INFORMATIVE MPC FORMULATIONS FOR ORTHONORMAL LAGUERRE SERIES MIMO MODELS

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Abstract

An informative model predictive control (MPC) algorithm is presented. The formulation builds upon recent advances in the field that extend certainty equivalence formulations with semi-definite constraints. The proposed adaptive control algorithm and the previous methods are presented in the context of fixed state transition dynamics generated by a finite orthonormal Laguerre basis. The development of an implementable convex relaxation is presented along with an illustrative example.

Keywords

Adaptive MPC, Dual Control, Information

Introduction

The control of uncertain systems require the consideration of conflicting tasks. Feldbaum (1961) formalized this problem with the term Dual Control in order to recognize the simultaneous need of system identification and control. Some of the recent developments in linear MPC under uncertainty have been aimed towards the robust stability of the controller with respect to model uncertainty (Zeilinger et al., 2014). On the other hand, strategies that aim to enforce excitation that leads to uncertainty reduction have been proposed by others (Marafioti et al., 2014; Larsson, 2014; Heirung et al., 2015). The scope of this article is the analysis of informative model predictive control formulations with linear state transition dynamics specified by a finite orthonormal series. First, we introduce the system model and the properties that enable an augmented problem definition that includes the propagation of information with respect to the uncertain parameters. With the explicit quantification of the exploratory value of the closed loop input sequence, constraints for an adaptive MPC formulation are defined. We conclude with the definition of a new constraint that combines distinct advantages of previous methods.

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System Model

We consider a MIMO system with n_y outputs and n_u inputs. Each MISO sub-system is given by the following output error (OE) LTI dynamics:

$$\begin{aligned} x_{t+1}^{ij} &= A^{ij} x_t^{ij} + b^{ij} u_t^j, \quad \forall j \in [1, 2, \dots, n_u] \\ y_t^i &= c^i(\theta^*) x_t^i + w_t^i, \quad \forall i \in [1, 2, \dots, n_y] \end{aligned} \tag{1}$$

where $x_t^i = \left[\left(x_t^{i1}\right)^\top, \left(x_t^{i2}\right)^\top, \dots, \left(x_t^{in_u}\right)^\top \right]$. The state transition matrix and the input vector for the j^{th} input and i^{th} output pair, y^i and u^j, are specified by a generating pole of a finite Laguerre series, a^{ij},:

$$A^{ij} = \begin{bmatrix} a^{ij} & 0 & \dots & 0 \\ (1 - {a^{ij}}^2) & a^{ij} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ (-a^{ij})^{n_x^{ij}-2}(1 - {a^{ij}}^2) & \dots & (1 - {a^{ij}}^2) & a^{ij} \end{bmatrix}$$

$$b^{ij} = \sqrt{1 - {a^{ij}}^2} \begin{bmatrix} 1 & -a^{ij} & \dots & (-a^{ij})^{n_x^{ij}-1} \end{bmatrix}^\top$$

The output is a linear function of the state given by a vector of unknown parameters, θ^*, that define all MISO state-output row vectors, c^i, and a white noise signal, w^i. By an appropriate arrangement of the model components for the MISO sub-systems, an overall MIMO state-space of the following form is constructed:

$$
\begin{aligned}
x_{t+1} &= Ax_t + Bu_t, \qquad x \in \mathbb{R}^{n_x}, n_x = \sum_{i=1}^{n_y}\sum_{j=1}^{n_u} n_x^{ij} \\
y_t &= C(\theta^*)x_t + w_t
\end{aligned} \tag{2}
$$

The control design in the following sections is given under this system definition. In general, it is applicable to any other orthonormal basis, such as Kautz series, that generate the state transition dynamics.

