Robust Optimization with Data Driven Asymmetric Uncertainty Set Construction

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Abstract

In this paper, we introduced a novel method for asymmetric uncertainty set construction based on the distributional information of sampling data. Deterministic robust counterpart optimization formulation is derived for D-norm induced uncertainty set with the proposed method. Furthermore, the asymmetric set induced robust optimization model is compared with the classical symmetric set induced robust optimization model. A numerical example and a reactor design problem are investigated. The results demonstrate that using asymmetric uncertainty set leads to less conservative robust solution.

Keywords

Robust optimization, Asymmetric uncertainty set, Data-driven method, Robust counterpart

Introduction

As a modeling framework for immunizing against uncertainty in mathematical optimization, robust optimization has received lots of attention in recent years. Robust optimization relies on appropriately defining an uncertainty set, and solving a deterministic robust counterpart, to ensure worst-case feasibility over the uncertainty set.

A general guideline for uncertainty set construction in robust optimization is that it should not lead to overly conservative or computationally challenging deterministic robust counterpart formulations. Traditionally, uncertainty set has been defined as symmetric type. Soyster (1973) introduced the interval based box type uncertainty set. Ben-Tal and Nemirovski (2000) introduced ellipsoidal type of uncertainty set for robust linear optimization. Bertsimas et al. (2004) introduced general norm induced uncertainty set for robust linear optimization. This is the most general type of symmetric uncertainty set, which can lead to various symmetric set under different type of norms. Li et al. (2011) and Li et al. (2012) made a comparative study of various symmetric set induced robust optimization models and their probabilistic guarantees. Yuan et al. (2016) studied robust linear optimization under correlated uncertainty, and demonstrated the advantage of introducing correlation information into the uncertainty set construction.

To capture the asymmetric distribution, Chen et al. (2007) introduced deviation measures to capture distributional asymmetry, however they assumed that the primitive uncertain parameters are independent. In this paper, we introduced a novel method for asymmetric uncertainty set construction based on the distributional information or sampling data of uncertain parameters. Correlation between primitive uncertain parameters can be captured in the proposed method. Deterministic robust counterpart formulation is derived based on the proposed uncertainty set. We also derived the specific robust formulation under D-norm, which leads to a linear optimization problem.

In the subsequent sections, we first present the general norm-induced symmetric uncertainty set and the robust counterpart. Then, we present the proposed asymmetric uncertainty set construction method and demonstrate some data-driven uncertainty set examples. Next, the robust counterpart optimization constraint is derived. Furthermore, the proposed asymmetric set induced robust optimization model is compared to the classical symmetric set induced model. A numerical ex-

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ample and a reactor design problem are investigated, both demonstrate that using asymmetric uncertainty set leads to less conservative robust solution.

Robust optimization with symmetric uncertainty set

Consider an optimization problem with uncertain linear constraints. Without loss of generality, assume there are p number of primitive uncertain parameters $(\xi_1, \cdots, \xi_p)$ in all the uncertain constraints. After rearrangement (and introducing auxiliary variable if necessary), we can get following general i-th linear constraint of an uncertain optimization optimization problem: $y_0^i + \sum_{k=1}^{p} \xi_k y_k^i \leq 0$. Consider the vector form of the constraint

$$y_0^i + \xi^T y^i \leq 0 \tag{1}$$

where $y^i = [y_1^i, \cdots, y_p^i]^T$, $\xi = [\xi_1, \cdots, \xi_p]^T$. The robust constraint is formulated as

$$y_0^i + \max_{\xi \in U} \xi^T y^i \leq 0 \tag{2}$$

to ensure worst-case feasibility under the uncertainty set U for uncertain parameters ξ.

Symmetric uncertainty set

A general norm-induced symmetric uncertainty set can be formulated as

$$U = \left\{\xi | \|M(\xi - \bar{\xi})\| \leq \Delta\right\} \tag{3}$$

where $M \in \mathbb{R}^{p \times p}$ is a invertible matrix, $\bar{\xi}$ represents the nominal value of the uncertain parameters, Δ is the set size. The norm $\|\cdot\|$ in the above formulation can be arbitrary vector norm. In the literature, l_p-norm has been a popular choice. For a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and a parameter p $(p \geq 1)$, the standard l_p-norm is defined as $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. Its special cases include: l_1-norm $\|x\|_1 = \sum_{i=1}^n |x_i|$; l_2-norm $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$; l_∞-norm: $\|x\|_\infty = \max_i |x_i|$. Another useful norm is the so-called D-norm introduced by Bertsimas and Sim (2004). For a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and a parameter Γ $(1 \leq \Gamma \leq n)$, D-norm is defined as:

$$\|x\|_\Gamma^D = \max_{\{S \cup \{t\} | S \subseteq N, |S| \leq \lfloor \Gamma \rfloor, t \in N \setminus S\}} \sum_{i \in S} |x_i| + (\Gamma - \lfloor \Gamma \rfloor)|x_t| \tag{4}$$

where $N = \{1, \cdots, n\}$. Note that D-norm can be reduced to special cases of l_p-norm: $\|x\|_{\Gamma=1}^D = \|x\|_\infty$, $\|x\|_{\Gamma=n}^D = \|x\|_1$.

Matrix M is used in the above uncertainty set model to scale the uncertain parameters. It should be selected as an invertible matrix. Typically it can be selected as follows:

- If ξ_i are bounded with $\xi_i \in [\bar{\xi}_i - d_i, \bar{\xi}_i + d_i]$, we can set $M = diag\left\{\frac{1}{d_1}, \cdots, \frac{1}{d_p}\right\}$
- If ξ_i are unbounded, but with known variance information, we can use the covariance matrix Σ of ξ and set $M = \Sigma^{-\frac{1}{2}}$

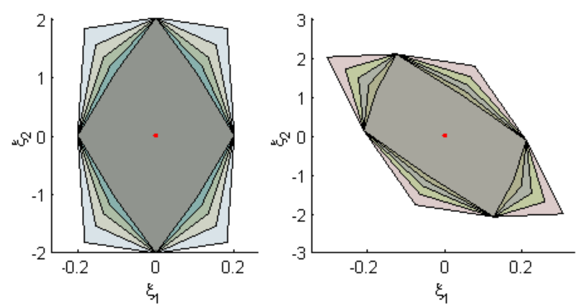

Figure 1. Symmetric uncertainty set (left: for independent uncertainty; right: for correlated uncertainty)

Figure 1 shows the D-norm induced uncertainty set (3), with $\bar{\xi} = [0, 0]$, set size $\Delta = 1$ and different Γ value $\{1.1, 1.3, 1.5, 1.7, 1.9\}$. The left figure is generated with $M = [5, 0; 0, 0.5]$ which corresponds to independent uncertainty with $d_1 = 0.2, d_2 = 2$, whereas the right figure is for $M = [5, 0.3; 0.3, 0.5]$, which corresponds to correlated uncertainty with covariance matrix $\Sigma = [0.451, -0.066; -0.066, 1.439]$. For both figures, the largest set on the figure is for $\Gamma = 1.1$, and the smallest set is for $\Gamma = 1.9$.

Robust counterpart constraint formulation

Property 1. *Under the general norm-induced uncertainty set* (3)*, the robust constraint* (2) *is equivalent to*

$$y_0^i + \bar{\xi}^T \mathbf{y}^i + \Delta \|M^{-T} \mathbf{y}^i\|^* \leq 0 \tag{5}$$

where $\|\cdot\|^*$ *is the dual norm.*

Proof. Define $s = \frac{M(\xi - \bar{\xi})}{\Delta}$, then $\xi = \Delta M^{-1} s + \bar{\xi}$, and the uncertainty set is equivalent to $\{s | \|s\| \leq 1\}$, the inner maximization problem in (2) can be evaluated as

$$\max_{\xi \in U} \xi^T y^i = \max_{s:\|s\| \leq 1} \Delta (M^{-1} s)^T y^i + \bar{\xi}^T y^i$$
$$= \bar{\xi}^T y^i + \Delta \max_{s:\|s\| \leq 1} s^T (M^{-T} y^i) = \bar{\xi}^T y^i + \Delta \|M^{-T} y^i\|^*$$

Note the last equality is based on the definition of dual norm. □

For l_p-norm, its dual norm is given as $\|x\|_p^* = \|x\|_q$, with $q = 1 + \frac{1}{p-1}$. Specially: $\|x\|_1^* = \|x\|_\infty$; $\|x\|_2^* = \|x\|_2$; $\|x\|_\infty^* = \|x\|_1$. The dual norm of D-norm is $\|x\|_\Gamma^{D*} = \max\left\{\|x\|_\infty, \frac{1}{\Gamma}\|x\|_1\right\}$, as shown in Bertsimas et al. (2004). For an uncertain linear constraint, when the uncertainty set is a polyhedron, the robust counterpart is linear. In this work, we focus on the norm that leads to polyhedral uncertainty set which is linear programming representable, such that the robust counterpart constraint is still linear. Using the D-norm (which include the l_1-norm and l_∞-norm as special cases), we can get the following robust counterpart formulation.

