathbb{E}\left[\sum_{i=k}^T x_i'W_ix_i + \nu_i u_i'U_iu_i|\mathcal{F}_k\right]. \tag{14}$$

which can be rewritten as follows by dynamic programming:

$$V_k(x_k) = \inf_{u_k} \mathbb{E}\left[x_k'W_kx_k + \nu_k u_k'U_ku_k + V_{k+1}(x_{k+1})|\mathcal{F}_k\right],$$

The following lemma can be proved

Lemma 2: The value function $V_k(x_k)$ is given by the following equation:

$$V_k(x_k) = \begin{cases} \mathbb{E}(x_k'S_kx_k|\mathcal{F}_k) + c_k & (\nu_{k-1} = 0) \\ \mathbb{E}(x_k'R_kx_k|\mathcal{F}_k) + d_k & (\nu_{k-1} = 1) \end{cases}, \tag{15}$$

where S_k, R_k, c_k, d_k can be defined recursively as

$$S_k = W_k + (1-\alpha)A'S_{k+1}A + \alpha A'R_{k+1}A - \alpha A'R_{k+1}B(U_k + B'R_{k+1}B)^{-1}B'R_{k+1}A, \tag{16}$$

$$R_k = W_k + \beta A'S_{k+1}A + (1-\beta)A'R_{k+1}A - (1-\beta)A'R_{k+1}B(U_k + B'R_{k+1}B)^{-1}B'R_{k+1}A, \tag{17}$$

$$\begin{aligned} c_k =& (1-\alpha)\text{tr}(S_{k+1}Q) + \alpha\text{tr}(R_{k+1}Q) + \\ &+ (1-\alpha)c_{k+1} + \alpha d_{k+1} + \\ &+ \alpha\text{tr}(A'R_{k+1}B(U_k+B'R_{k+1}B)^{-1}B'R_{k+1}AP_k), \end{aligned} \tag{18}$$

$$\begin{aligned} d_k =& \beta\text{tr}(S_{k+1}Q) + (1-\beta)\text{tr}(R_{k+1}Q) + \\ &+ \beta c_{k+1} + (1-\beta)d_{k+1} \\ &+ (1-\beta)\text{tr}(A'R_{k+1}B(U_k+B'R_{k+1}B)^{-1}B'R_{k+1}AP_k), \end{aligned} \tag{19}$$

with the following initial conditions

$$S_T = R_T = W_T, c_T = d_T = 0. \qquad (20)$$

Moreover, the optimal control u_k is

$$u_k = -(U_k + B'R_{k+1}B)^{-1} B'R_{k+1}A\hat{x}_k. \qquad (21)$$

Proof: The proof is similar to the ones presented by Costa et al. [15] and hence is omitted due to space limit. ∎

Since $J_T = \mathbb{E}(V_0(x_0))$, the following theorem can be proved:

Theorem 1: The optimal J_T for finite-horizon LQG problem (6) is given by the following equation:

$$J_T = \frac{1}{\alpha+\beta}\text{tr}\bigg[\beta\Sigma S_0 + \alpha\Sigma R_0 + \sum_{k=0}^{T-1}(\beta S_{k+1}Q + \alpha R_{k+1}Q) \\ + \sum_{k=0}^{T-1}(\alpha AR_{k+1}B(U_k + B'R_{k+1}B)^{-1}B'R_{k+1}AP_k)\bigg] \qquad (22)$$

where S_k, R_k and P_k are defined recursively in (16), (17) and (8) with initial condition $S_T = R_T = W_T, P_0 = \Sigma$. Moreover, the optimal control u_k is given by (21).

Proof: Since $J_T = \mathbb{E}(V_0(x_0))$, it follows that

$$\begin{aligned} J_T &= P(\nu_{-1} = 0)(\mathbb{E}(x_0'S_0x_0|\mathcal{F}_0) + c_0) \\ &\quad + P(\nu_{-1} = 1)(\mathbb{E}(x_0'R_0x_0|\mathcal{F}_0) + d_0) \\ &= \frac{1}{\alpha+\beta}\text{tr}(\beta\Sigma S_0 + \alpha\Sigma R_0) + \frac{1}{\alpha+\beta}(\beta c_0 + \alpha d_0). \end{aligned}$$