MPC Problem

The MPC problem is slightly modified from the definition by Larsson (2014) to include a quadratic cost for the terminal state. An objective function for tracking of piecewise linear output trajectories is given by

$$
\begin{aligned}
J(\hat{x}_0, \bar{u}) \coloneqq &\sum_{k=1}^{N-1} \mathbb{E}\left[\|y_k - r_k\|_Q^2\right] + \\
&\sum_{k=0}^{N-1} (\|\hat{u}_k\|_R^2 + \|\Delta\hat{u}_k\|_S^2) + \|\hat{x}_N - \hat{x}_N^r\|_P^2
\end{aligned} \tag{3}
$$

where $(\hat{\ })$ denotes the projected trajectories for the receding horizon window. According to the model definition in Eq. (2), the predicted output contribution to the objective function is nondeterministic since the state transition dynamics are fixed and noise free. Since future estimates are functions of output realizations that are not available in the present, expected output trajectories are defined as conditional expectations for a fixed parameter vector, θ, current state, $\hat{x}_0 = x_t$, and future input signals $\bar{u} \coloneqq \left[\hat{u}_0^\top \hat{u}_1^\top \dots \hat{u}_{N-1}^\top\right]^\top$:

$$
\hat{y}_k \coloneqq \mathbb{E}\left[y_k|\theta, \hat{x}_0, \bar{u}\right] = C(\theta)\hat{x}_k
$$

if the initial condition for the state is also uncertain, the current Kalman filter estimate, $\hat{x}_0 = \hat{x}_{t|t}$, can be used instead.

In order to define a deterministic optimization problem a decision must be made in how to model future output predictions. A certainty equivalence (CE) predictor, takes a parameter vector estimate as if it contained the actual values for the process. On the other hand, a cautious predictor includes the level of uncertainty defined by the current covariance of the parameter estimate as if it remained unchanged regardless of future realizations. Both of this limiting simplifications fail to capture important features of the actual behavior for parameter estimate distributions under sequential information gains.

Deterministic Propagation of Information

A model for the parameter-error is required for the expected value of the output contribution in Eq. (3):

$$
\begin{aligned}
\mathbb{E}\left[\|y_k\|^2\right] &\simeq \mathbb{E}\left[\|y_k\|^2|\theta, \hat{x}_0, \bar{u}\right] \\
&= \mathbb{E}\left[\|\hat{y}_k + y_k - \hat{y}_k\|^2|\theta, \hat{x}_0, \bar{u}\right] \\
&= \mathbb{E}\left[\|\hat{y}_k\|^2 + \|e_k\|^2 + \|w_k\|^2|\theta, \hat{x}_0, \bar{u}\right]
\end{aligned} \tag{4}
$$

where the model error is given by parameter and noise contributions (e_k and w_k respectively). The last equality in Eq. (4) holds if $\mathbb{E}\left[e_k\right] = \mathbb{E}\left[y_k - \hat{y}_k\right] = 0$. This requires a parameter estimate distribution such that $\mathbb{E}\left[\theta - \theta^*\right] = 0$. The maximum likelihood estimator (MLE), $\hat{\theta}_t$, which is equivalently obtained by recursive least squares (RLS) for Eq. (2), satisfies this condition. It holds asymptotically, as the number of recorded input-output pairs increases, that $\mathcal{I}_1^t(\theta^*)^{1/2}(\hat{\theta}_t - \theta^*) \to \mathcal{N}(0, I)$. The information matrix is defined in terms of the output sensitivities and the diagonal noise variance matrix Λ_w,

$$
\begin{aligned}
\mathcal{I}_1^t(\theta^*) &\coloneqq \sum_{k=1}^{t} \mathbb{E}\left[\left(\frac{\partial\hat{y}_k}{\partial\theta^*}\right)\Lambda_w^{-1}\left(\frac{\partial\hat{y}_k}{\partial\theta^*}\right)^\top\right] \\
\frac{\partial\hat{y}_k}{\partial\theta} &\coloneqq \left[\frac{\partial\hat{y}_k}{\partial\theta_1} \quad \frac{\partial\hat{y}_k}{\partial\theta_2} \quad \dots \quad \frac{\partial\hat{y}_k}{\partial\theta_{n_\theta}}\right]^\top
\end{aligned} \tag{5}
$$

and the sensitivities for a given parameter vector are obtained from uncertain contributions from the output matrix and the estimated trajectory for the states:

$$
\frac{\partial\hat{y}_k}{\partial\theta_l} = \frac{\partial C(\theta)}{\partial\theta_l}\hat{x}_k(\theta) + C(\theta)\frac{\partial\hat{x}_k(\theta)}{\partial\theta_l} = \frac{\partial C(\theta)}{\partial\theta_l}\hat{x}_k
$$

where $l = 1, 2, \dots, n_\theta$. For system Eq. (2), the uncertain state contribution is zero and the sensitivities reduce to $n_\theta \times n_y$ matrices of state vector components which evolve deterministically. Furthermore, they are not functions of the parameters but the parameter locations in $C(\theta)$ instead. This means that information propagates deterministically and the expected contribution of the prediction error in the MPC objective function can be approximated by the Cramer-Rao bound defined by the predicted information matrix.

