Scenario-based robust optimization of water flooding in oil reservoirs enjoys probabilistic guarantees

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Abstract: Model-based optimization of the water-flooding process in oil reservoirs suffers from high levels of uncertainty arising from strongly varying economic conditions and limited knowledge of the reservoir model parameters. To handle uncertainty, diverse robust optimization approaches that use an ensemble of uncertain parameter realizations (i.e., scenarios), have been adopted. However, in scenario-based approaches, the effect of considering a finite set of scenarios on the constraint violation and/or the performance degradation with respect to the unseen scenarios have not been studied. In this paper, we provide probabilistic guarantees on the worst-case performance degradation of a scenario-based solution. By using statistical learning, we analyze the impact of the number of scenarios on the probabilistic guarantees for the worst-case solution subject to both economic and geological uncertainties. For the economic uncertainty, we derive an explicit a-priori relationship between the probabilistic guarantee and the number of considered scenarios, while for the geological uncertainty, a-posteriori probabilistic upper bounds on the worst-case solution are given.

Keywords: water-flooding, scenario-based optimization, statistical learning.

1. INTRODUCTION

Water-flooding involves the injection of water in an oil reservoir to increase oil production. Various studies have shown that model-based dynamic optimization of the water-flooding process improves the economic life-cycle performance of oil fields, see e.g., Brouwer and Jansen [2004], Foss [2012]. In these studies, Net present value (NPV) is optimized as an economic objective. Besides computational complexity, induced due to complex dynamics and hence non-convexity, one of the key challenges in this model-based dynamic optimization is the high levels of uncertainty arising from the modeling process of water flooding and from strongly varying economic conditions. As a result, the potential advantages of optimization are usually not fully realized.

Typically, one of the first steps in optimization under uncertainty is the quantification of the uncertainty space. In a probabilistic setting, inexact knowledge of a parameter can be modeled via a probability distribution function, leading to stochastic objective function/constraints, Kali and Wallace [1994]. Alternatively, uncertainty can also be modeled as a variable within a deterministic bounded set, and the optimization seeks for a solution which is feasible for all possible realizations in the given set, see e.g., Ben-Tal et al. [2009]. In water-flooding optimization, motivated by the computational complexity, uncertainty is represented by an ensemble, consisting of a finite number of realizations of the uncertain parameters. In the petroleum engineering literature, these scenario (ensemble)-based robust approaches have been studied from various perspectives. In Van Essen et al. [2009], a so-called robust optimization approach has been introduced, which maximizes an average NPV over an ensemble of geological model realizations. A mean-variance optimization (MVO) approach honoring geological uncertainty, which maximizes the average NPV and minimizes the variance of the NPV distribution, has been implemented in Capolei et al. [2015b] and extended to consider economic uncertainty in Siraj et al. [2017]. Different risk-averse robust strategies, e.g., worst-case robust optimization, CVaR optimization, have been presented, e.g., in Xin and Albert [2015], Capolei et al. [2015a], Hanssen et al. [2015], Siraj [2017]. In all these approaches, the number of scenarios are chosen in an ad-hoc way and the effect of considering a *finite* set of scenarios, have not been studied in terms of violation of constraints, performance and robustness of the solution with respect to the *unseen scenarios.*

The scenario-based optimization is a randomized methodology for chance-constrained programs (Schwarm and Nikolaou [1999]) which has been generally characterized in Calafiore and Campi [2005] under the assumption that the sampled counterpart of the original optimization is convex. Based on convex optimization theory and statistical learning, the authors have shown an explicit lower bound on the number of scenarios to be sampled such that the robustly optimal solution to the sampled optimization problem is feasible (with high confidence) for the original chance-constrained program. Informally speaking, given an optimization problem with probabilistic constraints and a number of random samples of the uncertain variable, the theory of the scenario approach establishes how many samples shall be considered to ensure (with high confidence) a certain probability of constraint violation. In Campi and Garatti [2008], Calafiore [2010], the scenario approach theory has been refined in terms of minimum sample size in general; in Schildbach et al. [2013], Zhang et al. [2014, 2015], the scenario approach theory has been refined for sampled convex programs that exhibit specific structure in the constraint functions, and in Grammatico et al. [2014, 2016] the scenario approach has been extended to non-convex programs. All the sample size bounds in these papers are a-priori bounds, i.e., the number of scenarios that are necessary to achieve a desired probability of constraint violation is known independently on the outcome of the sampled optimization problem. Instead, an a-posteriori scenario approach theory, named "wait-and-judge" scenario optimization, has been recently developed in Campi and Garatti [2016], where the authors have formulated a probabilistic statement on the robustly optimal solution to the sampled program after the optimal solution has been computed and based on the number of scenarios used. Complementary to the scenario approach, statistical learning theory has also been applied to support probabilistic guarantees for optimization problems with nonconvex sampled counterpart Tempo et al. [2013]. While the domain of application is larger, the derived probabilistic statements are weaker than those of the scenario approach. The motivation is that the latter provides probabilistic guarantees for the optimal solution only, not for an entire subset of feasible solutions.

