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Finite Horizon H^{∞} Control for a Class of Linear Quantum Sampled-Data Measurement Systems: A Dynamic Game Approach ⋆

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Abstract: In this paper, the finite horizon H^{∞} control problem is solved for a class of linear quantum systems using a dynamic game approach for the case of sampled-data measurements. The methodology adopted involves a certain equivalence between the quantum problem and an auxiliary classical stochastic problem. Then, by solving the finite horizon H^{∞} control problem for the equivalent stochastic problem using some results from a corresponding deterministic problem following a dynamic game approach, the finite horizon H^{∞} control problem for the class of linear quantum systems under consideration is solved for the case of sampled-data measurements.

Keywords: Quantum control, game theory, finite horizon H^{∞} control, sampled-data measurements.

1. INTRODUCTION

The control of quantum systems is a rapidly growing and evolving field whose applications include quantum computing, control of molecular dynamics, design of semiconductor nanodevices, control of charged particles in beam accelerators; etc., see Pierce et al. [1988] and references therein. The most effective strategies in classical control applications involve feedback control and one of the major concentrated activities of control theory of the past three decades has been the development of the 'H^{∞}-optimal control theory', which addresses the issue of the worst-case controller design for classical linear plants subject to unknown disturbances and plant uncertainties.

Note that in many engineering problems (target maneuver, missile guidance, etc.), control over a limited period of time is needed. In such cases, the effect of the initial conditions is most important and infinite horizon H^{∞} methods cannot provide a satisfactory control strategy. The motivation for finite horizon H^{∞} control problems is then to consider the transient response of the system within the framework of H^{∞} control problems.

Within these perspectives, this paper solves the finite horizon H^{∞} control problem for a class of linear quantum systems using a dynamic game approach for the case of sampled-data measurements. Note that solving the finite horizon H^{∞} control problem for the case of sampled data measurements has a significance importance in the development of quantum control theory. In fact, practical and modern quantum control systems usually use digital computers as discrete-time controllers to control quantum continuous time systems.

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2. FORMULATION OF THE PROBLEM

2.1 The Plant Model

We consider a class of linear quantum dynamical systems described in the Heisenberg picture by a set of quantum stochastic differential equations; see James et al. [2008] and Nurdin et al. [2009].

The system is therefore described by the following continuous quantum stochastic differential equations (QSDEs) defined on the finite time interval $[0, t_f]$ and by the discrete time-varying quantum difference equation for the measured output defined at the jump time t_k.

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)du(t)dt + D(t)dw(t)dt \\ &\quad + G_v(t)dv(t); \\ y(t_k) &= C_d(t_k)x(t_k) + N_d(t_k)\beta_w(t_k) + L_d(t_k)\tilde{v}(t_k); \\ z(t) &= H(t)x(t) + G(t)\beta_u(t) + M(t)\beta_w(t); \end{aligned} \quad (1)$$

where

$$\begin{aligned} &H(t)^T G(t) = 0; \quad H(t)^T H(t) = Q(t); \quad G(t)^T G(t) = I; \\ &M(t) = 0. \end{aligned} \quad (2)$$

For the linear quantum systems under consideration, the continuous measurement $dy(t)$ is now replaced by a sampled-data measurement $y(t_k)$ where $\{t_k\}_{k\geq 1}$ is an increasing sequence of measurement time instants:

$$0 \leq t_1 < t_2 < \cdots < t_K < t_f.$$

The initial system variables $x(0) = x_0$ consist of operators (on an appropriate Hilbert space) satisfying the commutation relations:

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}$$

where Θ is a real antisymmetric matrix with components Θ_{jk}; see James et al. [2008].

Furthermore, we assume that the state of the quantum system is a Gaussian state with mean $\check{x}_0 \in \mathbb{R}^n$ and covariance matrix Y_0; e.g., see Meyer [1995]. Then $\langle x_0 \rangle = \check{x}_0$ and

$$Y_0 = \frac{1}{2}\left\langle (x_0 - \check{x}_0)(x_0 - \check{x}_0)^T + ((x_0 - \check{x}_0)(x_0 - \check{x}_0)^T)^T \right\rangle. \tag{3}$$

Here, $\langle . \rangle$ denotes quantum expectation; e.g., see Parthasarathy [1992]. In the sequel, we will fix Y_0 but $\check{x}_0$ will be taken as part of the disturbance.

Here, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times n_u}$, $D(t) \in \mathbb{R}^{n\times n_w}$, $G_v(t) \in \mathbb{R}^{n\times n_v}$ and (n, n_w, n_u and n_v are positive integers) for all $t \in [0, t_f]$. Also, $x(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of self-adjoint possibly noncommutative system variables; e.g., see James et al. [2008] for more details.

$C_d(t_k) \in \mathbb{R}^{n_{y_k}\times n_k}$, $N_d(t_k) \in \mathbb{R}^{n_{y_k}\times n_{w_k}}$, $L_d(t_k) \in \mathbb{R}^{n_{y_k}\times n_{v_k}}$ and (n_k, n_{w_k}, n_{y_k}, and n_{v_k} are positive integers) for all $k \in [0, K]$. The quantity $dw(t)$ represents the input variables or disturbances, $du(t)$ is the control input, $y(t_k)$ is the sampled measured output and $z(t)$ is the controlled output.

We assume that $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$ where $\tilde{w}(t)$ is the noise part of $w(t)$ and $\beta_w(t)$ is a square integrable classical disturbance signal. The set of all such $\beta_w(t)$ is denoted $\mathcal{W}$. The noise $\tilde{w}(t)$ is a vector of quantum Wiener processes with Ito table $F_{\tilde{w}}$ and commutation matrix $T_{\tilde{w}}$ which are defined below. Similarly, we also assume that $du(t) = \beta_u(t)dt + d\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ and $\beta_u(t)$ is a self-adjoint adapted process. The noise $\tilde{u}(t)$ is a quantum noise with Ito matrix $F_{\tilde{u}}$ and commutation matrix $T_{\tilde{u}}$. Also, the vector $dv(t)$ represents any additional quantum noise in the plant. It has an Ito matrix F_v and commutation matrix T_v.

The non-negative symmetric Ito matrices $F_{\tilde{w}}$, $F_{\tilde{u}}$ and F_v and the commutation matrices $T_{\tilde{w}}$, $T_{\tilde{u}}$ and $T_{\tilde{v}}$ are defined in Maalouf and Petersen [2010].