$$
\mathbb{E}\left[\|\hat{y}_k\|^2 + \|e_k\|^2|\hat{\theta}_t, \hat{x}_0, \bar{u}\right] \simeq \|C(\hat{\theta}_t)\hat{x}_k\|^2 + \|\hat{x}_k^\theta\|_{\hat{\Sigma}_k}^2 \tag{6}
$$

where $\hat{x}_k^\theta$ is the subset of elements of $\hat{x}_k$ that correspond to the uncertain parameter entries in C. The projected covariance bound, $\hat{\Sigma}_k$, has observed and projected contributions and is obtained as the inverse of the information matrix:

$$
\begin{aligned}
\hat{x}_k^{i,\theta} &\coloneqq \frac{\partial c^i(\theta)}{\partial\theta^i}\hat{x}_k^i, \; \hat{x}_k^\theta \coloneqq \left[(\hat{x}_k^{1,\theta})^\top \, (\hat{x}_k^{2,\theta})^\top \, \dots \, (\hat{x}_k^{n_y,\theta})^\top\right]^\top \\
\hat{\Sigma}_k &\coloneqq \left(\hat{\mathcal{I}}_1^{t+k}\right)^{-1} = \left(\mathcal{I}_1^t + \hat{\mathcal{I}}_{t+1}^{t+k}\right)^{-1}
\end{aligned}
$$

with θ^i defined as the subset of elements of θ contained in c^i. Note that for Eq. (2), the information matrix (and its inverse) has a block diagonal structure, with each block corresponds to the information for a MISO subsystem with respect to θ.

$$\left(\hat{\mathcal{I}}_1^{t+k}\right)_i = \left(\mathcal{I}_1^t\right)_i + \frac{1}{\lambda^i}\sum_{k'=1}^{k}\left(\hat{x}_{k'}^{i,\theta}\right)\left(\hat{x}_{k'}^{i,\theta}\right)^\top \tag{7}$$

λ^i is a diagonal element in Λ_w corresponding to the variance for w^i. The deterministic projected parameter error contribution is given by

$$\|\hat{x}_k^\theta\|^2_{\hat{\Sigma}_k} = \sum_{i=1}^{n_y}\left(\hat{x}_k^{i,\theta}\right)^\top\left(\hat{\Sigma}_k\right)_i\left(\hat{x}_k^{i,\theta}\right) \tag{8}$$

This term is nonconvex as it requires the nonlinear propagation of information given by the matrix inversion lemma applied to one-step recursions defined by Eq. (7). Heirung et al. (2015) introduced a reformulation that results on a quadratically constrained quadratic program (QCQP) by defining an auxiliary variable, $\hat{z}_k$. Under the framework developed here, an equivalent reformulation is presented:

$$\hat{z}_k^i = \left(\hat{\Sigma}_k\right)_i\left(\hat{x}_k^{i,\theta}\right),\ \hat{z}_k = \left[\left(\hat{z}_k^1\right)^\top\ \left(\hat{z}_k^2\right)^\top\ \ldots\ \left(\hat{z}_k^{n_y}\right)^\top\right]^\top$$