The main focus on this paper is to bridge the gap between scenario-based optimization and water-flooding optimization. We aim to address the question: for the worst-case robust water-flooding optimization with economic and geological uncertainty, can we provide probabilistic guarantees on the performance robustness, in terms of achieved optimal value, when an optimal or feasible solution is validated against the unseen scenarios? With this aim, we will define the notion of performance robustness probability and analyze the effect of the number of scenarios to achieve an upper bound on this probability. For economic uncertainty, we derive a-priori characterization on the relationship between the number of scenarios and performance robustness probability (§3), while for geological uncertainty, we provide an a-posteriori probabilistic upper bounds of the worst-case optimal solution (§4).

2. SCENARIO-BASED ROBUST OPTIMIZATION

2.1 Water-flooding optimization

In the model-based optimization of water-flooding process, the NPV to be maximized is typically represented as

$$J(\mathbf{u},\theta) = \sum_{k=1}^{K} \frac{\Delta t_k}{(1+b)^{t_k/\tau}} \left(r_k^{\text{oil}} q_k^{\text{oil}} - r_k^{\text{water}} q_k^{\text{water}} - r_k^{\text{inj}} q_k^{\text{inj}} \right) \tag{1}$$

where r_k^{oil}, r_k^{water}, r_k^{inj}, for all $k \in \{1, \dots, K\}$, and b are economic parameters, i.e., oil price r_k^{oil}, water production cost r_k^{water}, water injection cost r_k^{inj} in $\left[\$/m^3\right]$, and discount factor $b > 0$ for a certain reference time τ. In the definition of NPV, these economic parameters are typically considered as fixed over time, while in reality, these variables fluctuate with time and cannot be precisely predicted, hence become a source of uncertainty. Thus, we intend the parameters r_k^{oil}, r_k^{water}, r_k^{inj} and b as part of the uncertainty vector θ. K represents the production lifecycle, i.e., the total number of time steps k, and Δt_k is the time interval in days associated with one time step. The terms q_k^{oil} and q_k^{water} represent the total flow rate of produced oil and water at time instant k in $\frac{m^3}{\text{day}}$, respectively, and are the output variables. The input vector, $\mathbf{u}$, involves the total water injection rate q_k^{inj} and/or bottom-hole pressure defined at injection/production wells. The parameter vector θ also contains the porosities (the percentage of pore volume within rock that can contain fluids) and permeabilities (rock's ability to transmit fluids) in each grid cell, and other (uncertain) reservoir parameters, hence it affects the output variables q_k^{oil} and q_k^{water}, i.e., $q_k^{\text{oil}} = q_k^{\text{oil}}(\mathbf{u},\theta)$ and $q_k^{\text{water}} = q_k^{\text{water}}(\mathbf{u},\theta)$.

Various robust measures to handle uncertainty have been proposed in the petroleum engineering literature. In this work, we consider the scenario-based worst-case optimization (WCO) approach to handle both economic and geological uncertainty and to derive probabilistic guarantees with a finite number of scenarios in an ensemble.