Property 2. *Under D-norm $\|\cdot\|_\Gamma^D$ induced uncertainty set* (3)*, the robust counterpart* (2) *is equivalent to*

$$\begin{cases} y_0^i + \bar{\xi}^T \mathbf{y}^i + \Delta \cdot z \le 0 \\ z \ge u_k, k = 1, \cdots, p \\ z \ge \frac{1}{\Gamma}\sum_{k=1}^p u_k \\ -u_k \le t_k \le u_k, k = 1, \cdots, p \\ t = M^{-T}\mathbf{y}^i \end{cases} \tag{6}$$

Proof. Apply D-norm and introduce auxiliary variable z, the constraint (2) becomes

$$\begin{cases} y_0^i + \bar{\xi}^T \mathbf{y}^i + \Delta \cdot z \le 0 \\ z \ge \|t\|_\Gamma^{D*} \\ t = M^{-T}\mathbf{y}^i \end{cases}$$

Apply the dual D-norm $\|t\|_\Gamma^{D*} = \max\left\{\|t\|_\infty, \frac{1}{\Gamma}\|t\|_1\right\}$,

$$\begin{cases} y_0^i + \bar{\xi}^T \mathbf{y}^i + \Delta \cdot z \le 0 \\ z \ge \|t\|_\infty \\ z \ge \frac{1}{\Gamma}\|t\|_1 \\ t = M^{-T}\mathbf{y}^i \end{cases}$$

Apply the definition of l_1-norm, l_∞-norm, it leads to

$$\begin{cases} y_0^i + \bar{\xi}^T \mathbf{y}^i + \Delta \cdot z \le 0 \\ z \ge |t_k|, k = 1, \cdots, p \\ z \ge \frac{1}{\Gamma}\sum_{k=1}^p |t_k| \\ t = M^{-T}\mathbf{y}^i \end{cases}$$

Lastly, linearize the absolute value term $|t_k|$ using new variable u_k, then we get the robust counterpart. □

Asymmetric uncertainty set

While the uncertain parameter follow asymmetric distribution, an asymmetric uncertainty set is more appropriate for robust optimization. This will be shown by examples in the subsequent section. Two types of asymmetric uncertainty sets are presented in this section to address independent and correlated primitive uncertain parameters, respectively.

Independent primitive uncertainty

Consider a random vector $\xi = (\xi_1, \cdots, \xi_p) \in \mathbb{R}^p$, assume its nominal value is $\bar{\xi}$. Assume the parameters are independent. Define the positive and negative perturbation part of ξ respectively as:$\xi^+ = \max\{\xi - \bar{\xi}, 0\}, \xi^- = \max\{-(\xi - \bar{\xi}), 0\}$, then we have the following relationship $\xi - \bar{\xi} = \xi^+ - \xi^-$. This can be easily verified: if $\xi - \bar{\xi} \ge 0$, then $\xi^+ - \xi^- = \max\{\xi - \bar{\xi}, 0\} - \max\{-(\xi - \bar{\xi}), 0\} = \xi - \bar{\xi} - 0 = \xi - \bar{\xi}$; if $\xi - \bar{\xi} \le 0$, then $\xi^+ - \xi^- = \max\{\xi - \bar{\xi}, 0\} - \max\{-(\xi - \bar{\xi}), 0\} = 0 - (-(\xi - \bar{\xi})) = \xi - \bar{\xi}$. Consider the following norm-induced uncertainty set proposed by Chen et al. (2007)

$$U = \left\{ \xi \;\middle|\; \begin{array}{l} \xi = \bar{\xi} + (\xi^+ - \xi^-), \xi^+ \ge 0, \xi^- \ge 0 \\ \|P\xi^+ + Q\xi^-\| \le \Delta \end{array} \right\} \tag{7}$$

where $P \in \mathbb{R}^{p\times p}$, $Q \in \mathbb{R}^{p\times p}$, and $\|\cdot\|$ is a general norm. We are specially interested in the following selection of P and Q:

- If ξ_i are bounded with $\xi_i \in [\bar{\xi}_i - d_i^-, \bar{\xi}_i + d_i^+]$, that is, $0 \le \xi_i^+ \le d_i^+$ and $0 \le \xi_i^- \le d_i^-$, we set $P = diag\left\{\frac{1}{d_1^+}, \cdots, \frac{1}{d_p^+}\right\}, Q = diag\left\{\frac{1}{d_1^-}, \cdots, \frac{1}{d_p^-}\right\}$
- If ξ_i are unbounded, but with known variance information for ξ_i^+ and ξ_i^-. Assume the standard deviation for ξ_i^+ is σ_i^+ and the standard deviation for ξ_i^- is σ_i^-, then we can set $P = diag\left\{\frac{1}{\sigma_1^+}, \cdots, \frac{1}{\sigma_p^+}\right\}, Q = diag\left\{\frac{1}{\sigma_1^-}, \cdots, \frac{1}{\sigma_p^-}\right\}$

Correlated primitive uncertainty

Consider a random vector $\xi = (\xi_1, \cdots, \xi_p) \in \mathbb{R}^p$, assume its nominal value is $\bar{\xi}$, and covariance matrix is Σ. Define $\eta = \xi - \bar{\xi}$, then η also has covariance matrix Σ. Assume the unitary matrix of eigenvectors of matrix Σ is Φ, define $\mu = \Phi^T\eta$, then μ_i are independent and the covariance matrix of μ is a diagonal matrix with its main diagonal element being the eigenvalues of Σ. This is a de-correlation procedure on random vector η. Define the positive and negative perturbation part of μ respectively as: $\mu^+ = \max\{\mu, 0\}, \mu^- = \max\{-\mu, 0\}$, then we have the following relationship $\mu = \mu^+ - \mu^-$. Consider the following norm-induced uncertainty set

$$U = \left\{ \xi \;\middle|\; \begin{array}{l} \Phi^T(\xi - \bar{\xi}) = \mu^+ - \mu^-, \mu^+ \ge 0, \mu^- \ge 0 \\ \|P\mu^+ + Q\mu^-\| \le \Delta \end{array} \right\} \tag{8}$$

where $P \in \mathbb{R}^{p\times p}$, $Q \in \mathbb{R}^{p\times p}$, and $\|\cdot\|$ is a general norm. For the independent primitive uncertainty case, the selection of P and Q is based on ξ^+, ξ^-. Similarly to the construction of uncertainty set (7), we consider the following selection of P and Q based on the distribution of μ^+, μ^-. It is obvious that uncertainty set (8) is reduced to the special case of (7) for independent uncertainty with $\Phi = I$.

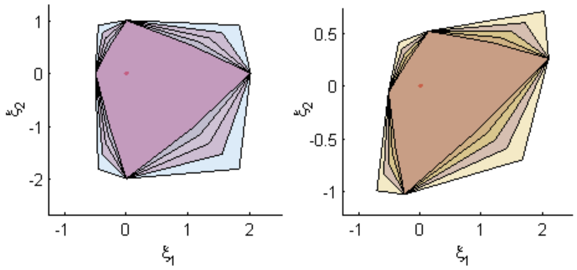

Figure 2. Asymmetric uncertainty set (left: for independent uncertainty; right: for correlated uncertainty

Figure 2 shows the D-norm induced asymmetric uncertainty set (7) and (8), with set size $\Delta = 1$ and Γ takes value $\{1.1, 1.3, 1.5, 1.7, 1.9\}$. The left figure is based on $d_1^+ = 2, d_2^+ = 1, d_1^- = 0.5, d_2^+ = 2$, whereas the right figure is based on $\Phi = [1, -0.25; -0.25, 2]$,$\sigma_1^+ = 2, \sigma_2^+ = 1, \sigma_1^- = 0.5, \sigma_2^+ = 2$.

Data-driven uncertainty set construction

While the exact distributional information of uncertainty may not be available in practical applications, data-driven method can be applied for uncertainty set construction. The following examples demonstrated the method.