We also assume that the vector $\tilde{v}(t_k)$ represents an additional quantum measurement noise. It has a covariance matrix $F_{\tilde{v}_k}$. The non-negative symmetric covariance matrix $F_{\tilde{v}_k}$ satisfies the following equation: $F_{\tilde{v}_k} = E\left(\tilde{v}(t_k)\tilde{v}(t_k)^T\right)$.

Let

$$\begin{aligned} G_{v_t}(t) &= [\, B(t)\;\; D(t)\;\; G_v(t) \,] \\ \text{and}\quad dv_t(t) &= [\, d\tilde{u}(t)\;\; d\tilde{w}(t)\;\; dv(t) \,]^T. \end{aligned}$$

Then equation (1) becomes

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)\beta_u(t)dt + D(t)\beta_w(t)dt \\ &\quad + G_{v_t}(t)dv_t(t); \\ y(t_k) &= C_d(t_k)x(t_k) + N_d(t_k)\beta_w(t_k) + L_d(t_k)\tilde{v}(t_k); \\ z(t) &= H(t)x(t) + G(t)\beta_u(t) + M(t)\beta_w(t); \end{aligned} \tag{4}$$

2.2 The Controller Model

We consider a sampled-data classical controller $\mathcal{K}$ of the following form defined by a differential equation with jumps on the finite time interval $[0, t_f]$:

$$\begin{aligned} d\psi(t) &= F_c(t)\psi(t)dt; \quad \psi(0) = \psi_0; \\ \psi(t_k^+) &= F_{c_d}(t_k)\psi(t_k^-) + G_{c_d}(t_k)y(t_k); \\ \beta_u(t) &= H_c(t)\psi(t) \end{aligned} \tag{5}$$

where $\psi(t)$ is the controller state. For all $t \in [0, t_f]$, $F_c(t) \in \mathbb{R}^{n_c\times n_c}$ and $H_c(t) \in \mathbb{R}^{n_u\times n_c}$ (n_c is a positive integer). Also, $F_{c_d}(t_k) \in \mathbb{R}^{n_{c_k}\times n_{c_k}}$ and $G_{c_d}(t_k) \in \mathbb{R}^{n_{c_k}\times n_{y_k}}$ (n_{c_k} is a positive integer) for all $k \in [0, K]$.

2.3 The Closed-Loop System

The closed-loop system is obtained by making the identification $\beta_u(t) = H_c(t)\psi(t)$ and interconnecting equations (4) and (5) to give a quantum-classical system described by the following stochastic differential equations with jumps

$$\begin{aligned} d\eta(t) &= \tilde{A}(t)\eta(t)dt + \tilde{D}(t)\beta_w(t)dt + \tilde{L}(t)dv_t(t); \\ \eta(t_k^+) &= \tilde{A}_d(t_k)\eta(t_k^-) + \tilde{D}_d(t_k)\beta_w(t_k) + \tilde{L}_d(t_k)\tilde{v}(t_k); \\ z(t) &= \tilde{H}(t)\eta(t) \end{aligned} \tag{6}$$

where

$$\eta(t) = \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix}, \eta(t_k^-) = \begin{bmatrix} x(t_k) \\ \psi(t_k^-) \end{bmatrix},$$

$$\eta(t_k^+) = \begin{bmatrix} x(t_k) \\ \psi(t_k^+) \end{bmatrix}, \tilde{A}(t) = \begin{bmatrix} A(t) & B(t)H_c(t) \\ 0 & F_c(t) \end{bmatrix},$$

$$\tilde{D}(t) = \begin{bmatrix} D(t) \\ 0 \end{bmatrix}, \tilde{L}(t) = \begin{bmatrix} G_{v_t}(t) \\ 0 \end{bmatrix},$$

$$\tilde{A}_d(t_k) = \begin{bmatrix} I & 0 \\ G_{c_d}(t_k)C_d(t_k) & F_{c_d}(t_k) \end{bmatrix},$$

$$\tilde{D}_d(t_k) = \begin{bmatrix} 0 \\ G_{c_d}(t_k)N_d(t_k) \end{bmatrix},$$

$$\tilde{L}_d(t_k) = \begin{bmatrix} 0 \\ G_{c_d}(t_k)L_d(t_k) \end{bmatrix}$$

$$\text{and}\quad \tilde{H}(t) = [\, H(t)\;\; G(t)H_c(t) \,].$$

2.4 The cost function

We take the overall disturbance as

$$\hat{\beta}_w(t) = (\check{x}_0, \beta_w(t), \{\tilde{v}(t_k)\}).$$

We therefore have to determine, whether, under the given measurement scheme, the upper value of the game with cost function

$$\begin{aligned}
&L_\gamma(\mathcal{K},\hat\beta_w)\\
&= \langle x(t_f)^T Q_f x(t_f)\rangle + \int_0^{t_f} \langle z(t)^T z(t)\rangle dt\\
&\quad -\gamma^2 \check{x}_0^T Q_0 \check{x}_0 - \gamma^2 \left\{ \int_0^{t_f} \beta_w(t)^T \beta_w(t) dt \right.\\
&\quad\quad \left. + \sum_{k=0}^{K} \left(\beta_w(t_k)^T \beta_w(t_k)\right) \right\}\\
&\quad -\gamma^2 \left\{ E\left(\tilde v(t_k)^T \tilde v(t_k)\right)\right\}\\
&= \langle x(t_f)^T Q_f x(t_f)\rangle + \int_0^{t_f} \langle x(t)^T Q(t) x(t)\rangle dt\\
&\quad + \int_0^{t_f} \langle \beta_u(t)^T \beta_u(t)\rangle dt - \gamma^2 \check{x}_0^T Q_0 \check{x}_0\\
&\quad -\gamma^2 \left\{ \int_0^{t_f} \beta_w(t)^T \beta_w(t) dt \right.\\
&\quad\quad \left. + \sum_{k=0}^{K} \left(\beta_w(t_k)^T \beta_w(t_k)\right) \right\}\\
&\quad -\gamma^2 \left\{ E\left(\tilde v(t_k)^T \tilde v(t_k)\right)\right\} \qquad (7)
\end{aligned}$$

is bounded, and if so, to obtain a corresponding min-sup controller

$$\beta_u(t) = \mu(t, y_1, y_2, \cdots, y_k)$$

where $t_k < t \le t_{k+1}$, $Q_f = Q_f^T \ge 0$, Q_0 is a weighting matrix taken to be positive definite, $Q(t) = Q(t)^T \ge 0$ and $\langle . \rangle$ represents the quantum expectation over all initial variables and noises; see Parthasarathy [1992].