2.2 Worst-case optimization

Worst-case robust optimization (WCO) is a deterministic paradigm, where the uncertainty is modeled as a variable θ that takes values in a deterministic set (Θ). It optimizes the worst possible case of the considered problem and solves a max-min problem. A typical unconstrained worst-case robust optimization problem can be stated as:

$$\max_{\mathbf{u}\in\mathcal{U}} \min_{\theta\in\Theta} J(\mathbf{u},\theta). \tag{2}$$

An equivalent formulation is the epigraph form with an auxiliary variable Ben-Tal et al. [2009]:

$$\text{WCO}: \begin{cases} \max\limits_{\mathbf{u}\in\mathcal{U}, z\in\mathbb{R}} & z \\ \text{s.t.} & z \leq \min\limits_{\theta\in\Theta} J(\mathbf{u},\theta) \end{cases} \tag{3}$$

Assumption 1. The optimization problem WCO in (3) is feasible. □

2.3 Scenario-based optimization

As the water-flooding optimization is a computationally expensive problem, a traditional approach in robust optimization of the water-flooding process is to sample the uncertainty space, i.e., to consider a finite number of realizations of the uncertain parameters, $\{\theta_1, \theta_2, \cdots, \theta_N\}$, where $\theta_i \in \Theta$ for all i, and to define the following scenario-based worst-case optimization problem:

$$\text{WCO}_N: \begin{cases} \max\limits_{\mathbf{u}\in\mathcal{U}, z\in\mathbb{R}} & z \\ \text{s.t.} & z \leq \min\limits_{i\in\{1,\dots,N\}} J(\mathbf{u},\theta_i) \end{cases} \tag{4}$$

Assumption 2. For all $\{\theta_1, \ldots, \theta_N\}$, the optimization problem in (4) is feasible and $\mathbf{u}_N^*$ denotes an optimal solution, with optimal NPV value being $J_N^* := J(\mathbf{u}_N^*)$. □

Assumption 3. The vectors $\theta_1, \ldots, \theta_N$ in (4) are i.i.d. samples from the probability measure $\mathcal{P}_\Theta$ on Θ. □

Since WCO_N is a sampled version of WCO, with a finite number of (not necessarily the worst-case) samples, the optimal solution $\mathbf{u}_N^*$ is super-optimal for WCO in (3). Due to this relaxation, a possible quantification of the robustness level achieved by the solution $\mathbf{u}_N^*$ of WCO_N is highly relevant. There are two main questions indeed:

(1) What can we claim about the achieved NPV value, when $\mathbf{u}_N^*$ is applied in the presence of the unseen scenarios for θ?
(2) Given a number of samples N, can we quantify, e.g. in probabilistic terms, the robustness of the performance, i.e, the measure of the subset of Θ such that $\text{J}(\mathbf{u}_N^*) \geq \max_{\theta \in \Theta} \text{J}(\mathbf{u}_N^*, \theta)$?

Intuitively, the effect of increasing the number of samples N is to improve the worst-case performance, since more samples implies more knowledge of the uncertainty set Θ. In order to address these questions, let us formally define a "probability of performance robustness".

Definition 1. (Performance robustness probability). Let $\mathbf{u}_N^*$ be an optimal solution to WCO_N in (4). The performance robustness probability of $\mathbf{u}_N^*$ in terms of NPV is defined as

$$V(\mathbf{u}_N^*) := \mathbb{P}_\Theta[J(\mathbf{u}_N^*) \geq J(\mathbf{u}_N^*, \theta \in \Theta)]. \quad (5)$$

□

We note that the definition above is equivalent to the "violation probability" for a chance constrained problem of the form $\min_x c^\top x$ s.t. $\mathbb{P}_\Theta[f(x, \theta \in \Theta) \leq 0] \geq 1 - \epsilon$ [Calafiore and Campi 2005, Def. 1].

The probability $V(\mathbf{u}_N^*)$ indicates the chance that the ensemble of scenarios used for WCO_N does not contain a worst-case realization. In order to quantify the robustness level of the solution $\mathbf{u}_N^*$, we define an $\epsilon-$level solution as in [Calafiore and Campi 2005, Def. 2].

Definition 2. ($\epsilon-$level solution). Let $\epsilon \in (0,1)$. $\mathbf{u}_N^* \in \mathcal{U}$ is an $\epsilon-$level robust solution to WCO in (3) if $V(\mathbf{u}_N^*) \leq \epsilon$. □

Let us emphasize, however, that given an optimal solution $\mathbf{u}_N^*$, $V(\mathbf{u}_N^*)$ cannot be directly computed since $\mathbb{P}_\Theta$ is unknown. Therefore, we shall rely on the following technical statement to upper bound the performance robustness probability $V(\mathbf{u}_N^*)$.