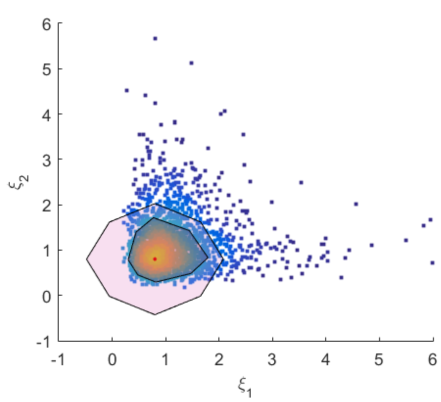

Figure 3. Data-driven symmetric and asymmetric set (for independent uncertainty, with marginal lognormal distributions)

In the first example, 2000 samples were generated from two random variables ξ_1 and ξ_2 using independent lognormal distribution $log\mathcal{N}(0, 0.25)$. Assume uncertainty sets are constructed around the nominal point $\bar{\xi} = [0.8, 0.8]$. The matrix M used for symmetric set, and Φ, P, Q used for asymmetric set are estimated from the data following the procedure introduced in the previous section. Figure 3 shows the D-norm (with $\|\cdot\|^D_{\Gamma=1.5}$) induced symmetric and asymmetric uncertainty with set size $\Delta = 2.0$.

Next, we consider a correlated distribution case. Assume ξ_1 follows a Gamma distribution $\Gamma(2, 1)$ and ξ_2 follows a t distribution with 5 degrees of freedom. Similarly, 2000 samples were generated. Assume uncertainty sets are to be constructed around the nominal point $\bar{\xi} = [1, -0.5]$. Figure 4 shows the D-norm (with $\|\cdot\|^D_{\Gamma=1.5}$) induced symmetric and asymmetric uncertainty sets with size $\Delta = 1.0$, respectively.

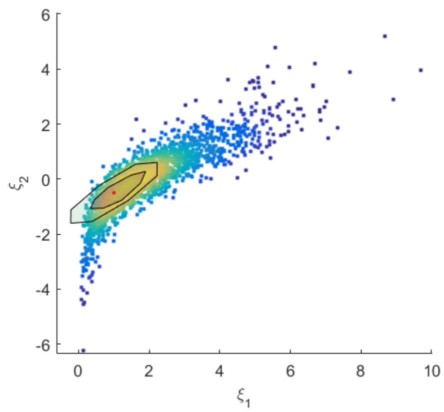

Figure 4. Data-driven symmetric and asymmetric set (for correlated uncertainty, with marginal Gamma and t distributions)

In the above figures, the polyhedra represent the uncertainty sets constructed. Colored scatter plot of the data was displayed to show the density of the distribution. The figures show that the asymmetric uncertainty sets better capture the correlation between two uncertain parameters. It is observed that the symmetric set is larger while using the same set size and Γ parameter of D-norm. Furthermore, while the symmetric uncertainty set unnecessarily covered some low density region, asymmetric uncertainty set fits better to the joint distribution. This will lead to less conservative robust solution, as demonstrated later by case studies.

Robust optimization with asymmetric uncertainty set

In this section, the deterministic robust counterpart based on the proposed new asymmetric uncertainty set (8) is derived. The uncertainty set under general norm is first studied, then the result is extended to the special D-norm case.

Property 3. *Under the uncertainty set* (8)*, the robust* *constraint* (2) *is equivalent to*

$$\begin{cases} y_0^i + (y^i)^T\bar{\xi} + \Delta\|t\|^* \leq 0 \\ t \geq P^{-1}\Phi^{-1}y^i \\ t \geq -Q^{-1}\Phi^{-1}y^i \\ t \geq 0 \end{cases} \quad (9)$$

Proof. First, the inner maximization problem $\max_{\xi\in U} \xi^T y^i$ is equivalent to $\max_{\xi\in U}(y^i)^T\xi$, or

$$\begin{aligned} \max_{\xi,\mu^+,\mu^-} \quad & (y^i)^T\xi \\ s.t. \quad & \Phi^T(\xi-\bar{\xi}) = \mu^+ - \mu^- \\ & \|P\mu^+ + Q\mu^-\| \leq \Delta \\ & \mu^+ \geq 0, \mu^- \geq 0 \end{aligned}$$

Notice that $\xi = \bar{\xi} + \Phi^{-T}(\mu^+ - \mu^-)$, so we have

$$\begin{aligned} \max_{\mu^+,\mu^-} \quad & (y^i)^T\bar{\xi} + (y^i)^T\Phi^{-T}\mu^+ - (y^i)^T\Phi^{-T}\mu^- \\ s.t. \quad & \|P\mu^+ + Q\mu^-\| \leq \Delta \\ & \mu^+ \geq 0, \mu^- \geq 0 \end{aligned}$$

