2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Control of a Swinging Juggling Robot

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***Abstract*— We present a control strategy for a robot that juggles a ball with a single actuated paddle that is attached to the tip of a pendulum-like mechanism. The robot juggles the ball from side-to-side by striking the ball with the paddle when the pendulum reaches its peak angles. Sustained juggling is only possible if the pendulum motion is synchronized to the ball motion. We propose adapting the paddle motion to achieve synchronization. Specifically, we exploit the dynamic coupling between the pendulum and the paddle, which is essentially a moving mass at the tip of the pendulum. Optimal control is used to compute paddle motions that synchronize the pendulum to the ball. Feedback is introduced with a lookup table that maps a measured state to an appropriate paddle motion. In experiments, the proposed feedback strategy enables the robot to juggle at various amplitudes.**

I. Introduction

The Swinging Blind Juggler (SBJ) can juggle balls with a single actuated paddle that swings from side-to-side and is attached to the tip of a pendulum. The paddle strikes the ball when the pendulum reaches its peak angle. Then, the pendulum swings to the other side in order to position the paddle for the next strike. A sketch of the SBJ is shown in Fig. 1; the juggling motion of the ball and the pendulum are shown in snapshots in Fig. 2, and a video of the system juggling is available in [1].

The SBJ can juggle without any sensors detecting the ball. In [2], we derived the nominal ball trajectory and showed that the trajectory is locally open-loop stable. Stability is provided by two key design parameters:
1) the parabolic, concave shape of the aluminum paddle, and
2) the decelerating motion of the paddle at ball impact.
These two stabilizing parameters were initially derived for a vertically juggling robot in [3].

For sustained side-to-side juggling, it is crucial to synchronize the swinging motion of the pendulum to the ball motion. In previous work [2], the synchronization was achieved with two electric motors attached to the pivot of the pendulum. In this paper, we propose to use the paddle motion to achieve synchronization. The key idea is to exploit the dynamic coupling between the motion of the paddle and the pendulum. Children on a swing exploit similar dynamics to control their amplitude.

The paddle performs the striking motion at the peak angles of the pendulum. Between strikes, the paddle motion is partially unconstrained and can be used for control. We use optimal control to compute paddle motions that bring the system to the right state at the right time in order to strike the ball. A straightforward feedback strategy is applied to compensate for deviations in pendulum amplitude and phase. For a bounded set of deviations, we precompute paddle motions that compensate for the deviations and bring the system back to the nominal trajectory. The precomputed paddle motions are stored in a lookup table. After striking, the deviation is measured and the appropriate paddle motion is selected from the lookup table and then applied. In experiments, the proposed feedback strategy allowed sustained juggling at various amplitudes and was able to compensate for disturbances.

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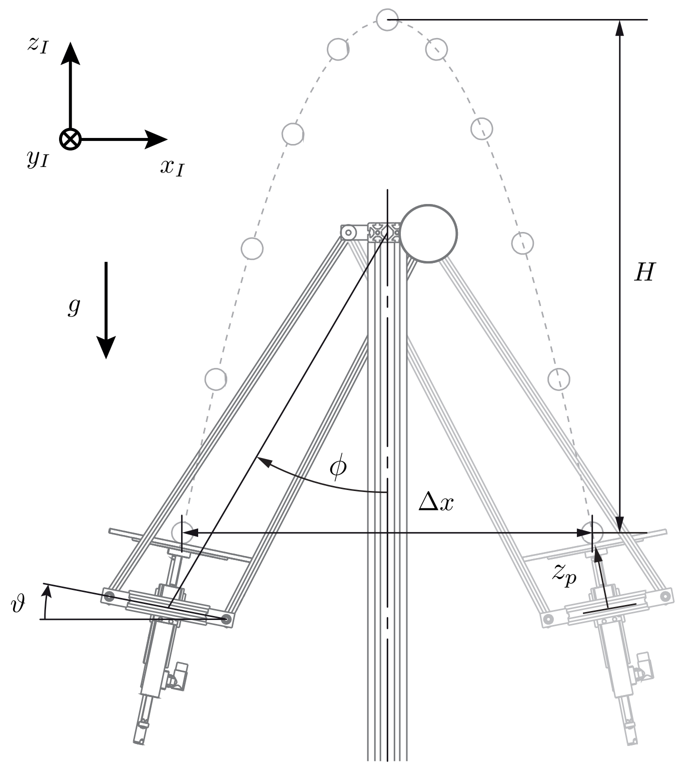

Fig. 1. The Swinging Blind Juggler striking the ball at the left peak angle (black) and the right peak angle (gray). Depicted are: the pendulum angle ϕ; the paddle angle ϑ; the paddle position z_p; The vertical juggling height H; the horizontal juggling distance Δx; gravity g; and the inertial coordinate system.