Lemma 1. ([Campi and Garatti 2008, Th. 1]). For all $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, let WCO_N in (4) be convex, with unique optimal solution $\mathbf{u}_N^* = \mathbf{u}_N^*(\boldsymbol{\theta}) \in \mathcal{U}$. Then, the performance robustness probability $V(\mathbf{u}_N^*)$ is upper bounded by a beta distribution, i.e.,

$$\mathbb{P}_\Theta^N[V(\mathbf{u}_N^*) > \epsilon] \leq \sum_{i=0}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} =: \beta, \quad (6)$$

where $n = \dim(\mathcal{U})$ is the number of decision variables. □

Explicit lower bounds on the number of scenarios N for given robustness level parameter $\epsilon \in (0,1)$ and confidence parameter $\beta \in (0,1)$ have been established in Campi and Garatti [2008], Calafiore [2010]. These bounds are given a-priori, so that N can be chosen for a desired level of robustness before the optimization is performed. Instead, in Campi and Garatti [2016], the wait-and-judge approach has been introduced which addresses probabilistic statements also for nonconvex problems. Within the latter approach, the robustness level ϵ of a solution $\mathbf{u}_N^*$ is evaluated a-posteriori, given the number of scenarios N used and the confidence parameter β.

Both a-priori and a-posteriori probabilistic statements are based on the notion of support constraint (SC) [Calafiore and Campi 2006, Def. 4], which we provide in the next definition.

Definition 3. (Support constraint (SC)). For each $j \in \{1, \ldots, N\}$, consider the sampled optimization problem

$$\text{WCO}_N^j : \begin{cases} \max\limits_{\mathbf{u} \in \mathcal{U}, z \in \mathbb{R}} & z \\ \text{s.t.} & z \leq \min\limits_{i \in \{1,\ldots,N\} \setminus \{j\}} J(\mathbf{u}, \theta_i) \end{cases} \quad (7)$$

A sample θ_j generates a support constraint, $z \leq J(\mathbf{u}, \theta_j)$ in (7), if the optimal value of WCO_N^j is strictly greater then that of WCO_N. □

In plain words, a sampled constraint is a support constraint for WCO_N if its removal would alter the optimal solution of the problem.

In the wait-and-judge approach, the robustness level is a function of the number of support constraints that are observed for the solution of scenario-based optimization problem WCO_N. Given this a-posteriori observation, an upper bound on the robustness level ϵ is provided in the next statement.

Lemma 2. ([Campi and Garatti 2016, Th. 2]). Let $\beta \in (0,1)$, s_N^* be the number of support constraints observed for $\mathbf{u}_N^*$, and, for all $k \in \{1, \ldots, N\}$, let $t(k)$ be the unique solution to the polynomial equation

$$\frac{\beta}{N+1} \sum_{m=k}^{N} \binom{m}{k} t^{m-k} - \binom{N}{k} t^{N-k} = 0. \quad (8)$$

Then, for $\epsilon(\cdot) := 1 - t(\cdot)$, it holds that

$$\mathbb{P}_\Theta^N[V(\mathbf{u}_N^*) > \epsilon(s_N^*)] \leq \beta. \quad (9)$$

□

2.4 Statistical learning theory

Note that the key assumption for the classic theory of the scenario approach is that the sampled version of the optimization problem has to be convex. Unfortunately, this feature cannot be verified in water-flooding optimization. Because of the inherent nonlinearity of the output functions $q^{\text{oil}}(\mathbf{u})$ and $q^{\text{water}}(\mathbf{u})$, the optimization problem WCO_N in (4) is nonconvex indeed. In this case, the globally optimal solution may be hard to compute. Thus, we shall rely on a notion of probability of performance robustness that is valid for an entire set, not just for the globally optimal solution.