At each time index k, the parameter error contribution from each MISO system in terms of $\hat{z}_k$ is fully defined by the following set of bilinear equations:

$$\begin{aligned}\|\hat{x}_k^\theta\|^2_{\hat{\Sigma}_k} &= \sum_{i=1}^{n_y}\left(\hat{x}_k^{i,\theta}\right)^\top\hat{z}_k\\ \left(\hat{\mathcal{I}}_1^{t+k}\right)_i\hat{z}_k^i &= \hat{x}_k^{i,\theta}\\ \left(\hat{\mathcal{I}}_1^{t+k}\right)_i &= \left(\hat{\mathcal{I}}_1^{t+k-1}\right)_i + \frac{1}{\lambda^i}\left(\hat{x}_k^{i,\theta}\right)\left(\hat{x}_k^{i,\theta}\right)^\top\end{aligned} \tag{9}$$

for $k = 1, 2, \ldots, N-1$ and $i = 1, 2, \ldots, n_y$. A deterministic objective function that explicitly accounts for the cost of uncertainty in the parameters related to Eq. (3) is formulated.

$$\begin{aligned}J^D(\hat{\theta}_t, \hat{x}_0, \bar{u}) := &\sum_{k=1}^{N-1}\|\hat{y}_k - r_k\|^2_Q +\\ &\sum_{k=1}^{N-1}\sum_{i=1}^{n_y}\left(\hat{x}_k^{i,\theta}\right)^\top\hat{z}_k^i +\\ &\sum_{k=0}^{N-1}(\|\hat{u}_k\|^2_R + \|\Delta\hat{u}_k\|^2_S) +\\ &\|\hat{x}_N - \hat{x}_N^r\|^2_P\end{aligned} \tag{10}$$

Note that the contribution from the process noise signal has been omitted since it is assumed to be a constant and does not affect the optimization result. The first three contributions define the exploitation, exploration, and caution components of dual control as originally described by Feldbaum (1961). The last term accounts for the cost-to-go which can be approximated by the solution of the LQR problem, via the discrete algebraic Riccati equation.

The minimization of Eq. (10) subject to the constraints Eq. (9) is nonconvex due to the bilinear equalities. Although it can be formulated and solved with global optimization algorithms such as BARON (Tawarmalani and Sahinidis, 2005), its solution is NP-hard and even under a feasible initialization with a CE solution, an optimal input sequence with respect to Eq. (10) is not guaranteed in polynomial time. The complexity of the problem, in terms of the number of equality constraints, increases quadratically with respect to the size of θ and linearly with the receding horizon window size N. In the following section, the formulation of convex relaxations derived from simplified strategies to guarantee informative closed-loop trajectories is presented and analyzed with respect to Eq. (2).

Informative Control Strategies

In order to simplify the analysis and provide a baseline for the performance of the informative formulations, a 'lifted' optimization problem with respect to the CE objective function is presented (Larsson, 2014):

$$\begin{aligned}J^{CE}(\theta, \hat{x}_0, \bar{u}) = &\sum_{k=1}^{N-1}\|\hat{y}_k - r_k\|^2_Q +\\ &\sum_{k=0}^{N-1}(\|\hat{u}_k\|^2_R + \|\Delta\hat{u}_k\|^2_S) + \|\hat{x}_N - \hat{x}_N^r\|^2_P\\ = &\,\bar{u}^\top\mathcal{Q}(\theta)\bar{u} + \eta_t(\theta)^\top\bar{u} + \text{constant}\end{aligned}$$

where matrix $\mathcal{Q}(\theta) \in \mathbb{S}^{Nn_u}$, and the vector $\eta_t(\theta) \in \mathbb{R}^{Nn_u}$ are defined by the receding horizon window size and the system dynamics:

$$\begin{aligned}\mathcal{Q}(\theta) &:= \Upsilon^\top\Xi(\theta)^\top(I_N \otimes Q)\Xi(\theta)\Upsilon + I_N \otimes R +\\ &\quad\ \mathcal{D}^\top(I_N \otimes S)\mathcal{D} + \Upsilon_N^\top P\Upsilon_N\\ \eta_t(\theta) &:= \left[2(\Xi(\theta)\Psi\hat{x}_0 - \bar{r}_t)^\top(I_N \otimes Q)\Xi\Upsilon -\right.\\ &\quad\ \left.2\bar{u}_{t-1}^\top(I_N \otimes S)\mathcal{D} + 2((A)^N\hat{x}_0 - \hat{x}_N^r)^\top P\Upsilon_N\right]^\top\end{aligned}$$