The dynamics of children on swings and, in particular, how they pump energy into their motion, has been the subject of several studies [4]–[6]. In [4], pumping of a swing has been analyzed from an optimal control point of view. Including the SBJ, there are a range of dynamic systems similar to children on swings, which are studied as challenging nonlinear control problems. Examples include: the swing up of the Acrobot [7]; damping oscillations of a pendulum that features an actuated mass and a translationally actuated pivot point [8]; and tracking reference trajectories with a variable

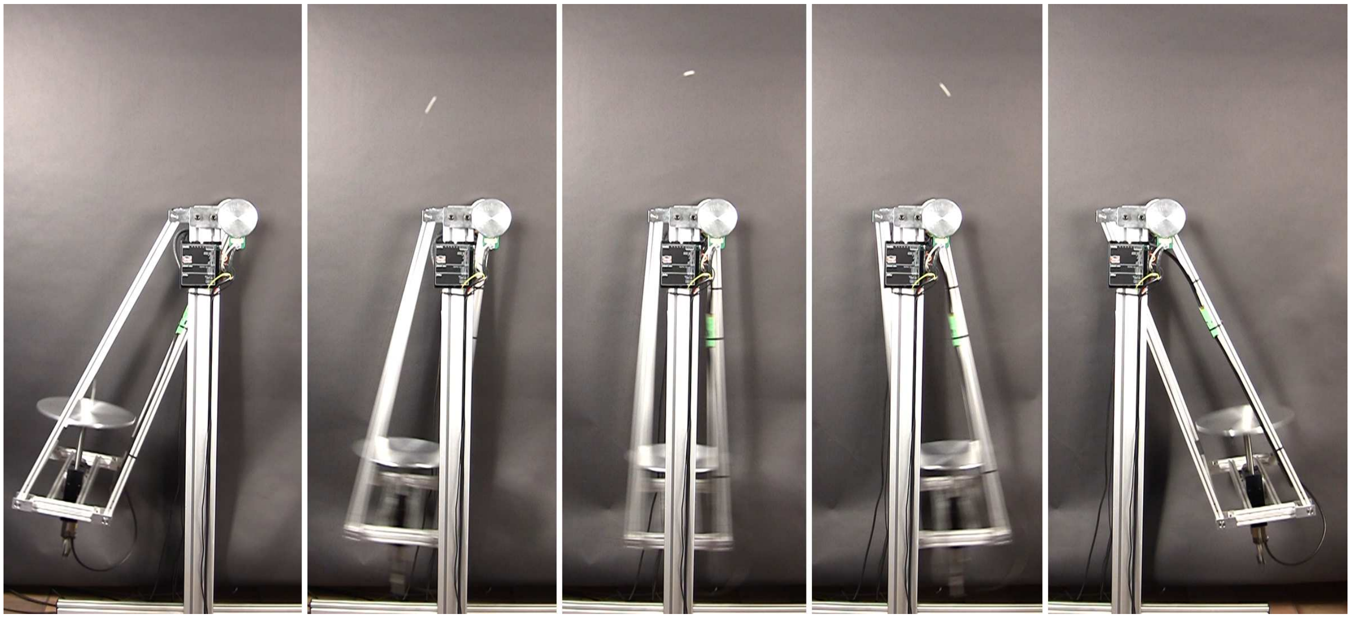

Fig. 2. The Swinging Blind Juggler bouncing a ball at an amplitude of $25°$. The round structures at the top are brushless motors that were used to control the pendulum motion in previous work [2].

length pendulum [9]. Juggling has been studied in robotics as a challenging dexterous task, see, for example, [10]. Similar to the SBJ is the Wiper robot, which consists of two actuated aluminum beams and juggles discs on a tilted air-hockey table [11].

The paper is organized as follows: First, we present the system modeling and identification in Section II. Then, the optimal control problem is introduced in Section III. The feedback strategy is presented in Section IV. Finally, experimental results are discussed in Section V.

II. Dynamic Model and System Identification

A key feature of the SBJ is the four-bar linkage that supports the paddle, see Fig. 1. The link lengths were optimized in [2] such that the resulting paddle angle ϑ is as perpendicular as possible to the ball impact velocity for all pendulum amplitudes Φ up to $30°$, and that the resulting pendulum period matches the flight time of the ball. In the following, we introduce the dynamic model of the SBJ and briefly discuss the system identification procedure.