Define $v = P\mu^+$, $w = Q\mu^-$, based on the fact that P and Q are both diagonal matrices with all positive diagonal elements, we have

$$\begin{aligned} \max_{v,w} \quad & (y^i)^T\bar{\xi} + (y^i)^T\Phi^{-T}P^{-1}v - (y^i)^T\Phi^{-T}Q^{-1}w \\ s.t. \quad & \|v + w\| \leq \Delta \\ & v \geq 0, w \geq 0 \end{aligned}$$

Rewrite the objective function, we get

$$\begin{aligned} \max_{v,w} \quad & (y^i)^T\bar{\xi} + (P^{-1}\Phi^{-1}y^i)^T v - (Q^{-1}\Phi^{-1}y^i)^T w \\ s.t. \quad & \|v + w\| \leq \Delta \\ & v \geq 0, w \geq 0 \end{aligned}$$

the optimal objective is $(y^i)^T\bar{\xi} + \Delta\|t\|^*$, with

$$t_j = \max\{(P^{-1}\Phi^{-1}y^i)_j, (-Q^{-1}\Phi^{-1}y^i)_j, 0\}$$

Finally the robust counterpart is obtained using the above results. □

Property 4. *Under D-norm* $\|\cdot\|_\Gamma^D$ *induced uncertainty set* (8), *the robust counterpart* (2) *is equivalent to*

$$\begin{cases} y_0^i + (y^i)^T\bar{\xi} + \Delta \cdot z \leq 0 \\ z \geq u_k, k = 1, \cdots, p \\ z \geq \frac{1}{\Gamma}\sum_{k=1}^p u_k \\ -u_k \leq t_k \leq u_k, k = 1, \cdots, p \\ t \geq P^{-1}\Phi^{-1}y^i \\ t \geq -Q^{-1}\Phi^{-1}y^i \\ t \geq 0 \end{cases} \quad (10)$$

The proof is similar to the proof of Property 2 after applying the D-norm and introducing auxiliary variable z. Detailed procedure is skipped for simplicity.

Case studies

Numerical example

Consider the following example

$$\begin{aligned} \min_{x_1\geq 0, x_2\geq 0} \quad & 2x_1 + 3x_2 \\ s.t. \quad & (2+\xi_1)x_1 + 6x_2 \geq 180 \\ & 3x_1 + (3.4 - \xi_2)6x_2 \geq 162 \\ & x_1 + x_2 \leq 100 \end{aligned}$$

The two uncertain constraints can be rearranged as $y_0^i + \sum_{k=1}^{2} \xi_k y_k^i \leq 0, i = 1, 2$ with

$$\begin{cases} y_0^{(1)} = 180 - 2x_1 - 6x_2, y_1^{(1)} = -x_1, y_2^{(1)} = 0 \\ y_0^{(2)} = 162 - 3x_1 - 20.4x_2, y_1^{(2)} = 0, y_2^{(2)} = 6x_2 \end{cases}$$

Assume ξ_1 follows a Gamma distribution $\Gamma(2,1)$,ξ_2 follows a t distribution with 5 degrees of freedom, and subject to correlation. Computational studies are performed on the robust counterpart optimization model under different set sizes for the two different uncertainty sets. Results in Figure 5 show that the symmetric set based solution is more conservative than the asymmetric set induced solution.

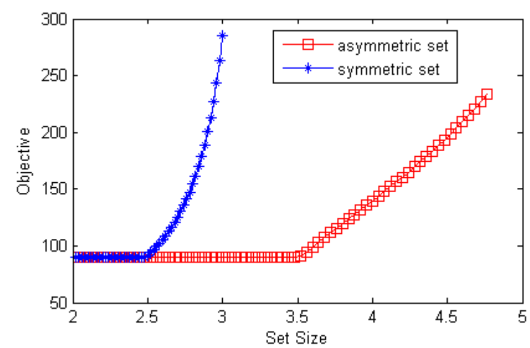

Figure 5. Results for numerical example

Reactor design problem

Consider a reaction-separation process shown in Figure 6. Material A is fed into a reactor where it reacts to materials B and C at an uncertain conversion ratio k. B and C are then separated to satisfy product demand D_B and D_C, which are both uncertain also. The nominal values for uncertain parameters $\xi = [k, D_B, D_c]$ are $\bar{\xi} = [0.6, 7, 4]$. Assume k follows independent normal distribution $\mathcal{N}(0.6, 0.01)$, D_B and D_C follows correlated distribution with marginal lognormal distribution $log\mathcal{N}(7, 0.01)$ and $log\mathcal{N}(4, 0.01)$, respectively.