where $\otimes$ denotes the Kronecker product. The dependence of η_t on the initial condition, $\hat{x}_0$, and the horizon tracking trajectory, $\bar{r}_t$, is made implicit by the subscript

t. Υ, Ψ, and Ξ are constructed from the sequence

$$\hat{y}_k = C(\theta)\left((A)^k \hat{x}_0 + \sum_{k'=1}^{k} (A)^{k-k'} B \hat{u}_{k'-1}\right) \tag{11}$$

for $k = 1, 2, \dots, N$. The matrix Υ_N corresponds to the last n_x rows of Υ, while the vector $\bar{u}_{t-1}$ and the matrix $\mathcal{D}$ define the vector of projected values for $\Delta\hat{u}_k$:

$$\begin{aligned}\Delta\bar{u} &\coloneqq \left[(\hat{u}_0 - u_{t-1})^\top\, (\hat{u}_1 - \hat{u}_0)^\top \dots (\hat{u}_N - \hat{u}_{N-1})^\top\right]^\top \\ &= \mathcal{D}\bar{u} - \bar{u}_{t-1}\end{aligned}$$

which leads to the definition of the following optimal control problem:

$$\begin{aligned}\min_{\bar{u}} \quad & \bar{u}^\top \mathcal{Q}(\theta)\bar{u} + \eta_t(\theta)^\top \bar{u} \\ \text{s.t.} \quad & \bar{u} \in \widehat{\mathcal{U}} \\ & \Upsilon\bar{u} + \Psi\hat{x}_0 \in \widehat{\mathcal{X}} \\ & \Xi(\theta)(\Upsilon\bar{u} + \Psi\hat{x}_0) \in \widehat{\mathcal{Y}}\end{aligned} \tag{12}$$

with input, state, and output box constraints defined by $\widehat{\mathcal{U}}$, $\widehat{\mathcal{X}}$, and $\widehat{\mathcal{Y}}$ respectively. The optimization Eq. (12) is reduced to the minimum number of free variables and is a quadratic program (QP) for linear inequality constraints.

One approach to generate information by the closed-loop trajectory defined by Eq. (12) would be to modify the reference signal, $\bar{r}_t$ to introduce the desired level of excitation. Here, we limit the control design problem to the case where this is fixed and does not provide sufficient excitation. Larsson (2014) and Marafioti et al. (2014) introduced two approaches applicable in this situation. Both designs share the inclusion of a bilinear matrix inequality (BMI) that ensures an informative measure. From a practical point of view, this is an indirect approach to steer the exploratory contribution in Eq. (10) to zero by producing input sequences with information content which result in the tightening of the parameter covariance. Unlike linear matrix inequalities (LMIs), BMIs describe sets that are not necessarily convex, which make them harder to handle computationally (VanAntwerp and Braatz, 2000). The informative BMI constraints are relaxed by the introduction of a symmetric matrix variable related to $\bar{u}$ via a LMI defined by the Schur complement lemma (Manchester, 2010):

$$U = \bar{u}\bar{u}^\top \text{ is relaxed to } U \succeq \bar{u}\bar{u}^\top$$

which in a Schur complement form, is equivalent to dropping the rank constraint from

$$\begin{bmatrix} U & \bar{u} \\ \bar{u}^\top & 1 \end{bmatrix} \succeq 0 \quad \text{and} \quad \text{rank}\begin{bmatrix} U & \bar{u} \\ \bar{u}^\top & 1 \end{bmatrix} = 1$$

MPC-X (Larsson, 2014)

The projected information blocks in Eq. (7) for a horizon N_I is rewritten in terms of $\bar{u}$:

$$\begin{aligned}\left(\hat{\mathcal{I}}_{t+1}^{t+N_I}\right)_i = \frac{1}{\lambda^i}\frac{\partial c^i(\theta)}{\partial\theta^i}\sum_{k=1}^{N_I}\Big(&\Psi_k^i\hat{x}_0^i\hat{x}_0^{i\top}\Psi_k^{i\top} + \Psi_k^i\hat{x}_0^i\bar{u}_I^\top\Upsilon_k^{i\top} \\ &+\Upsilon_k^i\bar{u}_I\hat{x}_0^{i\top}\Psi_k^{i\top} + \Upsilon_k^i\bar{u}_I\bar{u}_I^\top\Upsilon_k^{i\top}\Big)\frac{\partial c^i(\theta)}{\partial\theta^i}^\top\end{aligned}$$