2.5 Explicit Expression for L_γ

For the quantum closed-loop system (6), we define the covariance matrix P given by

$$P(t) = \frac{1}{2}\left\langle \eta(t)\eta(t)^T + \left(\eta(t)\eta(t)^T\right)^T \right\rangle. \qquad (8)$$

Note that $P_0 = P(0) = \mathrm{diag}(Y_0 + \check{x}_0\check{x}_0^T, 0)$. We calculate

$$\begin{aligned}
dP(t) &= \frac{1}{2}\left\{ \langle d\eta(t)\eta(t)^T\rangle + \left\langle \left(d\eta(t)\eta(t)^T\right)^T\right\rangle \right\}\\
&+ \frac{1}{2}\left\{ \langle \eta(t)d\eta(t)^T\rangle + \left\langle \left(\eta(t)d\eta(t)^T\right)^T\right\rangle \right\}\\
&+ \frac{1}{2}\left\{ \langle d\eta(t)d\eta(t)^T\rangle + \left(\langle d\eta(t)d\eta(t)^T\rangle\right)^T \right\}.
\end{aligned}$$

An expression for $dP(t)$ using the quantum Ito rule (see James et al. [2008], Parthasarathy [1992]) is given by

$$\begin{aligned}
dP(t) &= \tilde A(t)P(t)dt + P(t)\tilde A(t)^T dt + \tilde D(t)\beta_w(t)\\
&\quad \langle \eta(t)^T\rangle dt + \langle \eta(t)\rangle \beta_w(t)^T \tilde D(t)^T dt\\
&\quad + \tilde L(t) S_v(t) \tilde L(t)^T dt \qquad (9)
\end{aligned}$$

where

$$S_v(t)dt = \frac{1}{2}\left\langle dv_t(t)dv_t(t)^T + \left(dv_t(t)dv_t(t)^T\right)^T\right\rangle.$$

Thus, we obtain the matrix differential equation with jumps

$$\begin{aligned}
\dot P(t) &= \tilde A(t)P(t) + P(t)\tilde A(t)^T + \tilde D(t)\beta_w(t)\langle \eta(t)^T\rangle\\
&\quad + \langle \eta(t)\rangle \beta_w(t)^T \tilde D(t)^T + \tilde L(t) S_v(t) \tilde L(t)^T; \qquad (10)\\
P(t_k^+) &= \tilde A_d(t_k) P(t_k^-) \tilde A_d(t_k)^T\\
&\quad + \tilde D_d(t_k)\beta_w(t_k)\beta_w(t_k)^T \tilde D_d(t_k)^T\\
&\quad + \tilde L_d(t_k) S_{v_k} \tilde L_d(t_k)^T\\
&\quad + \tilde A_d(t_k)\langle \eta(t_k^-)\rangle \beta_w(t_k)^T \tilde D_d(t_k)^T\\
&\quad + \tilde D_d(t_k)\beta_w(t_k)\langle \eta(t_k^-)^T\rangle \tilde A_d(t_k)^T \qquad (11)
\end{aligned}$$

where $S_{v_k} = \frac{1}{2}E\left(\tilde v(t_k)\tilde v(t_k)^T\right)$.

Now, we find an expression for L_γ. In fact,

$$\begin{aligned}
\eta(t_f) &= \begin{bmatrix} x(t_f) \\ \psi(t_f) \end{bmatrix}\\
\text{so that}\quad Q_f^{1/2}x(t_f) &= \begin{bmatrix} Q_f^{1/2} & 0 \end{bmatrix} \begin{bmatrix} x(t_f) \\ \psi(t_f) \end{bmatrix}\\
&= \begin{bmatrix} Q_f^{1/2} & 0 \end{bmatrix} \eta(t_f)\\
\text{and}\quad x(t_f)^T Q_f^{1/2} &= \eta(t_f)^T \begin{bmatrix} Q_f^{1/2} \\ 0 \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle x(t_f)^T Q_f x(t_f)\rangle\\
&= \left\langle x(t_f)^T Q_f^{1/2} Q_f^{1/2} x(t_f)\right\rangle\\
&= \left\langle \eta(t_f)^T \begin{bmatrix} Q_f^{1/2} \\ 0 \end{bmatrix} \begin{bmatrix} Q_f^{1/2} & 0 \end{bmatrix} \eta(t_f)\right\rangle\\
&= \left\langle \eta(t_f)^T \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} \eta(t_f)\right\rangle\\
&= \frac{1}{2} tr \left\langle \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} \left(\eta(t_f)\eta(t_f)^T\right.\right.\\
&\quad\quad \left.\left. + \left(\eta(t_f)\eta(t_f)^T\right)^T\right)\right\rangle\\
&= tr\left(\begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix} P(t_f)\right)\\
&= tr\left(\hat Q_f P(t_f)\right)
\end{aligned}$$

where $\hat Q_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$.

On the other hand,

$$\langle x(t)^T Q(t) x(t)\rangle = tr\left(\hat Q(t) P(t)\right)$$

where $\hat Q(t) = \begin{bmatrix} Q(t) & 0 \\ 0 & 0 \end{bmatrix}$ and

$$\langle \beta_u(t)^T \beta_u(t)\rangle = tr(\hat H(t) P(t))$$

where $\hat H(t) = \begin{bmatrix} 0 & 0 \\ 0 & H_c(t)^T H_c(t) \end{bmatrix}$.

Let $R(t) = \hat Q(t) + \hat H(t) = \begin{bmatrix} Q(t) & 0 \\ 0 & H_c(t)^T H_c(t) \end{bmatrix}$.

Hence,

$$
\begin{aligned}
&L_\gamma(\mathcal{K},\hat{\beta}_w)\\
&\quad = tr\left(\hat{Q}_f P(t_f)\right) + \int_0^{t_f} tr\left(R(t)P(t)\right)dt\\
&\quad -\gamma^2\left[\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right]\\
&\quad -\gamma^2\left[\sum_{k=1}^{K}\left(\beta_w(t_k)^T\beta_w(t_k) + E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right)\right]
\end{aligned}
\tag{12}
$$

where $\hat{Q}_0 = \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\check{\eta}_0 = \langle \eta_0 \rangle$.