Since the Markov process is assumed to be irreducible and stationary, $P(\nu_{-1} = 0) = \beta/(\alpha + \beta)$ and $P(\nu_{-1} = 1) = \alpha/(\alpha + \beta)$ and then

$$J_T = \frac{1}{\alpha+\beta}\text{tr}(\beta\Sigma S_0 + \alpha\Sigma R_0) + \frac{1}{\alpha+\beta}(\beta c_0 + \alpha d_0).$$

By (18) and (19), we have

$$\begin{aligned} \beta c_0 + \alpha d_0 &= \text{tr}(\beta S_1 Q + \alpha R_1 Q + \\ &\quad + \alpha A'R_1B(U_1 + B'R_1B)^{-1}B'R_1AP_0) + \beta c_1 + \alpha d_1 \\ &= \sum_{k=0}^{T-1}\text{tr}(\beta S_{k+1}Q + \alpha R_{k+1}Q + \\ &\quad + \alpha AR_{k+1}B(U_k + B'R_{k+1}B)^{-1}B'R_{k+1}AP_k). \end{aligned}$$

As a result, the optimal J_T is given by (22). ∎

C. Infinite-Horizon LQG and Convergence

This subsection tackles the the convergence of the infinite-horizon LQG problem. Throughout this subsection, it is assumed that $W_0 = \ldots = W_T = \ldots = W$ and $U_0 = \ldots = U_T = \ldots = U$. Since $Q > 0$ is assumed to be strictly positive definite, the covariance matrix P_k of the Kalman filter converges to an unique positive definite matrix P, regardless of the initial condition Σ. Therefore, let us define

$$P \triangleq \lim_{k\to\infty} P_k. \qquad (23)$$

Moreover let us also define the infinite-horizon cost function as

$$J_\infty = \limsup_{T\to\infty} J_T/T.$$

In order to simplify notations, we will also introduce the following functions

$$\begin{aligned} g(X, Y) &\triangleq W + (1 - \alpha)A'XA + \alpha A'YA \qquad (24) \\ &\quad - \alpha A'YB(U + B'YB)^{-1}B'YA, \\ h(X, Y) &\triangleq W + \beta A'XA + (1 - \beta)A'YA \qquad (25) \\ &\quad - (1 - \beta)A'YB(U + B'YB)^{-1}B'YA. \end{aligned}$$

and

$$\begin{aligned} g^k(X, Y) &\triangleq g(g^{k-1}(X, Y), h^{k-1}(X, Y)), \qquad (26) \\ h^k(X, Y) &\triangleq h(g^{k-1}(X, Y), h^{k-1}(X, Y)). \qquad (27) \end{aligned}$$

Please note that the latter functions are such that $S_k = g(S_{k+1}, R_{k+1})$ and $R_k = h(S_{k+1}, R_{k+1})$ and moreover $S_k = g^{T-k}(S_T, R_T)$, $R_k = h^{T-k}(S_T, R_T)$.

Theorem 2: If two positive semidefinite matrices $\bar{X} \geq 0$ and $\bar{Y} \geq 0$ exist such that

$$\bar{X} \geq g(\bar{X}, \bar{Y}), \bar{Y} \geq h(\bar{X}, \bar{Y}). \qquad (28)$$

Then, the following hold:
1) The following equations

$$X = g(X, Y), Y = h(X, Y),$$

has a unique positive definite solution X_*, Y_*.
2) For all positive semidefinite matrices $X_0 \geq 0$, $Y_0 \geq 0$, the following holds

$$\lim_{k\to\infty} g^k(X_0, Y_0) = X_*, \lim_{k\to\infty} h^k(X_0, Y_0) = Y_*.$$

On the contrary, if there do not exist positive semidefinite matrices $\bar{X}, \bar{Y}$, such that (28) holds, then for all positive semidefinite matrices $X_0 \geq 0$, $Y_0 \geq 0$,

$$\lim_{k\to\infty} g^k(X_0, Y_0) = \infty, \lim_{k\to\infty} h^k(X_0, Y_0) = Y_*.$$

Proof: The proof is reported in the appendix for the sake of readability. ∎

By exploiting the latter result the following theorem can be proved:

Theorem 3: Suppose that there exist two positive semidefinite matrices $X \geq 0$ and $Y \geq 0$, such that (28) holds, then the optimal J_∞ is given by

$$J_\infty = \frac{\text{tr}(\beta SQ + \alpha RQ + \alpha ARB(U + B'RB)^{-1}B'RAP_\infty)}{\alpha + \beta}, \qquad (29)$$

where S, R are the unique solutions of the following equations

$$S = g(S, R), R = h(S, R). \qquad (30)$$

The optimal control law is given by

$$u_k = -(U + B'RB)^{-1} B'RA\hat{x}_k. \qquad (31)$$

On the contrary, if there does not exist $X \geq 0$, $Y \geq 0$, such that (28) holds, then

$$J_\infty = \infty. \qquad (32)$$

As it is well known, when A is asymptotically stable, the LQG cost is always finite. For the case of unstable A, it can be easily seen that the finiteness of LQG control depends on functions g and h, which further depend on α and β. In the following theorem, we give a necessary condition on α for the convergence of the LQG control. Such a condition is also sufficient if B is invertible.

Theorem 4: Suppose that A is unstable, then the following condition is necessary for the LQG control cost J_∞ to be finite:

$$\alpha > 1 - \frac{1}{\rho^2}, \tag{33}$$

where ρ is the spectral radius of A matrix. Moreover, if B is invertible, then (33) is also sufficient.

Proof: Since J_∞ is finite, by Theorem 3 there exist matrices $X > 0, Y > 0$, such that

$$X \geq g(X,Y) = W + (1-\alpha)A'XA + \alpha A'YA$$
$$- \alpha A'YB(U + B'YB)^{-1}B'YA$$

By using the matrix inversion lemma $Y - YB(U + B'YB)^{-1}B'Y = (Y^{-1} + BU^{-1}B')^{-1}$, it can be seen that

$$X \geq W + (1-\alpha)A'XA + \alpha A'(Y^{-1} + BU^{-1}B')^{-1}A$$
$$\geq W + (1-\alpha)A'XA.$$

By the properties of Lyapunov equation, we know that to be true $\sqrt{1-\alpha}A$ must be strictly stable, which implies that $\alpha > 1 - \rho^{-2}$, which concludes the first part of the proof. For the second part , if B is invertible, then as a consequence:

$$g(X,Y) = W + (1-\alpha)A'XA + \alpha A'(Y^{-1} + BU^{-1}B')^{-1}A$$
$$\leq W + \alpha A'(BU^{-1}B')^{-1}A + (1-\alpha)A'XA,$$

and

$$h(X,Y) \leq W + \beta A'XA + (1-\beta)A'(BU^{-1}B')^{-1}A.$$

Since we assume that $\alpha > 1 - \rho^{-2}$, we could always find $X > 0$, such that

$$X \geq W + \alpha A'(BU^{-1}B')^{-1}A + (1-\alpha)A'XA \geq g(X,Y).$$

Then we choose

$$Y = W + \beta A'XA + (1-\beta)A'(BU^{-1}B')^{-1}A \geq h(X,Y).$$

Finally by Theorem 3, J_∞ is finite, which concludes the proof. ∎

Remark 1: It is interesting to note that, for the sake of convergence, the role of the parameters α and β and in general the recursions $X = g(X,Y)$ and $Y = h(X,Y)$ is not the same. In particular it is possible to see that α and $X = g(X,Y)$ are in a sense "more critical" than β and $Y = h(X,Y)$. The reason is that α is strictly connected to the probability to have long "bursts" of packet losses which is a critical condition for the convergence of the cost.

IV. GENERALIZATION TO SENSORS PACKET LOSSES

The seen results can be summarized by the following theorem:

Theorem 5: Let the system (1)-(3) with A, C detectable and A, B stabilizable. Let the information set (4) be available to the controller. Then, considering the problem of minimizing (6)

- **(Finite Horizon)** The optimal control law is (21) a **linear** function depending only on the estimated system state $\hat{x}$ and where R_k is recursively obtained by (16)-(17) . The *separation principle* holds true since the optimal estimator is independent of the control input u_k and the two can be designed separately. The optimal estimator is the Kalman filter (8) and the optimal cost is (22).
- **(Infinite Horizon)** The optimal cost J_∞ is finite **if and only if** there exist two positive semidefinite matrices $X \geq 0$ and $Y \geq 0$, such that (28) holds true. In this case, the optimal cost is given by (29) where S, R are the unique solutions (30). The optimal control law is linear (31).