with the matrices Ψ^i, Υ^i, given by the rows of Eq. (11) corresponding to a single output, $\hat{y}^i$. $\bar{u}_I$ corresponds to the first $N_I n_u$ elements of $\bar{u}$. An experiment design constraint in Eq. (12) is constructed from the projected contribution defined above, the observed information matrix, and a target matrix-valued measure

$$\left(\mathcal{I}_1^t\right)_i + \left(\hat{\mathcal{I}}_{t+1}^{t+N_I}\right)_i \succeq \kappa_t\mathcal{I}_i^* \tag{13}$$

the time varying constant on the right hand is defined such that $\kappa_t \to 1$ for a finite $t = T_I$ (*e.g.* $(t+N_I)/T_I$). $\mathcal{I}^*$ can be derived from a probabilistic measure obtained by an approximation of the Hessian of a predefined application cost function. Defining $N_\theta \coloneqq \max\text{rank}\,(\mathcal{I}_i^*)$, feasibility of Eq. (13) requires $N_\theta \le N_I \le N$.

PE-MPC (Marafioti et al., 2014)

The mechanism for the persistent excitation (PE) formulation is to guarantee sufficiently rich inputs. This is achieved by sequentially subtracting rank 1 contributions of previous inputs from a matrix known to satisfy an excitation measure. A constraint that ensures $\hat{u}_0$ compensates for this subtraction is then introduced in the CE formulation. Define a vector of past inputs of order N_θ and its corresponding sufficient input richness matrix of order $N_\Omega \ge N_\theta$:

$$\phi_t \coloneqq \left[u_t^\top u_{t-1}^\top \dots u_{t-(N_\theta-1)}^\top\right]^\top, \quad \Omega_t \coloneqq \sum_{p=0}^{N_\Omega-1}\phi_{t-p}\phi_{t-p}^\top$$

for $\Omega_{t-1} \succeq \rho I_{N_\theta n_u}$,

$$\Omega_t \succeq \rho I_{N_\theta n_u} \iff \Gamma_t + \beta_t u_t^\top + u_t\beta_t^\top + \alpha_t u_t u_t^\top \succeq \rho I_{n_u}$$

where the PE parameters (matrix Γ_t, the vector β_t, and the scalar α_t) are defined by the Schur complement of Ω_t with respect to its upper principal minor corresponding to the recorded input sequence. Finally, with a suitable initialization of the parameters, the following PE BMI constraint is enforced at each t:

$$\Gamma_t + \beta_t\hat{u}_0^\top + \hat{u}_0\beta_t^\top + \alpha_t\hat{u}_0\hat{u}_0^\top \succeq \rho I_{n_u} \tag{14}$$

Informative MPC

An implementable constraint is formulated for both approaches in terms of Eq. (12) by including the relaxed version of the corresponding BMI. Note that Eq. (13) and Eq. (14) share a similar structure with the right hand side fixed by a desired performance measure and the left hand side is a function of constant, linear, and bilinear contributions in terms of $\bar{u}$. The X formulation only requires the storage of the observed information and yields a set of n_y BMIs with dimension $N_I n_u$ each. The PE formulation requires the additional storage of $N_\theta + N_\Omega - 1$ previous controls while the BMI dimension corresponds to the number of inputs in the system, n_u. The higher problem size is the cost of explicitly accounting for a multi-step filtering of the inputs through the state transition dynamics. Conversely, the PE approach requires an arbitrary parameter initialization which defines the closed-loop trajectory.