2.6 The Finite Horizon H^∞ problem

We will consider, as the standard problem, the case where x_0 is a part of the unknown disturbance. Let

$$(\check{x}_0, \beta_w(.), \tilde{v}(t_k)) := \hat{\beta}_w(.) \in \Omega_q = \mathbb{R}^n \times \mathcal{W} \times \mathcal{V}_k. \tag{13}$$

The set of admissible controllers $\mathcal{K}$, which will be denoted by $\mathfrak{M}$, are controllers which are of the form given by (5) and under which the problem defined by (4) and (5) has a unique solution for every $\hat{\beta}_w(.) \in \Omega_q$.

$L_\gamma(\mathcal{K},\hat{\beta}_w)$ defined in (7) can be written in terms of the closed-loop system variable $\eta(t)$ as

$$
\begin{aligned}
L_\gamma(\mathcal{K},\hat{\beta}_w) &= \left\langle \eta(t_f)^T\hat{Q}_f\eta(t_f)\right\rangle + \left\langle \int_0^{t_f}\eta(t)^T R(t)\eta(t)\right\rangle dt\\
&\quad -\gamma^2\left(\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right)\\
&\quad -\gamma^2\left(\sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k)\right)\right.\\
&\quad \left. +E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right)\\
&= tr\left(\hat{Q}_f P(t_f)\right) + \int_0^{t_f} tr\left(R(t)P(t)\right)dt\\
&\quad -\gamma^2\left(\check{\eta}_0^T\hat{Q}_0\check{\eta}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right)\\
&\quad -\gamma^2\left(\sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k)\right)\right.\\
&\quad \left. +E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right).
\end{aligned}
$$

On the other hand, we define, $J_\gamma(\mathcal{K},\hat{\beta}_w) = -L_\gamma(\mathcal{K},\hat{\beta}_w)$.

The disturbance attenuation problem to be solved is the following:

Problem $\mathcal{P}_\gamma$. Determine necessary and sufficient conditions on γ such that the quantity

$$\inf_{\mathcal{K}\in\mathfrak{M}} \sup_{\hat{\beta}_w\in\Omega_q} L_\gamma(\mathcal{K},\hat{\beta}_w)$$

is finite, and for each such γ find a controller $\mathcal{K}$ that achieves the minimum. The infimum of all γ's that satisfy these conditions will be denoted by γ_q^*.

3. AUXILIARY CLASSICAL STOCHASTIC AND DETERMINISTIC SYSTEMS

The Auxiliary Classical Stochastic System

We define the following classical linear stochastic system with sampled data measurements

$$
\begin{aligned}
d\xi(t) &= A(t)\xi(t)dt + B(t)\beta_u(t)dt + D(t)\beta_w(t)dt\\
&\quad + G_{v_t}(t)S_v^{1/2}(t)dv_t(t); \quad t \geq 0;\\
y(t_k) &= C_d(t_k)\xi(t_k) + N_d(t_k)\beta_w(t_k) + L_d(t_k)S_{v_k}^{1/2}\tilde{v}(t_k);\\
z(t) &= H(t)\xi(t) + G(t)\beta_u(t)
\end{aligned}
\tag{14}
$$

where equations (2) are satisfied and $\xi(0) = \xi_0$ is a Gaussian random variable with mean $\check{x}_0$ and covariance matrix Y_0.

The vector $dv_t(t)$ represents a stochastic noise in the plant and $v_t(t)$ is a Wiener process. We also assume that the vector $\tilde{v}(t_k)$ represents a stochastic measurement noise.

3.1 Closed-Loop System

The classical controller $\mathcal{K}$ is given by (5) and the corresponding closed-loop stochastic system is obtained by making the identification $\beta_u(t) = H_c(t)\psi(t)$ and interconnecting equations (14) and (5) to give:

$$
\begin{aligned}
d\mu(t) &= \tilde{A}(t)\mu(t)dt + \tilde{D}(t)\beta_w(t)dt + \tilde{L}_s(t)dv_t(t);\\
\mu(t_k^+) &= \tilde{A}_d(t_k)\mu(t_k^-) + \tilde{D}_d(t_k)\beta_w(t_k) + \tilde{L}_{d_s}(t_k)\tilde{v}(t_k);\\
z(t) &= \tilde{H}(t)\mu(t)
\end{aligned}
\tag{15}
$$

where $\mu(t) = \begin{bmatrix} \xi(t) \\ \psi(t) \end{bmatrix}$, $\mu(t_k^+) = \begin{bmatrix} \xi(t_k) \\ \psi(t_k^+) \end{bmatrix}$, $\mu(t_k^-) = \begin{bmatrix} \xi(t_k) \\ \psi(t_k^-) \end{bmatrix}$, $\tilde{L}_s(t) = \tilde{L}(t)S_v^{1/2}(t)$ and $\tilde{L}_{d_s}(t_k) = \tilde{L}_d(t_k)S_{v_k}^{1/2}$.

3.2 Cost Function

We define the classical cost function

$$\hat{L}(\mathcal{K},\beta_w) = E\left(\xi(t_f)^T Q_f \xi(t_f)\right) + \int_0^{t_f} E\left(z(t)^T z(t)\right)dt$$

where $Q_f = Q_f^T \geq 0$ and

$$
\begin{aligned}
&\hat{L}_\gamma(\mathcal{K},\hat{\beta}_w)\\
&\quad = E\left(\xi(t_f)^T Q_f\xi(t_f)\right) + \int_0^{t_f} E\left(z(t)^T z(t)\right)dt\\
&\quad -\gamma^2\left\{\check{x}_0^T Q_0\check{x}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right\}\\
&\quad -\gamma^2\left\{\sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k) + E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right)\right\}
\end{aligned}
\tag{16}
$$

where $E(.)$ denotes the stochastic expectation.

3.3 An Explicit Expression for the Closed-Loop Cost Function

For the stochastic closed-loop system (15), we define the covariance matrix

$$\tilde{P}(t) = E(\mu(t)\mu(t)^T). \tag{17}$$

Note that $\tilde{P}(0) = \tilde{P}_0 = P(0) = \text{diag}(Y_0 + \check{x}_0\check{x}_0^T, 0)$.