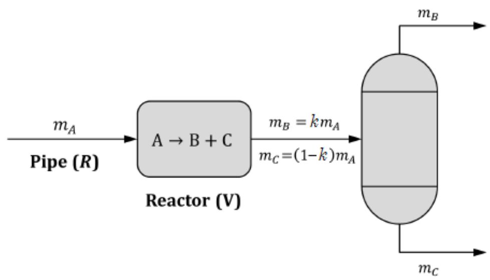

Figure 6. Reactor-separator process

The reactor design problem is formulated as

$$\begin{aligned} \min_{m_A,V,R>0} \quad & 5V + R \\ s.t. \quad & 0.2V \leq m_A \leq V \\ & m_A - R \leq 0 \\ & -30 - m_A + 0.8V + 1.2R \leq 0 \\ & -km_A + D_B \leq 0 \\ & -(1-k)m_A + D_C \leq 0 \end{aligned}$$

where the first three constraints are enforced for flow conditions, and the last two constraints are enforced for product demands. The objective is to minimize the cost.

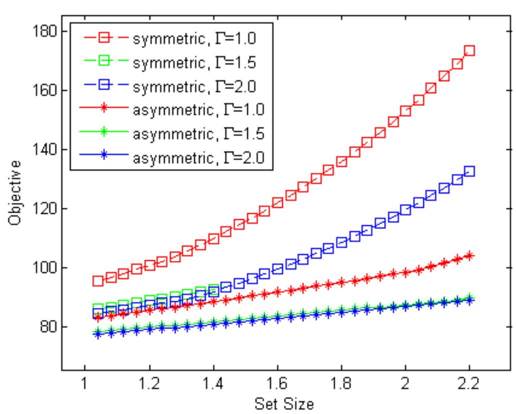

Figure 7. Results for reactor design problem

For this example, both symmetric and asymmetric uncertainty sets are first generated based on sampled data from the distributions. Then the corresponding robust optimization models are solved. The robust solution obtained under different set size and different Γ parameters of D-norm are summarized in Figure 7. From the plot, it is observed that the symmetric set leads to solutions with larger cost (i.e., more conservative) than asymmetric set under various cases.

Conclusions

For robust optimization, uncertainty set construction is a key step since it affects the conservativeness and computational tractability of the deterministic robust counterpart. In this work, we proposed novel uncertainty set construction method based on data, and compared the performance of symmetric and asymmetric uncertainty set induced robust optimization. The proposed method can be applied to general uncertainty distributions with or without correlation. Future work will include investigating the probabilistic guarantee of the robust solution.

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References

Soyster, A. L. (1973). Technical noteconvex programming with set-inclusive constraints and applications to inexact linear programming. Operations research, 21(5), 1154-1157.

Ben-Tal, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical programming*, 88(3), 411-424.

Bertsimas, D., & Sim, M. (2004). The price of robustness. *Operations research*, 52(1), 35-53.

Bertsimas, D., Pachamanova, D., & Sim, M. (2004). Robust linear optimization under general norms. *Operations Research Letters*, 32(6), 510-516.

Chen, X., Sim, M., & Sun, P. (2007). A robust optimization perspective on stochastic programming. *Operations Research*, 55(6), 1058-1071.

Li, Z., Ding, R., & Floudas, C. A. (2011). A comparative theoretical and computational study on robust counterpart optimization: I. Robust linear optimization and robust mixed integer linear optimization. *Industrial & engineering chemistry research*, 50(18), 10567-10603.

Li, Z., Tang, Q., & Floudas, C. A. (2012). A comparative theoretical and computational study on robust counterpart optimization: II. Probabilistic guarantees on constraint satisfaction. *Industrial & engineering chemistry research*, 51(19), 6769-6788.

Yuan, Y., Li, Z., & Huang, B. (2016). Robust optimization under correlated uncertainty: Formulations and computational study. *Computers & Chemical Engineering*, 85, 58-71.