A. Dynamics

For modeling, the SBJ structure is reduced to three rigid bodies, which are sketched in Fig. 3. Both side-links of the four-bar linkage are treated as a single body K_1. The lower link and the static parts of the linear motor are combined into body K_2, and the paddle and moving parts of the linear motor are represented by body K_3.

We use Lagrange equations of the second kind to derive the system dynamics. Here, only the key points are stated. A Mathematica file providing the full derivation may be found in [1].

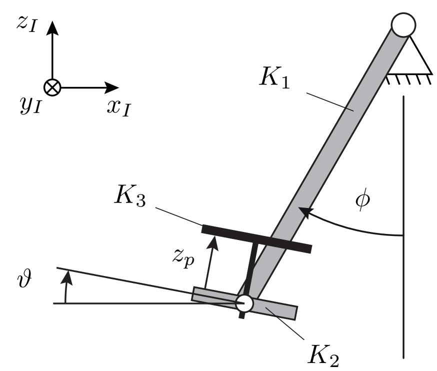

Fig. 3. The reduced model of the SBJ, featuring three rigid bodies.

1) Minimal coordinates: The set of minimal coordinates that describes the system shown in Fig. 3 is $\{\phi(t), \vartheta(t), z_p(t)\}$. In the following, we omit the time dependency for clarity. The four-bar linkage shown in Fig. 1 was designed to keep the paddle perpendicular to the ball velocity at impact. We find that for pendulum angles up to $\phi = 30°$, the relation

$$\vartheta = c\phi, \tag{1}$$

where $c = 0.36$, is a reasonable approximation: The maximal error between the approximated ϑ and the actual angle realized by the four-bar linkage is $0.2°$. With this holonomic constraint, we can remove ϑ from the set of minimal coordinates.

The PD controller that controls the paddle motion runs at a much higher frequency than the pendulum's eigenfrequency. In addition, the linear motor provides enough control authority to track trajectories independent of the pendulum's state. This was confirmed in experiments. In the following, we therefore assume that the paddle is able to perfectly track feasible reference trajectories. Therefore, we treat $\ddot{z}_p$ as a control input and we remove z_p from the set of minimal coordinates. This leaves us with only one minimal coordinate

$$q = \phi, \tag{2}$$

to describe the swinging motion.

2) Lagrange and Nonpotential Torques: The Lagrange equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial E_{kin}}{\partial \dot{q}}\right) - \frac{\partial E_{kin}}{\partial q} + \frac{\partial E_{pot}}{\partial q} = f_{np}, \tag{3}$$

where E_{kin} is the sum of the kinetic energies of the three bodies, and E_{pot} is the sum of the potential energy of the three bodies. The only nonpotential torque is damping

$$f_{np} = c_d \dot{\phi}, \tag{4}$$

that we introduce in order to account for friction in the joints. Solving (3) for $\ddot{\phi}$, we obtain the nonlinear dynamics of the pendulum angle:

$$\ddot{\phi} = f_\phi\left(\phi, \dot{\phi}, z_p, \dot{z}_p, \ddot{z}_p\right). \tag{5}$$

B. System Identification

In order to identify the parameters of the pendulum dynamics (5), i.e. masses, lengths, inertias, and friction, we perform a system identification step. We optimize the parameters to best fit the model to data obtained in experiments with the SBJ. The optimization is seeded with values for the parameters that we obtained from the CAD model of the SBJ. The identification was performed using a nonmoving paddle at different positions z_p; however, we also verified the parameters in an experiment with a moving paddle. The identified parameters are available in [1].

III. The Optimal Control Problem

In order to juggle the ball, the pendulum motion must be synchronized to the flying ball. In this section, we formulate the goal of finding a feasible paddle motion that brings the SBJ to the right state at the right time to strike the ball.

A. Dynamics

We define the system state $s(t)$ and the control input $u(t)$ as

$$s(t) := \begin{bmatrix} \phi(t) \\ \dot{\phi}(t) \\ z_p(t) \\ \dot{z}_p(t) \end{bmatrix}, \quad u(t) := \ddot{z}_p(t). \tag{6}$$

Given the dynamics of the pendulum angle (5), the nonlinear dynamics of the SBJ are

$$\dot{s}(t) = f_s(s(t), u(t)) = \begin{bmatrix} \dot{\phi}(t) \\ f_\phi\left(\phi(t), \dot{\phi}(t), z_p(t), \dot{z}_p(t), u(t)\right) \\ \dot{z}_p(t) \\ u(t) \end{bmatrix}. \tag{7}$$

B. Paddle Constraints

The paddle movement is subject to constraints, which are introduced in this section. We limit the maximal achievable acceleration to

$$|u(t)| \leq 20\,\mathrm{m\,s^{-2}}. \tag{8}$$

We choose a conservative bound such that the perfect trajectory tracking assumption holds for the paddle. The limited stroke of the linear motor limits the paddle position to

$$0\,\mathrm{m} \leq z_p(t) \leq 0.18\,\mathrm{m}. \tag{9}$$

C. Juggling Constraints

In this section, we derive the state constraints that are imposed by juggling a ball.