Using the classical Ito rule, we can write

$$\begin{aligned} d\tilde{P}(t) &= E(d\mu(t)\mu(t)^T) + E(\mu(t)d\mu(t)^T) + E(d\mu(t)d\mu(t)^T) \\ &= \tilde{A}(t)\tilde{P}(t)dt + \tilde{P}(t)\tilde{A}(t)^T dt + \tilde{D}(t)\beta_w(t)E(\mu(t)^T)dt \\ &\quad + E(\mu(t))\beta_w(t)^T\tilde{D}(t)^T dt + \tilde{L}(t)S_v(t)\tilde{L}(t)^T dt; \\ \tilde{P}(t_k^+) &= \tilde{A}_d(t_k)\tilde{P}(t_k^-)\tilde{A}_d(t_k)^T \\ &\quad + \tilde{D}_d(t_k)\beta_w(t_k)\beta_w(t_k)^T\tilde{D}_d(t_k)^T \\ &\quad + \tilde{L}_d(t_k)S_{v_k}\tilde{L}_d(t_k)^T \\ &\quad + \tilde{A}_d(t_k)E\left(\mu(t_k^-)\right)\beta_w(t_k)^T\tilde{D}_d(t_k)^T \\ &\quad + \tilde{D}_d(t_k)\beta_w(t_k)E\left(\mu(t_k^-)^T\right)\tilde{A}_d(t_k)^T. \end{aligned} \tag{18}$$

We have that

$$Q_f^{1/2}\xi(t_f) = \begin{bmatrix} Q_f^{1/2} & 0 \end{bmatrix}\mu(t_f)$$
$$\text{and} \quad \xi(t_f)^T Q_f^{1/2} = \mu(t_f)^T \begin{bmatrix} Q_f^{1/2} \\ 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} &E\left(\xi(t_f)^T Q_f \xi(t_f)\right) \\ &= E\left(\xi(t_f)^T Q_f^{1/2} Q_f^{1/2}\xi(t_f)\right) \\ &= E\left(\mu(t_f)^T \begin{bmatrix} Q_f^{1/2} \\ 0 \end{bmatrix} \begin{bmatrix} Q_f^{1/2} & 0 \end{bmatrix}\mu(t_f)\right) \\ &= E\left(\mu(t_f)^T \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}\mu(t_f)\right) \\ &= tr\left(\begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}\tilde{P}(t_f)\right) \\ &= tr\left(\hat{Q}_f\tilde{P}(t_f)\right). \end{aligned} \tag{19}$$

On the other hand, $E\left(z(t)^T z(t)\right) = tr\left(\tilde{H}(t)^T\tilde{H}(t)\tilde{P}(t)\right)$. Thus,

$$\hat{L}(\mathcal{K}, \hat{\beta}_w) = tr\left(\hat{Q}_f\tilde{P}(t_f)\right) + \int_0^{t_f} tr\left(\tilde{H}(t)^T\tilde{H}(t)\tilde{P}(t)\right)dt$$

and

$$\begin{aligned} &\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\ &= tr\left(\hat{Q}_f\tilde{P}(t_f)\right) + \int_0^{t_f} tr\left(\tilde{H}(t)^T\tilde{H}(t)\tilde{P}(t)\right)dt \\ &\quad -\gamma^2\left[\check{\mu}_0^T\hat{Q}_0\check{\mu}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right] \\ &\quad -\gamma^2\sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k) + E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right) \end{aligned} \tag{20}$$

where $\check{\mu}_0 = E(\mu_0)$.

3.4 Equivalence Between $P(.)$ and $\tilde{P}(.)$

Theorem 1. The covariance matrices $P(.)$ given by (8) and $\tilde{P}(.)$ given by (17) are equal.

As a consequence of Theorem 1, $L_\gamma(\mathcal{K}, \hat{\beta}_w) \equiv \hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ for all classical linear controllers $\mathcal{K}$ of the form (5) and for all disturbance inputs $\beta_w(t)$. This equivalence between the cost functions of the quantum closed-loop system $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ and the corresponding cost functions of the stochastic closed-loop system $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ is in the sense that applying the same controller $\mathcal{K}$ given by (5) to the quantum system (4) and the stochastic system (14), then the resulting quantum closed-loop system (6) and the resulting stochastic closed-loop system (15) will have the same cost functions values for all disturbance inputs $\beta_w(t)$, i.e; $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ will have the same value as $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$.

3.5 Reformulation of the Auxiliary Classical Stochastic Closed-Loop System

In this subsection, we reformulate the stochastic worst case performance problem for the closed-loop system. The closed-loop system (15) can also be rewritten as:

$$\begin{aligned} d\mu(t) &= \tilde{A}(t)\mu(t)dt + \tilde{D}(t)\beta_w(t)dt + dv_n(t); \\ \mu(t_k^+) &= \tilde{A}_d(t_k)\mu(t_k^-) + \tilde{D}_d(t_k)\beta_w(t_k) + \hat{v}(t_k) \\ z(t) &= \tilde{H}(t)\mu(t) \end{aligned} \tag{21}$$

where $dv_n(t) = \tilde{L}_s(t)dv_t(t)$ and $\hat{v}(t_k) = \tilde{L}_{d_s}(t_k)\tilde{v}(t_k)$.

We now assume that the initial condition random variable $\mu(0) = \mu_0$ for the closed-loop system (21) is normal with mean m and covariance matrix R_0. The stochastic process $v_n(t)$ has zero mean and covariance matrix $R_1(t)$. We assume that the process $v_n(t)$ is independent of μ_0 and that the matrices R_0 and $R_1(t)$ are symmetric and nonnegative definite for all $t \in [0, t_f]$. We assume as well that $\hat{v}(t)$ has a zero mean and covariance matrix $R_{\hat{v}}(t)$.

Reformulating the Closed-Loop Performance Index We rewrite the performance index for the closed-loop system as:

$$\begin{aligned} &\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\ &= E\left(\mu(t_f)^T\hat{Q}_f\mu(t_f)\right) + \int_0^{t_f} E\left(\mu(t)^T R(t)\mu(t)\right)dt \\ &\quad -\gamma^2\left\{\check{\mu}_0^T\hat{Q}_0\check{\mu}_0 + \int_0^{t_f}\beta_w(t)^T\beta_w(t)dt\right\} \\ &\quad -\gamma^2\left\{\sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k) + E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right)\right\} \end{aligned} \tag{22}$$

where $\hat{Q}_f = \hat{Q}_f^T \geq 0$, $R(t) = R(t)^T \geq 0$.