Remark 2: Please note that ν_k does not explicitly appears in the control law. Nevertheless, this information cannot be dropped from the information set, since it is used in the estimation. In fact, by resorting to the same arguments presented in [12], it can be proved that with the information set $\mathcal{G}_k \triangleq (y_0, \ldots, y_{k-1}, u_0, \ldots, u_{k-1}, x_0)$ the separation principle does not hold anymore.

Interestingly enough, the above results can be generalized also to the case where observation packets are lost according to a stochastic variable γ_k such that $\gamma_k = 1$ if the measurements packets are correctly received and $\gamma_k = 0$ otherwise. The only difference is that the Kalman filter update (8) becomes

$$\hat{x}_k = \hat{x}_{k|k-1} + \gamma_k K_k(y_k - C\hat{x}_{k|k-1}), \tag{34}$$
$$P_k = \gamma_k P_{k|k-1} - K_k C P_{k|k-1}, \tag{35}$$

and that the cost depends on $\mathbb{E}[P_k]$ rather than on P_k as shown below for the finite horizon case

$$J_T = \frac{1}{\alpha + \beta} \text{tr} \Bigg[\beta \Sigma S_0 + \alpha \Sigma R_0 + \sum_{k=0}^{T-1} (\beta S_{k+1}Q + \alpha R_{k+1}Q) $$
$$+ \sum_{k=0}^{T-1} \big(\alpha A R_{k+1}B(U_k + B'R_{k+1}B)^{-1}B'R_{k+1}A\mathbb{E}[P_k] \big) \Bigg] \tag{36}$$

and for the infinite horizon case

$$J_\infty = \frac{\text{tr}(\beta SQ + \alpha RQ + \alpha ARB(U + B'RB)^{-1}B'RA\mathbb{E}P_\infty)}{\alpha + \beta}. \tag{37}$$

Clearly in this case, in order to have a finite J_∞ cost, also $E[P_\infty]$ has to be finite. Conditions to obtain a finite $E[P_\infty]$ in the case that γ_k is modeled as a Markov processes can be found in [16], [17].

V. NUMERICAL EXAMPLES

We consider the following system

$$A = \begin{bmatrix} 1.05 & 0 \\ 0 & -1.05 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with W = Q = R = I , U = 1. Figure V shows the region with finite optimal LQG cost, with respect to the possible values of α and β. Note that the unstable region is actually not connected. The system is chosen in such a way that (A^2, B) is not controllable. Therefore, if $\alpha = \beta = 1$, then the system can only be controlled at either odd or at even times, causing the loss of controllability, thus explaining why the upper right corner of the plot is unstable.

Moreover, intuitively speaking, a "good" channel is a channel with a small β and a large α. However, this may not be the case for LQG control, as is shown in Figure V, where increasing α may result in instability.
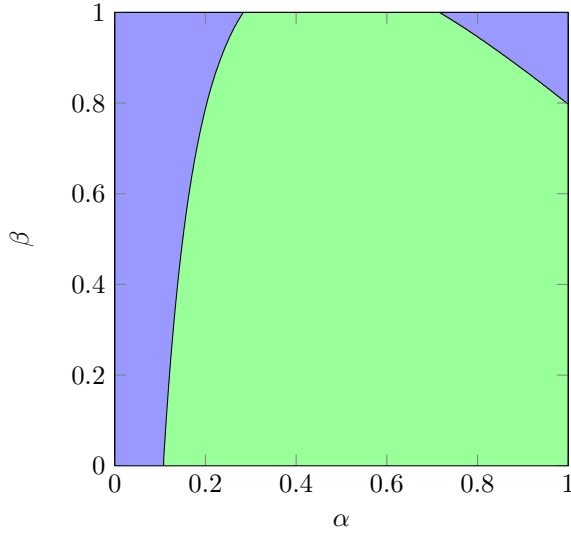

Fig. 2. Convergence Region. The green region corresponds to finite J_∞ and the blue region corresponds to infinite J_∞.

VI. CONCLUSIONS

In this paper we considered the problem of LQG optimal control in the case control and observation packets may be lost according to a particular Gilbert-Elliot channel model. It has been shown that, interestingly enough, under the assumption of a TCP-like acknowledgment mechanism, the separation principle still holds and that the optimal controller is still a linear function of the state. The problem of infinite horizon LQG has also been investigated, showing how, in this case, differently from the memoryless one, stability depends on the parameters of the Markov chain.