Definition 4. (Set performance robustness probability). Let $\mathcal{F}$ be the feasible subset of WCO_N in (4). The performance robustness probability of $\mathcal{F}$ in terms of NPV is defined as

$$V(\mathcal{F}) := \mathbb{P}_\Theta \left[\inf_{v \in \mathcal{F}} J(v) - J(v, \theta \in \Theta) \geq 0 \right]. \qquad (10)$$

□

Relative to an entire feasible set, statistical learning theory [Erdogan and Iyengar 2006, Sec. 3.2], [Alamo et al. 2009, Sec. IV, V] provides probabilistic statements based on the so-called Vapnik–Chervonenkis (VC) dimension [Tempo et al. 2013, Def. 10.2], which characterizes the structure of the sampled optimization problem.

Lemma 3. ([Anthony and Biggs 1992, Th. 8.4.1]). Let $\mathcal{F}$ be a feasible subset for WCO_N in (4). Then, the performance robustness probability $V(\mathcal{F})$ is upper bounded as follows:

$$\mathbb{P}_\Theta^N \left[V(\mathcal{F}) > \epsilon\right] \leq 2 \left(\tfrac{2N\epsilon}{\nu}\right)^\nu \exp\left(-\epsilon N/2\right),$$

where ν is the VC dimension. □

The VC dimension is in general hard to compute, or even infinite Grammatico et al. [2016]. However, we will show that for the water-flooding optimization problem with economic uncertainty, the cost function has a special structure for which we can explicitly characterize it.

In the next sections, we show that the probabilistic statements mentioned above are applicable to the considered water-flooding optimization. We address two cases: the WCO subject to economic uncertainty as presented in Siraj [2017], and the WCO with geological uncertainty as in Xin and Albert [2015], Siraj [2017]. Our main message is that ensemble-based optimization approaches as conceived in the literature do enjoy probabilistic guarantees on the performance robustness. This observation allows us to theoretically (that is, not just empirically) analyze the effect of the number of scenarios used in the optimization.

3. ECONOMIC UNCERTAINTY

Oil reservoirs typically have a long life cycle that ranges from 10 to 100 years. The economic variables that govern the NPV, especially the oil price, r_k^{oil}, which in the classic definition of NPV it is considered fixed over time, can vary drastically over time. Moreover, accurate predictions of the oil price cannot be available in practice. The unknown fluctuations of oil prices are in fact the key source of economic uncertainty. Therefore, in this section, let us consider oil prices only as economic uncertainty.

Furthermore, we assume that some (inaccurate) predictions are available. In reality, different models, e.g. the Prospective Outlook on Long-term Energy Systems (POLES) in the European Union, the National Energy Modeling System (NEMS) for energy markets developed by the U.S. Department of Energy and the Energy Information Administration (EIA) in the United States, can be used to generate predictions for energy prices, see Bhattacharyya and Timilsina [2010], Birol [2010] for details. In Siraj [2017], a simplified Auto-Regressive-Moving-Average model (ARMA) model is used to generate oil price scenarios. Thus, in this section, we assume that the uncertain variable represents the oil price, i.e., $\theta = [r_1^{\text{oil}}, \ldots, r_K^{\text{oil}}]^\top$, and that some potential oil price scenarios, $\theta_1, \ldots, \theta_N$, are available.

In Siraj [2017], the WCO for the oil price ensemble has been presented and the results compared in terms of NPV distribution with the mean-optimization approach in Van Essen et al. [2009]. Our contribution here is show that the scenario-based WCO enjoys probabilistic guarantees on the performance robustness probability (Definition 1). With this aim, first, we note that for the WCO with economic uncertainty only, the NPV function J in (1) takes the form

$$J(\mathbf{u}, \theta) = \theta^\top B q^{\text{oil}}(\mathbf{u}) - \left(r^{\text{water}}\right)^\top B q^{\text{water}}(\mathbf{u}) - \left(r^{\text{water}}\right)^\top B \mathbf{u}, \qquad (11)$$

where $B := \text{diag}\left(\frac{\Delta t_1}{(1+b)^{t_1/\tau}}, \ldots, \frac{\Delta t_K}{(1+b)^{t_K/\tau}}\right)$.

We can now apply the recent scenario approach theory to derive an upper bound on the number of support constraints and consequently, an a-priori probabilistic guarantee.

Lemma 4. Suppose that, for all $\theta = r^{\text{oil}} \in \Theta$, the function $\mathbf{u} \mapsto J(\mathbf{u}, \theta)$ in (11) is concave. Then, the number of support constraints in the optimization problem WCO_N in (4) is no more than $K + 1$. □

Proof. It follows from [Zhang et al. 2015, Lemma 1].