We note here that the closed-loop performance index (22) is equivalent to the closed-loop cost function (20). In fact, $E\left(\mu(t_f)^T\hat{Q}_f\mu(t_f)\right) = tr\left(\hat{Q}_f\tilde{P}(t_f)\right)$ where $\tilde{P}(t) = E\left(\mu(t)\mu(t)^T\right)$ and $E\left(\mu(t)^T R(t)\mu(t)\right) = tr\left(R(t)\tilde{P}(t)\right)$ with $R(t) = \tilde{H}(t)^T\tilde{H}(t)$ having $G(t)^T G(t) = I$, $H(t)^T H(t) = Q(t)$ and $H(t)^T G(t) = 0$ from (2). Let

$$\begin{aligned}
&\hat{J}_\gamma(\mathcal{K}, \hat{\beta}_w) \\
&\quad = -\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\
&\quad = E\left(\mu(t_f)^T \tilde{Q}_f \mu(t_f)\right) + \int_0^{t_f} E\left(\mu(t)^T \tilde{R}(t)\mu(t)\right)dt \\
&\qquad +\gamma^2 \left\{\check{\mu}_0^T \hat{Q}_0 \check{\mu}_0 + \int_0^{t_f} \beta_w(t)^T \beta_w(t) dt\right\} \\
&\qquad +\gamma^2 \left\{\sum_{k=0}^{K} \left(\beta_w(t_k)^T \beta_w(t_k) + E\left(\tilde{v}(t_k)^T \tilde{v}(t_k)\right)\right)\right\}
\end{aligned} \tag{23}$$

where $\tilde{Q}_f = \tilde{Q}_f^T = -\hat{Q}_f \leq 0$ and $\tilde{R}(t) = \tilde{R}(t)^T = -R(t) \leq 0$.

We want to minimize $\hat{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ which is equivalent to maximizing $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$.

Using Theorem 1, $P(.)$ and $\tilde{P}(.)$ are equal. Thus, minimizing $\hat{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ is equivalent to maximizing $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$.

By taking $\check{x}_0$ as a part of the unknown disturbance, the quantum cost function $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ defined in Problem P_γ is equal to the stochastic cost function $\hat{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ since $P(.)$ and $\tilde{P}(.)$ are equal.

Hence, minimizing $\hat{J}_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ is equivalent to maximizing $L_\gamma(\mathcal{K}, \hat{\beta}_w)$ over $\hat{\beta}_w(.)$ in Problem P_γ.

The Auxiliary Classical Deterministic System

We now consider a deterministic system corresponding to the auxiliary classical stochastic system (14) defined as:

$$\begin{aligned}
\dot{\xi}(t) &= A(t)\xi(t) + B(t)\beta_u(t) + D(t)\beta_w(t); \quad \xi_0 = \check{x}_0; \\
y(t_k) &= C_d(t_k)\xi(t_k) + N_d(t_k)\beta_w(t_k); \\
z(t) &= H(t)\xi(t) + G(t)\beta_u(t)
\end{aligned} \tag{24}$$

where equations (2) are satisfied.

The standard problem we consider is the case where $\check{x}_0$ is a part of the unknown disturbance. The set of admissible controllers $\mathcal{K}$ will be denoted by $\mathfrak{M}$. These controllers are of the form given by (5) and such that the problem defined by (24) and (5) has a unique solution for every ξ_0 and every $\beta_w(.) \in \mathcal{W}$.

We also introduce the extended performance index

$$\begin{aligned}
&\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\
&\quad = \xi(t_f)^T Q_f \xi(t_f) + \int_0^{t_f} z(t)^T z(t) dt \\
&\qquad -\gamma^2 \left(\int_0^{t_f} \beta_w(t)^T \beta_w(t) dt + \check{x}_0^T Q_0 \check{x}_0\right) \\
&\qquad -\gamma^2 \left(\sum_{k=0}^{K} \beta_w(t_k)^T \beta_w(t_k)\right) \\
&\quad = \xi(t_f)^T Q_f \xi(t_f) + \int_0^{t_f} \left(\xi(t)^T Q(t) \xi(t)\right) dt \\
&\qquad + \int_0^{t_f} \left(\beta_u(t)^T \beta_u(t)\right) dt \\
&\qquad -\gamma^2 \left(\int_0^{t_f} \beta_w(t)^T \beta_w(t) dt + \check{x}_0^T Q_0 \check{x}_0\right) \\
&\qquad -\gamma^2 \left(\sum_{k=0}^{K} \beta_w(t_k)^T \beta_w(t_k)\right)
\end{aligned}$$

where Q_0 is a weighting matrix, taken to be positive definite and $\gamma > 0$.

Also, $\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$ can be rewritten in terms of the closed-loop variable $\mu(t)$ as

$$\begin{aligned}
&\tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w) \\
&\quad = \mu(t_f)^T \hat{Q}_f \mu(t_f) + \int_0^{t_f} \mu(t)^T R(t) \mu(t) dt \\
&\qquad -\gamma^2 \left(\int_0^{t_f} \beta_w(t)^T \beta_w(t) dt + \check{\mu}_0^T \hat{Q}_0 \check{\mu}_0\right) \\
&\qquad -\gamma^2 \left(\sum_{k=0}^{K} \beta_w(t_k)^T \beta_w(t_k)\right) \\
&\quad = tr\left(\hat{Q}_f \tilde{P}(t_f)\right) + \int_0^{t_f} tr\left(\tilde{H}(t)^T \tilde{H}(t) \tilde{P}(t)\right) dt \\
&\qquad -\gamma^2 \left(\int_0^{t_f} \left(\beta_w(t)^T \beta_w(t)\right) dt + \check{\mu}_0^T \hat{Q}_0 \check{\mu}_0\right) \\
&\qquad -\gamma^2 \left(\sum_{k=0}^{K} \beta_w(t_k)^T \beta_w(t_k)\right)
\end{aligned} \tag{25}$$

where $\hat{Q}_0 = \begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix}$. The corresponding disturbance attenuation problem to be solved is the following:

Problem $\tilde{\mathcal{P}}_\gamma$. Determine necessary and sufficient conditions on γ such that the quantity

$$\inf_{\mathcal{K} \in \mathfrak{M}} \sup_{\hat{\beta}_w \in \Omega_q} \tilde{L}_\gamma(\mathcal{K}, \hat{\beta}_w)$$

is finite, and for each such γ find a controller $\mathcal{K}$ (or family of controllers) that achieves the minimum. The infimum of all γ's that satisfy these conditions will be denoted by γ_c^*.