The lower bound on N_I in Eq. (13) is the implicit requirement of an increment of all eigenvalues which can only occur with at least N_θ control actions. This is problematic not only because it results in high ordered BMIs, but most importantly because the informative content in the implemented input is not guaranteed. By modifying the right side of the BMI to a rank 1 matrix, a one-step informative constraint is defined:

$$\frac{\partial c^i(\theta)}{\partial \theta^i}\left(A^i \hat{x}_0^i \hat{x}_0^{i\top} A^{i\top} + A^i \hat{x}_0^i \hat{u}_0^\top B^{i\top} + B^i \hat{u}_0 \hat{x}_0^{i\top} A^{i\top} + B^i \hat{u}_0 \hat{u}_0^\top B^{i\top}\right) \frac{\partial c^i(\theta)}{\partial \theta^i}^\top \succeq \lambda^i \kappa \left(\nu_t \nu_t^\top\right) \tag{15}$$

where ν_t iterates among the vectors of a basis for $\mathbb{R}^{N_\theta}$ (*e.g.* the columns of an identity matrix) and the matrices A^i and B^i generate the state transition for the corresponding MISO system. Assuming that only input box constraints are present yields the following optimization problem

$$\begin{aligned}
\min_{\bar{u},U} \quad & \mathrm{Tr}(\mathcal{Q}(\theta)U) + \eta_t(\theta)^\top \bar{u} \\
\text{s.t.} \quad & \begin{bmatrix} U & \bar{u} \\ \bar{u}^\top & 1 \end{bmatrix} \succeq 0 \\
& U_{kj,kj} - u^j_{min}\bar{u}_{kj} - u^j_{max}\bar{u}_{kj} \succeq u^j_{min} u^j_{max} \\
& \text{for } kj = 0, 1, \ldots, N n_u \\
& \text{Eq. (15)} \\
& \text{for } i = 0, 1, \ldots, n_y
\end{aligned} \tag{16}$$

where the objective function and the constraints have been reformulated in terms of U. The operator Tr() is in the form of a general real-valued linear function on $\mathbb{S}^{N n_u}$ (Boyd and Vandenberghe, 2004). Since LMI constraints define convex sets, Eq. (16) is a convex problem. Furthermore, it is a linear conic optimization problem over the linear cone and the semidefinite cones defined by the dimension of their respective LMI. Output and state constraints can be equivalently formulated according to the expressions in Eq. (12). This new constraint set has the advantage of taking into account the state transition dynamics with n_y BMIs of order n_u, achieving the appealing features of the previously discussed approaches. In practical terms, it enforces the systematic tightening of the parameter covariance for all MISO blocks one direction at a time. A simplified sketch for the associated adaptive control algorithm is listed below

Algorithm

1. Define prior knowledge $(\hat{x}_0, \mathcal{I}_1^1, \hat{\theta}_1)$, MPC problem formulation $(P, Q, R, S, N, \bar{r}_1)$, system dynamics (A, B, C), and informative parameters $(\kappa, \nu, \frac{\partial c^i(\theta)}{\partial \theta^i})$. Set $1 \to t$
2. Solve Eq. (16), if infeasible solve Eq. (12)
3. Implement $\hat{u}_0$ and measure y_{t+1}. Update the state, the predicted output, and the sensitivities.
4. Update $\hat{\theta}_t$ and $\mathcal{I}_1^t$ with RLS,
$$\hat{\theta}_{t+1} = \hat{\theta}_t + K_{t+1}\left(y_{t+1} - C(\hat{\theta}_t)x_{t+1}\right)$$
$$K_{t+1} = \left(\mathcal{I}_1^{t+1}\right)^{-1} \frac{\partial \hat{y}_{t+1}}{\partial \theta^*}$$
$$\mathcal{I}_1^{t+1} = \mathcal{I}_1^t + \left(\frac{\partial \hat{y}_{t+1}}{\partial \theta^*}\right) \Lambda_w^{-1} \left(\frac{\partial \hat{y}_{t+1}}{\partial \theta^*}\right)^\top$$
5. Update P, $\bar{r}_t$, and set $t + 1 \to t$. Return to 2.