1) Paddle Striking Motion: First, we introduce the paddle motion required for juggling the ball, i.e. the striking motion. The parameters derived in the following depend on the specific juggling amplitude Φ. However, we omit this dependency in the notation for clarity. The natural pendulum period of the SBJ is T. The resulting nominal ball flight time between two impacts is

$$t_F = \frac{1}{2}T \tag{10}$$

and the nominal apex height is

$$H = \frac{1}{2}g\left(\frac{t_F}{2}\right)^2, \tag{11}$$

where $g = 9.8\,\mathrm{m\,s^{-2}}$ is the gravitational acceleration. The constant, nominal horizontal ball velocity during flight is

$$\dot{x}_b = \frac{\Delta x}{t_F}, \tag{12}$$

where Δx is the horizontal distance that the ball travels between impacts, see Fig. 1. At the maximal specified swing amplitude of $30°$ for the SBJ, the ball travels $\Delta x \approx 1\,\mathrm{m}$. The vertical ball velocity just before impact is

$$\dot{z}_b = -\frac{t_F}{2}g. \tag{13}$$

The magnitude of the impact velocity is

$$|v| = \sqrt{\dot{x}_b^2 + \dot{z}_b^2}. \tag{14}$$

We model damping losses at impact with Newton's impact law

$$|v^+| = e_z|v^-| + (1 + e_z)|\dot{z}_{p,i}|, \tag{15}$$

where $\dot{z}_{p,i}$ is the paddle velocity at impact and the superscripts $^{+}, ^{-}$ denote post- and pre-impact, respectively. The coefficient of restitution e_z was determined experimentally in [12]. In (15), we further assume that the ratio of the ball mass to the paddle mass is negligible. Given that nominally, $|v^+| = |v^-| = |v|$ and $\dot{z}_b < 0$, the paddle velocity at impact is

$$\dot{z}_{p,i} = |v|\frac{1 - e_z}{1 + e_z}. \tag{16}$$

D. Juggling State Constraints

Next, we derive the full system state during the striking motion. In the following, we consider a single juggling motion of the system, where the pendulum swings from the left side ($\phi = \Phi$) to the right side ($\phi = -\Phi$), see Fig. 1. The reverse motion can be derived analogously. We define the nominal impact time on the left side as $t_{i,l}$ and on the right side as $t_{i,r}$. The relation between the two impact times is

$$t_{i,r} = t_{i,l} + t_F = t_{i,l} + \frac{T}{2}. \tag{17}$$

The system states at nominal impact are defined as

$$s_{i,l} := s\left(t_{i,l}\right) = \begin{bmatrix} \Phi \\ 0 \\ z_{p,i} \\ \dot{z}_{p,i} \end{bmatrix}, \quad s_{i,r} := s\left(t_{i,r}\right) = \begin{bmatrix} -\Phi \\ 0 \\ z_{p,i} \\ \dot{z}_{p,i} \end{bmatrix}. \tag{18}$$

The paddle height $z_{p,i}$ may be chosen freely as long as the paddle constraints (8) and (9) are satisfied. The paddle velocity $\dot{z}_{p,i}$ can be calculated with (10)–(16).

In order to account for balls that come in contact with the paddle too early or too late, the striking motion is performed over a time of 2τ. The motion starts τ before the nominal impact and ends τ afterwards. We use $\tau = 0.05\,\mathrm{s}$, which is based on ball impact time statistics as described in [3]. We define the starting times of the left and right striking motions as $t_{i,l}^-$ and $t_{i,r}^-$, and the end times as $t_{i,l}^+$ and $t_{i,r}^+$, respectively. The relation between these times and the nominal impact times is:

$$t_{i,l}^- = t_{i,l} - \tau, \qquad t_{i,l}^+ = t_{i,l} + \tau \tag{19}$$

$$t_{i,r}^- = t_{i,r} - \tau, \qquad t_{i,r}^+ = t_{i,r} + \tau. \tag{20}$$

We showed in [2] that the ball trajectory is locally open-loop stable if the paddle has a specific parabolic shape and if it is decelerating with $\ddot{z}_p = -g/2$ while performing the striking motion. The system state $s(t)$ during the striking motion can be fully defined by this paddle deceleration, the impact states (18), the time τ, and the dynamics (7). We obtain the system state after striking on the left side, and before striking on the right side

$$s_{i,l}^+ := s(t_{i,l}^+), \qquad s_{i,r}^- := s(t_{i,r}^-), \tag{21}$$

by integrating the dynamics (7) forward and backward in time with initial conditions $s_{i,l}$ and $s_{i,r}$, respectively.