APPENDIX: PROOF OF THEOREM 2

This section is devoted to proving Theorem 2, which requires several intermediate results.

First let us define function $\varphi(K, X, Y)$ and $\phi(K, X, Y)$ as follows:

$$\varphi(K, X, Y) \triangleq W + (1-\alpha)A'XA + \alpha(F'YF + K'UK),$$
$$\phi(K, X, Y) \triangleq W + \beta A'XA + (1-\beta)(F'YF + K'UK),$$

where $F = A + BK$. Moreover, we define φ^k, ϕ^k as

$$\varphi^k(K, X, Y) \triangleq \varphi(K, \varphi^{k-1}(X, Y), \phi^{k-1}(X, Y)), \quad (38)$$
$$\phi^k(K, X, Y) \triangleq \phi(K, \varphi^{k-1}(X, Y), \phi^{k-1}(X, Y)). \quad (39)$$

Now we have the following lemma:

Lemma 3: The following statements on functions g, h, φ, ϕ are true:

1) φ and ϕ are monotonically increasing with respect to X and Y.

2) For any K, the following inequalities hold:

$$g(X, Y) \leq \varphi(K, X, Y), \ h(X, Y) \leq \phi(K, X, Y).$$

Furthermore, the equalities hold if

$$K = K_Y = -(U + B'YB)^{-1}B'YA.$$

3) g and h are monotonically increasing with respect to X and Y.

Proof:

1) It directly follows from the structure of $\varphi(K, X, Y)$ and $\phi(K, X, Y)$.

2) It is enough to notice that by construction $\varphi(K, X, Y)$ and $\phi(K, X, Y)$ are such that

$$\varphi(K, X, Y) = g(X, Y)$$
$$+ \alpha(K - K_Y)'(B'YB + U)(K - K_Y),$$
$$\phi(K, X, Y) = h(X, Y)$$
$$+ (1-\beta)(K - K_Y)'(B'YB + U)(K - K_Y),$$

3) Suppose that $X_1 \geq X_2$ and $Y_1 \geq Y_2$, we know that

$$g(X_1, Y_1) = \varphi(K_{Y_1}, X_1, Y_1) \geq \varphi(K_{Y_1}, X_2, Y_2)$$
$$\geq g(X_2, Y_2).$$

By the same argument, we know that h is also monotonically increasing.

∎

Lemma 4: Suppose there exist positive semidefinite matrices $\bar{X} \geq 0, \bar{Y} \geq 0$ and some matrix K, such that

$$\bar{X} \geq \varphi(K, \bar{X}, \bar{Y}), \ \bar{Y} \geq \phi(K, \bar{X}, \bar{Y}). \quad (40)$$

Then there exists $X_* > 0, Y_* > 0$, such that for all $X_0 \geq 0, Y_0 \geq 0$, we have

$$\lim_{k \to \infty} \varphi^k(K, X_0, Y_0) = X_*, \lim_{k \to \infty} \phi^k(K, X_0, Y_0) = Y_*.$$

Proof: First let us consider $X_k = \varphi^k(K, 0, 0)$ and $Y_k = \phi^k(K, 0, 0)$. Since $Y_0 = 0$ and $X_0 = 0$, then $X_0 < X_1$ and $Y_0 < Y_1$. By applying φ and ϕ recursively, we know that the sequences of X_i, Y_i is increasing, i.e.,

$$0 = X_0 < X_1 \leq \cdots \leq X_k, \ 0 = Y_0 < Y_1 \leq \cdots \leq Y_k.$$

By the same argument, the following sequences are decreasing

$$\bar{X} \geq \varphi(K, \bar{X}, \bar{Y}) \geq \cdots \geq \varphi^k(K, \bar{X}, \bar{Y})$$
$$\bar{Y} \geq \phi(K, \bar{X}, \bar{Y}) \geq \cdots \geq \phi^k(K, \bar{X}, \bar{Y})$$

Now since $\bar{X}, \bar{Y} \geq 0$, by monotonicity of φ, ϕ, we have

$$X_k \leq \varphi^k(K, \bar{X}, \bar{Y}) \leq \bar{X}, \; Y_k \leq \phi^k(K, \bar{X}, \bar{Y}) \leq \bar{Y}.$$

Therefore, $\{X_k\}$ and $\{Y_k\}$ are bounded and monotone, which implies that they converge to positive definite matrices X_* and Y_* respectively.