Proposition 1. Suppose that, for all $\theta = r^{\text{oil}} \in \Theta$, the function $\mathbf{u} \mapsto J(\mathbf{u}, \theta)$ in (11) is concave. Let $\mathbf{u}_N^*$ be the optimal solution to WCO_N in (4), with

$$N \geq \tfrac{2}{\epsilon}\left(K + \ln\left(\tfrac{1}{\beta}\right)\right).$$

Then, it holds that $\mathbb{P}_\Theta^N[V(\mathbf{u}_N^*) > \epsilon] \leq \beta$. □

Proof. It follows from Lemma 1 and [Grammatico et al. 2016, Equ. (5)].

We conclude the section on the economic uncertainty by dropping the assumption of convexity of r^{oil} in objective function. Next, we provide probabilistic guarantees that are more generally applicable, at the price of increased sample size.

Lemma 5. The VC dimension for the optimization problem WCO_N in (4) with function J as in (11) is $K + 1$. □

Proof. It follows from [Anthony and Biggs 1992, Example 8.4.4].

Proposition 2. Let $\mathcal{F}$ be a feasible set for the optimization problem WCO_N in (4) with function J as in (11) and

$$N \geq \tfrac{4}{\epsilon}\left((K+1)\ln\left(\tfrac{12}{\epsilon}\right) + \ln\left(\tfrac{2}{\beta}\right)\right).$$

Then, it holds that $\mathbb{P}_\Theta^N\left[V(\mathcal{F}) > \epsilon\right] \leq \beta$. □

Proof. See [Anthony and Biggs 1992, Th. 8.4.1] and [Grammatico et al. 2016, Equ. (6)].

4. GEOLOGICAL UNCERTAINTY

Reservoir dynamics are nonlinear in nature and typically represented by large-scale partial differential equations, with a number of state variables in the order of $10^4 - 10^6$ and a similar number of the model parameters. Model uncertainty, in terms of the model structure and parameters, is one of the key sources of uncertainty in model-based optimization of the water-flooding process. It arises mainly due to the lack of knowledge of subsurface geology that defines the reservoir. For dealing with uncertainty, scenario-based optimization is typically used on a set of sampled models, $\mathcal{M}(\theta_1), \ldots, \mathcal{M}(\theta_N)$. The resulting ensemble of models has been used in various robust approaches, e.g. mean-optimization and worst-case optimization, see Van Essen et al. [2009], Xin and Albert [2015], Capolei et al. [2015b], Siraj [2017].

We emphasize that the probabilistic bounds derived in the presence of economic uncertainty cannot be directly extended to the case with geological uncertainty, as the uncertain model parameters do not affect the NPV objective function linearly. Instead, the NPV is a function of total output flow q^{oil} over time, which in turn is a highly nonlinear function of the decision variables. Therefore, in the presence of geological uncertainty, the resulting scenario optimization problem WCO_N in (4) is inherently nonconvex, and in turn an a-priori probabilistic guarantee cannot be stated in general. In this case, a-posterior bounds on the performance robustness probability shall be instead applied by observing the number of support constraints for the WCO_N problem. Alternatively, we can observe the number of actives constraints, which is in general an upper bound on the number of support constraints. In this section, we take the a-posteriori approach on a simulation example with an ensemble of oil reservoir models, and derive probabilistic bounds based on the number of observed active constraints.

In the following, we consider a simulation example subject to geological uncertainty, where the only source of uncertainty is from the unknown model parameters, while the economic parameters are considered as fixed. We perform that numerical experiment using the MATLAB Reservoir Simulation Toolbox (MRST) Lie et al. [2012].

We use an ensemble of finite number of geological realizations of the standard egg model, see Jansen et al. [2014], with 100 realizations, i.e., $N = N_{\text{geo}} = 100$. Each model is a three-dimensional realization of a channelized reservoir produced under water-flooding conditions, with eight water injectors and four producers, based on the original egg model proposed in Van Essen et al. [2009]. The true permeability field is considered to be the unknown parameter. The life cycle of each reservoir model is assumed to be 3600 days. The absolute-permeability field of the first realization in the set is shown in Fig. 1. For illustration purposes, Fig. 2 shows the permeability fields of six randomly chosen realizations of the standard egg model in the assumed ensemble. Each realization in the set is considered as equi-probable.