E. The Optimal Control Problem

We use optimal control to find a feasible system trajectory that connects the system state after striking on the left side $s_{i,l}^+$ to before striking on the right side $s_{i,r}^-$, while not violating any constraints. Using the dynamics (7), the boundary constraints (21), and the paddle constraints (8) and (9), we formulate the optimal control problem:

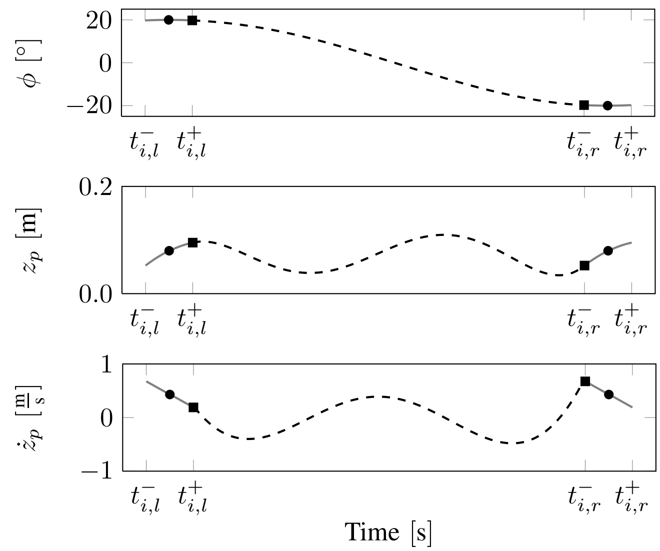

Fig. 4. Nominal trajectory $\bar{s}(t)$ for juggling at $\Phi = 20°$, featuring the striking motion (——) with the nominal impact (●) and the optimized transition (- - -) connecting the boundary constraints (■).

$$\begin{aligned} \underset{u(t)}{\text{minimize}} \quad & \int_{t_{i,l}^+}^{t_{i,r}^-} u(t)^2\, dt \\ \text{subject to} \quad & \dot{s}(t) = f_s\left(s(t), u(t)\right) \\ & s(t_{i,l}^+) = s_{i,l}^+ \\ & s(t_{i,r}^-) = s_{i,r}^- \\ & |u(t)| \le 20\,\mathrm{m\,s^{-2}} \\ & 0\,\mathrm{m} \le z_p(t) \le 0.18\,\mathrm{m}. \end{aligned} \tag{22}$$

To avoid large peaks in the paddle acceleration, we choose to minimize a quadratic cost on the control input. We solve (22) with a direct transcription method implemented according to [13]. We solve the resulting nonlinear program with the sequential quadratic programming package SNOPT [14]. All Matlab files used to solve the problem can be found in [1]. The resulting nominal trajectory, which features the striking motion and the optimized transition trajectory is denoted as $\bar{s}(t)$ and is shown in Fig. 4.

IV. A Feedback Strategy

A feedback strategy is required to compensate for modeling errors and disturbances acting on the system. In this section, we present a straightforward feedback strategy. The main idea is to measure the system state at impact and then choose a precomputed paddle motion that brings the system back to the nominal trajectory $\bar{s}(t)$. For the paddle, we do not consider deviations since we assume that the control loop can perfectly track a given reference trajectory. For the pendulum, we consider deviations at nominal impact time in pendulum angle $\Delta\phi$ and angular velocity $\Delta\dot{\phi}$. Based on empirical data, we consider deviations in the intervals

$$\Delta\phi \in \left[-0.5°, 0.5°\right], \quad \Delta\dot{\phi} \in \left[-3\,°/\mathrm{s}, 3\,°/\mathrm{s}\right]. \tag{23}$$