Example

Consider a MIMO system with $n_y = n_u = 2$. The state transition dynamics are generated by $a^{11} = 0.6, a^{12} = -0.1, a^{21} = 0.2$, and $a^{22} = -0.5$, each SISO subsystem has two states. The MISO output matrix is given by an uncertain parameter vector, $\theta^* \in \mathbb{R}^8$:

$$c^1 = [\theta_1^* \quad \theta_2^* \quad \theta_3^* \quad \theta_4^*] = [0.65 \quad -0.35 \quad 0.45 \quad 0.25]$$
$$c^2 = [\theta_5^* \quad \theta_6^* \quad \theta_7^* \quad \theta_8^*] = [-0.70 \quad 0.30 \quad 0.50 \quad 0.40].$$

A tracking experiment ($r_t^1 = 0.5, r_t^2 = -0.5 \forall t$) is formulated for 100 steps with $N = 20, Q = I_2, R = S = 0.01Q$. Outputs are assumed to have known noise variances

$\lambda^1 = \lambda^2 = 0.001$. Upper and lower input bounds defined by $u^1_{max} = 3, u^2_{max} = 2, u^1_{min} = -1, u^2_{min} = -2$,. The excitation vector, ν_t, iterates among the columns of I_4. Common initial state vector, noise sequence, and guess for the parameters are used for all experiments.

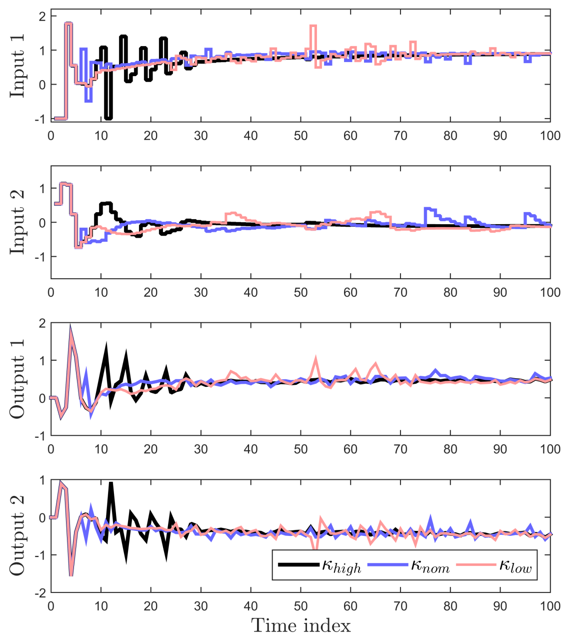

Figure 1. Example input/output trajectories

The resulting optimal control problem solved at each time index is a linear conic optimization with 40 linear variables, and 3 semi-definite constraints solved with MOSEK (MOSEKApS, 2015). The solution time is less than a second for the default tolerance for all feasible instances. When infeasible, the QP is solved to define a control action. The input/output performance of the informative algorithm for 3 different levels of information ($\kappa_{low} = 1 \times 10^{-5}, \kappa_{nom} = 1 \times 10^{-4}, \kappa_{high} = 1 \times 10^{-3}$) is shown in Figure 1. Figure 2 displays the solution feasibility of Eq. (16).

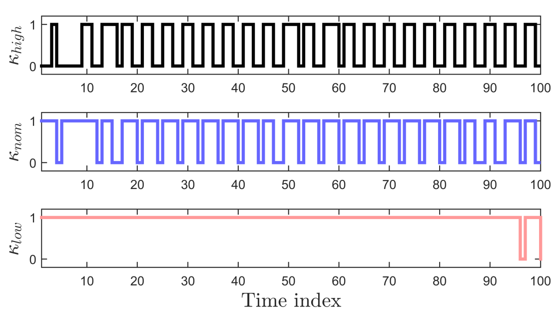

Figure 2. Feasibility of Eq. (16)

As expected, the number of infeasible instances increases with κ. The QP solution was implemented in a total of 2, 29, and 50 steps for the low, nominal, and high levels respectively.

Conclusions

Based on the observed number of infeasible instances, the results suggest the need for adaptation of the information level parameter κ. The minimum time formulation by Larsson (2014) addresses this observation by maximizing the informative content of the input signal with respect to a fixed deterioration of the QP objective function. The informative MPC algorithm presented here can be equivalently modified.

Given the fixed state-transition dynamics, excitation could be maximized constrained by a pre-defined Lyapunov function in terms of the states instead. The exact approach on how to accomplish this will be the subject of future work.

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