Now consider the case of an arbitrary $X_0 \geq 0$ and $Y_0 \geq 0$. Since X_* and Y_* are strictly positive definite, we could find a scalar $q > 0$, such that $X_* \geq qX_0$ and $Y_* \geq qY_0$. At this point, due to the fact φ and ϕ are increasing w.r.t. X and Y we have:

$$\varphi^k(K, 0, 0) \leq \varphi^k(K, qX_0, qY_0) \leq \varphi^k(K, X_*, Y_*) = X_*,$$
$$\phi^k(K, 0, 0) \leq \phi^k(K, qX_0, qY_0) \leq \phi^k(K, X_*, Y_*) = Y_*.$$

By taking the limit on both sides, and exploiting the above result we have that:

$$\lim_{k \to \infty} \varphi^k(K, qX_0, qY_0) = X_*,$$
$$\lim_{k \to \infty} \phi^k(K, qX_0, qY_0) = Y_*.$$

To complete the proof it is enough to remark that the following equations holds true

$$\varphi^k(K, qX_0, qY_0) - \varphi^k(K, 0, 0) = q(\varphi^k(K, X_0, Y_0) - \varphi^k(K, 0, 0)),$$
$$\phi^k(K, qX_0, qY_0) - \phi^k(K, 0, 0) = q(\phi^k(K, X_0, Y_0) - \phi^k(K, 0, 0)).$$

and that by taking the limit on both sides again we obtain

$$\lim_{k \to \infty} \varphi^k(K, X_0, Y_0) = X_*,$$
$$\lim_{k \to \infty} \phi^k(K, X_0, Y_0) = Y_*,$$

which ends the proof. ∎

Using the latter Lemmas it is now possible to prove Theorem 2. *Proof:* [Proof of Theorem 2] Assume that there exist $\bar{X} \geq 0$ and $\bar{Y} \geq 0$, such that

$$\bar{X} \geq g(\bar{X}, \bar{Y}), \; \bar{Y} \geq h(\bar{X}, \bar{Y}).$$

Let us define $X_k = g^k(0, 0)$ and $Y_k = h^k(0, 0)$. By following the exact same arguments of the proof of Lemma 4, it results that the sequences X_k and Y_k converge to $X_* > 0$ and $Y_* > 0$ respectively, which implies that

$$X_* = g(X_*, Y_*) = \varphi(K_*, X_*, Y_*),$$
$$Y_* = h(X_*, Y_*) = \phi(K_*, X_*, Y_*),$$

where $K_* = -(U + B'Y_*B)^{-1}B'Y_*A$. By choosing arbitrary $X_0 \geq 0$ and $Y_0 \geq 0$, and exploiting Lemma 3 we can write the following inequalities

$$g^k(0, 0) \leq g^k(X_0, Y_0) \leq \varphi^k(K_*, X_0, Y_0),$$
$$h^k(0, 0) \leq h^k(X_0, Y_0) \leq \phi^k(K_*, X_0, Y_0).$$

Finally, by taking the limit on the above terms, we have

$$X_* \leq \lim_{k \to \infty} g^k(X_0, Y_0) \leq X_*,$$
$$Y_* \leq \lim_{k \to \infty} h^k(X_0, Y_0) \leq Y_*,$$

which proves that

$$\lim_{k \to \infty} g^k(X_0, Y_0) = X_*, \; \lim_{k \to \infty} h^k(X_0, Y_0) = Y_*.$$

To prove the necessity of the stated condition, suppose matrices $\bar{X} \geq 0$ and $\bar{Y} \geq 0$, such that

$$\bar{X} \geq g(\bar{X}, \bar{Y}), \; \bar{Y} \geq h(\bar{X}, \bar{Y}).$$

do not exist. Then the following monotone sequences must be unbounded:

$$g(0, 0) \leq g^2(0, 0) \leq \ldots, \; h(0, 0) \leq h^2(0, 0) \leq \ldots.$$

Since $g^k(X_0, Y_0) \geq g^k(0, 0)$ and $h^k(X_0, Y_0) \geq h^k(0, 0)$, it results that for all initial condition X_0, Y_0,

$$\lim_{k \to \infty} g^k(X_0, Y_0) = \infty, \; \lim_{k \to \infty} h^k(X_0, Y_0) = \infty.$$

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