4. AN EQUIVALENT DETERMINISTIC WORST CASE PERFORMANCE PROBLEM FOR THE CLOSED-LOOP SYSTEM

4.1 The Closed-Loop System

In the deterministic case, the closed-loop system corresponding to (24) and (5) is given by:

$$\begin{aligned}
\dot{\mu}(t) &= \tilde{A}(t)\mu(t) + \tilde{D}(t)\beta_w(t); \quad \mu_0 = m \\
\mu(t_k^+) &= \tilde{A}_d(t_k)\mu(t_k^-) + \tilde{D}_d(t_k)\beta_w(t_k); \\
z(t) &= \tilde{H}(t)\mu(t).
\end{aligned} \tag{26}$$

4.2 The Performance Index

The closed-loop deterministic performance index is given by:

$$\tilde{J}_\gamma(\mathcal{K},\beta_w) = \mu(t_f)^T\tilde{Q}_f\mu(t_f) + \int_0^{t_f}\left(\mu(t)^T\tilde{R}(t)\mu(t)\right)dt + \gamma^2\int_0^{t_f}\left(\beta_w(t)^T\beta_w(t)\right)dt + \gamma^2\sum_{k=0}^{K}\beta_w(t_k)^T\beta_w(t_k) \quad (27)$$

with $\tilde{Q}_f = \tilde{Q}_f^T \leq 0$ and $\tilde{R}(t) = \tilde{R}(t)^T \leq 0$.

4.3 Solution to the Deterministic Worst Case Performance Problem

To determine the worst case closed-loop cost, we assume that the admissible disturbance strategies are such the value of the disturbance signal is a deterministic function of time. The deterministic worst case performance problem can be stated as follows:

Problem: Consider the closed-loop deterministic system described by (26). Find an admissible strategy $\beta_w(.)$ such that the criterion (27) is minimal.

We define the following Riccati equation with jumps

$$\dot{Z}(t) + Z(t)\tilde{A}(t) + \tilde{A}(t)^T Z(t) + \tilde{R}(t) - \gamma^{-2}Z(t)\tilde{D}(t)\tilde{D}(t)^T Z(t) = 0; \quad Z(t_f) = \tilde{Q}_f; \quad (28)$$

$$Z(t_k^-) = \tilde{A}_d(t_k)^T Z(t_k^+)\tilde{A}_d(t_k) - \tilde{A}_d(t_k)^T Z(t_k^+)\tilde{D}_d(t_k)\left(\gamma^2 I + \tilde{D}_d(t_k)^T Z(t_k^+)\tilde{D}_d(t_k)\right)^{\#}\tilde{D}_d(t_k)^T Z(t_k^+)\tilde{A}_d(t_k) \quad (29)$$

where # denotes the Moore-Penrose pseudo-inverse.

Lemma 2. Assume that the Riccati equation (28)-(29) has a solution on the interval $0 \leq t \leq t_f$. Consider any square integrable disturbance signal $\beta_w(t)$ defined on $[0, t_f]$ and let $\mu(t)$ be a corresponding solution of the differential equation (26). Assume that there are K jumps in the interval $[0, t_f]$. Then

$$\mu(t_f)^T\tilde{Q}_f\mu(t_f) + \int_0^{t_f}\left(\mu(t)^T\tilde{R}\mu(t)\right)dt + \gamma^2\int_0^{t_f}\left(\beta_w(t)^T\beta_w(t)\right)dt + \gamma^2\sum_{k=0}^{K}\beta_w(t_k)^T\beta_w(t_k)$$
$$= \mu_0^T Z_0\mu_0 + \gamma^2\sum_{k=0}^{K-1}\int_{t_k+}^{t_{(k+1)^-}}\left(\beta_w(t) + \gamma^{-2}\tilde{D}(t)^T Z(t)\mu(t)\right)^T\left(\beta_w(t) + \gamma^{-2}\tilde{D}(t)^T Z(t)\mu(t)\right)dt$$
$$+\gamma^2\int_{t_{K+}}^{t_f}\left(\beta_w(t) + \gamma^{-2}\tilde{D}(t)^T Z(t)\mu(t)\right)^T\left(\beta_w(t) + \gamma^{-2}\tilde{D}(t)^T Z(t)\mu(t)\right)dt$$
$$+\gamma^2\sum_{k=0}^{K}\left[\beta_w(t_k) + \Gamma(t_k)\right]^T\left(\gamma^2 I - \tilde{D}_d(t_k)^T Z(t_k^+)\tilde{D}_d(t_k)\right)\left[\beta_w(t_k) + \Gamma(t_k)\right]. \quad (30)$$

where $\Gamma(t_k) = \left(\gamma^2 I + \tilde{D}_d(t_k)^T Z(t_k^+)\tilde{D}_d(t_k)\right)^{\#}\tilde{D}_d(t_k)^T Z(t_k^+)\tilde{A}_d(t_k)\mu(t_k^-)$.

Theorem 3. Assume that the Riccati equation (28)-(29) has a solution on $[0, t_f]$. Assume as well that there are K jumps in the interval $[0, t_f]$. Then, the optimal linear solution of the deterministic worst case performance problem (26), (27) is such that the deterministic signal $\beta_w(t)$ is given by the worst case distrurbance

$$\beta_w^*(t) = -G_0(t)\mu(t); \quad \forall t \neq t_k; \quad t \in [0, t_f];$$
$$\beta_w^*(t_k) = -\left(\gamma^2 I + \tilde{D}_d(t_k)^T Z(t_k^+)\tilde{D}_d(t_k)\right)^{\#}\tilde{D}_d(t_k)^T Z(t_k^+)\tilde{A}_d(t_k)\mu(t_k^-) \quad (31)$$

where

$$G_0(t) = \gamma^{-2}\tilde{D}(t)^T Z(t). \quad (32)$$

Here $Z(t)$ is the solution of the matrix Riccati equation (28)-(29). The minimal value of the criterion function is given by $\tilde{J}_\gamma(\mathcal{K}, \beta_w^*) = m^T Z_0 m$.

Theorem 4. The deterministic linear quadratic control problem has a finite solution, for every initial system variable $\mu_0 = m$, if and only if the Riccati differential equation with jumps (28)-(29) has a symmetric solution $Z(.)$ on $[0, t_f]$.