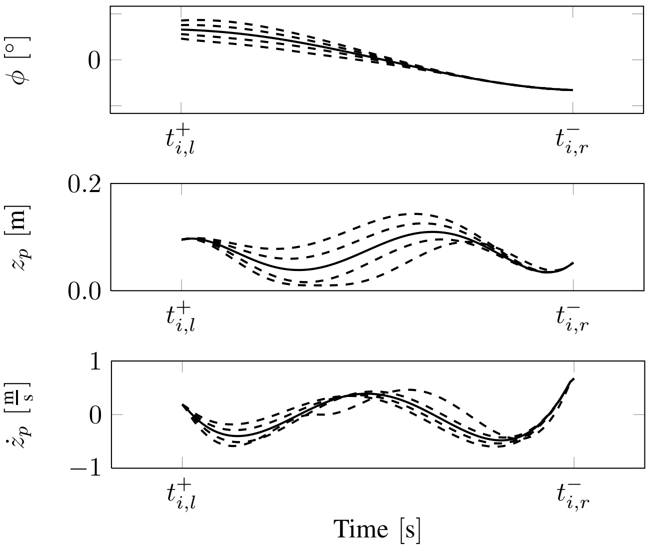

Fig. 5. Selection of trajectories for perturbed initial conditions (- - -) pushing the system back to the nominal trajectory. (——).

We precompute the paddle motions for a discrete set of deviations. Therefore, we discretize the intervals (23):

$$\begin{aligned} \Delta\phi(a) &= -0.5^\circ + a \cdot \frac{1}{6}^\circ \qquad a \in \{0,1,\dots,6\} \\ \Delta\dot{\phi}(b) &= -3\,^\circ/\mathrm{s} + b \cdot 1^\circ/\mathrm{s} \quad\; b \in \{0,1,\dots,6\}. \end{aligned} \tag{24}$$

We define the perturbed system state at nominal impact time using (18):

$$\tilde{s}_{i,l}\,(a,b) := \begin{bmatrix} \Phi + \Delta\phi(a) \\ \Delta\dot{\phi}(b) \\ z_{p,i} \\ \dot{z}_{p,i} \end{bmatrix}. \tag{25}$$

For every combination of $\Delta\phi(a)$ and $\Delta\dot{\phi}(b)$ in (24), we precompute a paddle motion that brings the system from the perturbed state $\tilde{s}_{i,l}\,(a,b)$ back to the nominal trajectory $\bar{s}(t)$. First, we compute the perturbed state at the end of the left striking motion $\tilde{s}^+_{i,l}(a,b)$ by integrating the dynamics (7) starting at $\tilde{s}_{i,l}\,(a,b)$ for τ. In order to find a feasible paddle motion that links $\tilde{s}^+_{i,l}(a,b)$ to the nominal $s^-_{i,r}$, we reformulate the optimal control problem (22) with the new initial condition $s(t^+_{i,l}) = \tilde{s}^+_{i,l}\,(a,b)$. The resulting system trajectory $s(\Delta\phi(a), \Delta\dot{\phi}(b), t)$ is stored in a lookup table. A selection of trajectories is shown in Fig. 5.

For some combinations of $\Delta\phi(a)$ and $\Delta\dot{\phi}(b)$ the final state constraint cannot be satisfied, i.e. no feasible solution to the optimal control problem (22) can be found. In this case, we relax the final state constraint. A small error in pendulum state is acceptable and does not cause the ball to fall off the SBJ. However, we still constrain the final paddle state in order to obtain smooth paddle trajectories and accurately perform the striking motion on the right side. With this approach, we may not find a paddle motion that pushes the system back onto the nominal trajectory in one swing. However, the tracking errors reduce from swing to swing, until they reach an initial state from which the nominal trajectory can be reached again.

We define the final error in the pendulum state as

$$\Delta e_f := \begin{bmatrix} \phi(t^-_{i,r}) \\ \dot{\phi}(t^-_{i,r}) \end{bmatrix} - \begin{bmatrix} \phi^-_{i,r} \\ \dot{\phi}^-_{i,r} \end{bmatrix}, \tag{26}$$

where $\phi^-_{i,r}$ and $\dot{\phi}^-_{i,r}$ are the nominal pendulum angle and velocity at the beginning of the right striking motion. The partial relaxation of the final state constraint on the pendulum angle and angular velocity is achieved with a quadratic cost on Δe. The adapted optimal control problem reads

$$\begin{aligned} \underset{u(t)}{\text{minimize}} \quad & \Delta e_f^\intercal Q_f \Delta e_f + \int_{t^+_{i,l}}^{t^-_{i,r}} u(t)^2\, dt \\ \text{subject to} \quad & \dot{s}(t) = f_s\,(s(t), u(t)) \\ & s(t^+_{i,l}) = \tilde{s}^+_{i,l}\,(a,b) \\ & z_p(t^-_{i,r}) = z^-_{p,i,r} \\ & \dot{z}_p(t^-_{i,r}) = \dot{z}^-_{p,i,r} \\ & |u(t)| \le 20\,\mathrm{m\,s^{-2}} \\ & 0\,\mathrm{m} \le z_p(t) \le 0.18\,\mathrm{m}, \end{aligned} \tag{27}$$