We consider the numerical setup described next. The WCO problem is considered with non-discounted NPV, i.e., with discount factor $b = 0$. The economic parameters, oil price r_k^{oil}, water injection r_k^{inj} and production cost r_k^{water} are chosen as $126\frac{\$}{m^3}$, $6\frac{\$}{m^3}$ and $19\frac{\$}{m^3}$, respectively. The control input, that is, the decision variable of the optimization problem, $\mathbf{u}$, involves injection flow rate trajectories for the eight injection wells. The minimum and the maximum rate for each injection well are set to $0.2\frac{m^3}{\text{day}}$ and $79.5\frac{m^3}{\text{day}}$, respectively. The production wells operate at a constant bottom-hole pressure of 395 bar. For each of the eight injection wells, the control input is parametrized into ten time periods of 360 days each, during which the injection rate is intended to be constant. Thus, the decision variable has dimension $n = 80$. For numerical results in terms of optimal NPV values and a comparison with Van Essen et al. [2009], we refer the interested reader to Siraj [2017].

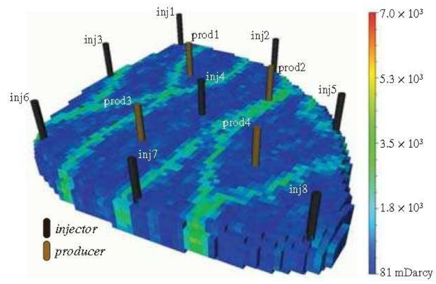

Fig. 1. Illustration of the permeability field of one model realization Van Essen et al. [2009].

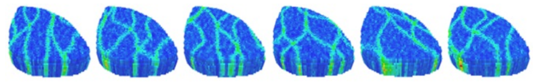

Fig. 2. Illustration of the permeability fields of six randomly chosen geological realizations (Van Essen et al. [2009], Jansen et al. [2014].

To derive the upper bound ϵ in (9), we select a confidence level β of 0.01. Then, in order to evaluate the robustness level of the obtained scenario-based optimal solution $\mathbf{u}_N^*$, we a-posteriori evaluate the number of active constraints, which is always an upper bound for the numbers of support constraints. The motivation for this choice is that assessing which constraints are active at the optimum is computationally lighter than removing one constraint j at the time and solving WCO_N^j in (7). In our numerical simulation, for the obtained solution $\mathbf{u}_N^*$, we observe 7 active constraints only. In view of Equation (8), we plot $\epsilon(s_N^*)$ for the chosen confidence level β in Fig. 3. In particular, we observe that $\epsilon(7) = 0.09$. It shows that, for this particular example, 100 realizations are sufficient to obtain low upper-bound on performance robustness. Increasing the number of realizations will further improve the worst-case performance.

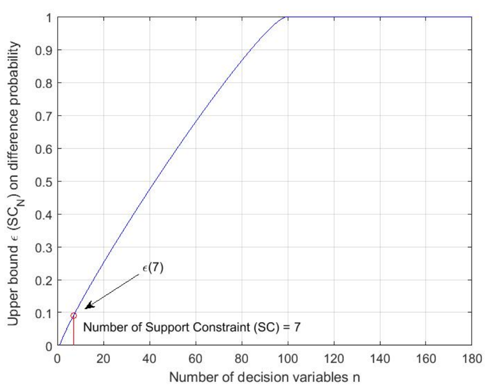

Fig. 3. For the case of geological uncertainty, upper-bound ϵ as a function of SC and the confidence level $1 - \beta) = 0.99$ on performance robustness probability.

5. CONCLUSION AND OUTLOOK

Scenario-based approaches to robust optimization represent an attractive computational approach that, surprisingly, come with theoretical, probabilistic guarantees. Since scenario-based robust optimization approaches are very common in water-flooding optimization for oil reservoirs, we have shown that scenario-based optimal control solutions do enjoy probabilistic guarantees. Specifically, in case of economic uncertainty, probabilistic statements hold depending on the number of scenarios used. For geological uncertainty, probabilistic statements shall be derived based on the computed optimal solution. We believe that these observations can be extended to constrained problems in the water-flooding optimization.

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