where $z^-_{p,i,r}$ and $\dot{z}^-_{p,i,r}$ are the nominal paddle position and velocity at the beginning of the right striking motion, respectively. Q_f is a matrix for tuning the trade-off between minimizing the control effort and achieving the nominal pendulum state. We found that

$$Q_f = \begin{bmatrix} 10^4 & 0 \\ 0 & 10^4 \end{bmatrix} \tag{28}$$

resulted in an acceptable trade-off.

In practice with the SBJ, we measure the deviation in angle $\Delta\hat{\phi}$ and angular velocity $\Delta\dot{\hat{\phi}}$ at nominal impact. Then we select the precomputed paddle motion from the lookup table, which brings the system back to the nominal trajectory $\bar{s}(t)$. We only precompute paddle motions for a discrete set of deviations (24). Therefore, we apply the precomputed trajectory $s(\Delta\phi(\hat{a}), \Delta\dot{\phi}(\hat{b}), t)$ according to

$$\begin{aligned} \hat{a} &= \arg \min_{a \in \{0,1,\dots,6\}} |\Delta\hat{\phi} - \Delta\phi(a)| \\ \hat{b} &= \arg \min_{b \in \{0,1,\dots,6\}} |\Delta\dot{\hat{\phi}} - \Delta\dot{\phi}(b)|. \end{aligned} \tag{29}$$

We considered only paddle motions where the system swings from the left side to the right side. However, the problem is symmetric and the same paddle motions can by applied on the swing back from the right side to the left side.

V. Experimental Results And Discussion

The performance of the proposed feedback strategy is evaluated in experiments. In the following, we present and discuss the experimental data.

A. Tracking Performance

The feedback strategy presented in this paper achieved robust tracking of the nominal trajectory $\bar{s}(t)$. Precise tracking at ball impact is crucial for juggling. Therefore, we evaluate the tracking error at nominal impact for $\Phi = 20^\circ$ over $60\,\mathrm{s}$, i.e. 64 impacts. The mean of the error in pendulum angle is $+0.05°$ with a standard deviation of $0.18°$. The mean of the error in angular velocity is $-0.36\,\frac{°}{s}$ with a standard deviation of $1.01\,\frac{°}{s}$. In experiments, the resulting tracking performance enabled the SBJ to continuously juggle a ball.

B. Juggling At Various Amplitudes

The presented feedback strategy is not limited to a single juggling amplitude Φ. We precomputed the lookup table presented in Section IV for the following amplitudes:

$$\Phi \in \{20°, 21°, \ldots, 28°, 28.5°, \ldots, 30°\}. \tag{30}$$

It was not only possible to juggle at the various amplitudes in (30), but to change the amplitude reference while juggling. To change the amplitude, we simply activated the lookup table for the new Φ. In experiments, the feedback strategy handled the step in the reference signal without problems. We adapted Φ every 10th impact in small increments according to (30). The quasi-static change in the amplitude reference ensured that the ball was able to follow and did not fall off the robot. We found that at larger amplitudes $\Phi > 28°$, we had to reduce the step size from $1°$ to $0.5°$ in order to sustain juggling.

C. Disturbance Rejection

In experiments without a ball, the feedback strategy was further able to handle larger disturbances, which we tested by manually pushing the pendulum. Experimental data is shown in Fig. 6. The deviations after introducing the disturbance were $\Delta\phi = 3.9°$ and $\Delta\dot{\phi} = -53.5\,\frac{°}{s}$. Although the deviations were outside the interval (23) for which we precomputed paddle motions, the proposed feedback strategy was able to push the system back to the nominal trajectory. A video showing the disturbance rejection can be found in [1].

D. Swing Up

We also tested the proposed feedback strategy to swing up from the hanging position. We found that for small pendulum amplitudes, the strategy could not control the pendulum. We suspect that this is mainly due to modeling errors in combination with limited control authority because the dynamic coupling between the paddle and the pendulum is a second-order effect. For small pendulum amplitudes Φ, the resulting angular velocities are small. For small angular velocities, the dynamic coupling between the pendulum and the paddle motion is weak. We will address the swing up with the paddle in future work.

For the experiments presented here, we used the brushless motors at the pivots, see Fig. 2, to swing up to $\Phi = 20°$. Once the desired amplitude was reached, the brushless motors at the pivots were disabled and the swinging motion was controlled solely by the paddle.

VI. Acknowledgements

The authors would like to thank Matthew Donovan for his great help with the mechanical design of the Swinging Blind Juggler.

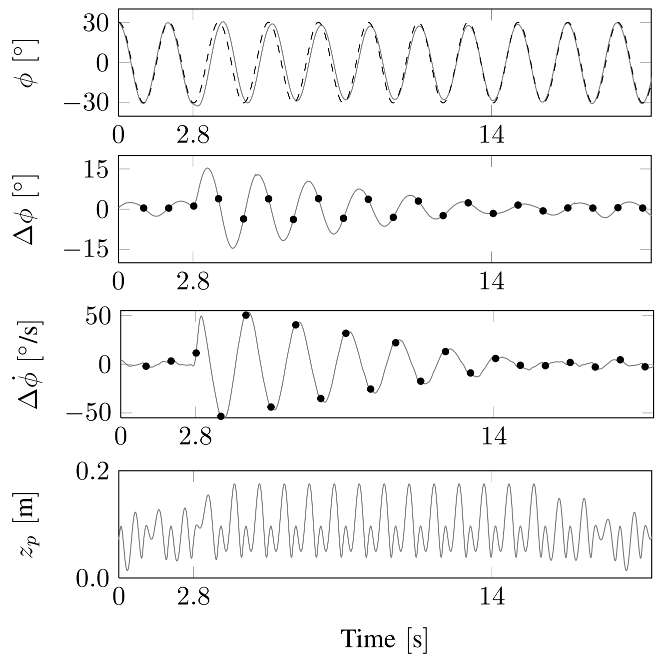

Fig. 6. The SBJ reacting to a push at 2.8 s. The paddle motion adapts to bring the measured state (——) back to the nominal trajectory (– – –). The nominal impact is depicted with (•).

References

[1] "Swinging Blind Juggler Website." [Online]. Available: http://www.blindjuggler.org/the-swinging-blind-juggler/

[2] P. Reist and R. D'Andrea, "Design of the Pendulum Juggler," in *Proc. IEEE Conf. Robotics and Automation (ICRA)*, 2011, pp. 5154 – 5159.

[3] ——, "Design and Analysis of a Blind Juggling Robot," *IEEE Transactions on Robotics*, vol. 28, no. 6, pp. 1228–1243, 2012.

[4] B. Piccoli and J. Kulkarni, "Pumping a swing by standing and squatting: do children pump time optimally?" *Control Systems, IEEE*, vol. 25, no. 4, pp. 48–56, 2005.

[5] J. Burns, "More on pumping a swing," *American Journal of Physics*, vol. 38, pp. 920–922, 1970.

[6] W. B. Case, "The pumping of a swing from the standing position," *American Journal of Physics*, vol. 64, pp. 215–220, 1996.

[7] M. Spong, "The swing up control problem for the acrobot," *Control Systems, IEEE*, vol. 15, no. 1, pp. 49–55, 1995.

[8] S. Okanouchi, K. Yoshida, I. Matsumoto, and H. Kawabe, "Damping Angular Oscillations of a Pendulum under State Constraints," in *18th IFAC World Congress*, Milano, Italy, 2011, pp. 7735–7742.

[9] X. Xin, T. Shinji, T. Yamasaki, and C. Sun, "Trajectory Tracking Control of Pendulum with Variable Length by Partial Energy Shaping," in *18th IFAC World Congress*, Milano, Italy, 2011, pp. 10 679–10 684.

[10] M. Buehler, D. E. Koditschek, and P. J. Kindlmann, "Planning and Control of Robotic Juggling and Catching Tasks," *I. J. Robotic Res.*, vol. 13, no. 2, pp. 101–118, 1994.

[11] R. Ronsse, P. Lefevre, and R. Sepulchre, "Rhythmic Feedback Control of a Blind Planar Juggler," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 790 –802, 2007.

[12] P. Reist and R. D'Andrea, "Bouncing an Unconstrained Ball in Three Dimensions with a Blind Juggling Robot," in *Proc. IEEE Conf. Robotics and Automation (ICRA)*, 2009, pp. 1774 – 1781.

[13] J. T. Betts, *Practical methods for optimal control and estimation using nonlinear programming*, 2nd ed. Society for Industrial and Applied Mathematics, 2010.

[14] P. E. Gill, W. Murray, and M. A. Saunders, "SNOPT: An SQP Algorithm for Large-Scale Constrained Optimization," *SIAM Review*, vol. 47, no. 1, pp. 99–131, 2005.