5. SOLUTION TO THE STOCHASTIC WORST CASE PERFORMANCE PROBLEM

To determine the worst case closed-loop cost, we assume that the admissible disturbance strategies are such the value of the disturbance signal is a deterministic function of time. The stochastic worst case performance problem can be stated as follows:

Problem: Consider the closed-loop stochastic system described by (21). Find an admissible strategy $\beta_w(.)$ such that the following criterion is minimal

$$\check{J}_\gamma(\mathcal{K},\beta_w) = \hat{J}_\gamma(\mathcal{K},\hat{\beta}_w) - \gamma^2\check{\mu}_0^T\hat{Q}_0\check{\mu}_0$$
$$= E\left(\mu(t_f)^T\tilde{Q}_f\mu(t_f)\right) + \int_0^{t_f}E\left(\mu(t)^T\tilde{R}(t)\mu(t)\right)dt + \gamma^2\left\{\int_0^{t_f}\beta_w(t)^T\beta_w(t)dt + \sum_{k=0}^{K}\left(\beta_w(t_k)^T\beta_w(t_k)\right)\right\} + \gamma^2\left\{E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right)\right\}. \quad (33)$$

Lemma 5. Assume that the Riccati equation (28)-(29) has a solution on the interval $0 \leq t \leq t_f$. Assume as well that there are K jumps in the interval $[0,t_f]$. Consider any square integrable disturbance signal $\beta_w(t)$ defined on $[0,t_f]$ and let $\mu(t)$ be a corresponding solution of the stochastic differential equation (21). Then

$$\begin{aligned}
&\mu(t_f)^T\tilde{Q}_f\mu(t_f)+\int_0^{t_f}\Big(\mu(t)^T\tilde{R}\mu(t)+\gamma^2\beta_w(t)^T\beta_w(t)\Big)dt\\
&+\gamma^2\sum_{k=0}^{K}\beta_w(t_k)^T\beta_w(t_k)\\
&=\mu_0^TZ_0\mu_0\\
&\quad+\gamma^2\sum_{k=0}^{K-1}\int_{t_{k+}}^{t_{(k+1)^-}}\Big(\beta_w(t)+\gamma^{-2}\tilde{D}(t)^TZ(t)\mu(t)\Big)^T\\
&\quad\Big(\beta_w(t)+\gamma^{-2}\tilde{D}(t)^TZ(t)\mu(t)\Big)dt\\
&\quad+\gamma^2\int_{t_{K+}}^{t_f}\Big(\beta_w(t)+\gamma^{-2}\tilde{D}(t)^TZ(t)\mu(t)\Big)^T\\
&\quad\Big(\beta_w(t)+\gamma^{-2}\tilde{D}(t)^TZ(t)\mu(t)\Big)dt\\
&\quad+\gamma^2\sum_{k=0}^{K}[\beta_w(t_k)+\Gamma(t_k)]^T\\
&\quad\Big(\gamma^2I+\tilde{D}_d(t_k)^TZ(t_k^+)\tilde{D}_d(t_k)\Big)\\
&\quad[\beta_w(t_k)+\Gamma(t_k)]\\
&\quad+\sum_{k=0}^{K-1}\left(\int_{t_{k+}}^{t_{(k+1)^-}}dv_n(t)^TZ(t)\mu(t)\right)\\
&\quad+\sum_{k=0}^{K-1}\left(\int_{t_{k+}}^{t_{(k+1)^-}}\mu(t)^TZ(t)dv_n(t)\right)\\
&\quad+\sum_{k=0}^{K-1}\left(\int_{t_{k+}}^{t_{(k+1)^-}}tr(Z(t)R_1(t))dt\right)\\
&\quad+\int_{t_{K+}}^{t_f}dv_n(t)^TZ(t)\mu(t)+\int_{t_{K+}}^{t_f}\mu(t)^TZ(t)dv_n(t)\\
&\quad+\int_{t_{K+}}^{t_f}tr(Z(t)R_1(t))dt+\sum_{k=0}^{K}\big(\mu(t_k^-)^T\ \tilde{A}_d(t_k)^T\\
&\quad Z(t_k^+)\hat{v}(t_k)\big)\\
&\quad+\sum_{k=0}^{K}\Big(\beta_w(t_k)^T\tilde{D}_d(t_k)^TZ(t_k^+)\hat{v}(t_k)\Big)\\
&\quad+\sum_{k=0}^{K}\Big(\hat{v}(t_k)^TZ(t_k^+)\tilde{A}_d(t_k)\mu(t_k^-)\Big)\\
&\quad+\sum_{k=0}^{K}\Big(\hat{v}(t_k)^TZ(t_k^+)\tilde{D}_d(t_k)\beta_w(t_k)\\
&\qquad+\hat{v}(t_k)^TZ(t_k^+)\hat{v}(t_k)\Big).
\end{aligned}$$

Using Lemma 5, the following theorem provides a solution to the stochastic worst case performance problem for the closed-loop system.

Theorem 6. Assume that the Riccati equation (28)-(29) has a solution on $[0,t_f]$. Assume as well that there are K jumps in the interval $[0,t_f]$.Then, the minimal value of the criterion function in the stochastic worst case performance problem (21), (33) satisfies

$$\min_{\beta_w\in\mathcal{W}}\check{J}_\gamma(\mathcal{K},\beta_w)\geq m^TZ(0)m+\alpha$$

where

$$\begin{aligned}
\alpha=&\,tr(Z(0)R_0)+\sum_{k=0}^{K-1}\int_{t_{k+}}^{t_{(k+1)^-}}tr(R_1(t)Z(t))dt\\
&+\int_{t_{K+}}^{t_f}tr(R_1(t)Z(t))dt+\sum_{k=0}^{K}tr(Z(t_k^+)R_{\hat{v}}(t_k))\\
&+\gamma^2\sum_{k=0}^{K}E\left(\tilde{v}(t_k)^T\tilde{v}(t_k)\right).
\end{aligned}\qquad(34)$$

Here $Z(t)$ is the solution of the matrix Riccati equation (28)-(29).

6. A RELATIONSHIP BETWEEN $\check{J}_\gamma(\mathcal{K},\beta_W)$ AND $\tilde{J}_\gamma(\mathcal{K},\beta_W)$

The following theorem shows the relationship between the optimum values of the stochastic cost function $\check{J}_\gamma(\mathcal{K},\beta_w)$ and the deterministic cost function $\tilde{J}_\gamma(\mathcal{K},\beta_w)$ where $m\in\mathbb{R}^{(n+n_c)}$ defines the initial condition of the deterministic system (26) and the mean of the initial condition in the stochastic system (21). Let

$$\check{V}(m)=\inf_{\beta_w\in\mathcal{W}}\check{J}_\gamma(\mathcal{K},\beta_w)\qquad(35)$$

and

$$\tilde{V}(m)=\inf_{\beta_w\in\mathcal{W}}\tilde{J}_\gamma(\mathcal{K},\beta_w).\qquad(36)$$

Theorem 7. Given any $m\in\mathbb{R}^{(n+n_c)}$, the infimum $\check{V}(m)$ in the stochastic case is related to the corresponding infimum $\tilde{V}(m)$ in the deterministic case by the following equation

$$\check{V}(m)=\tilde{V}(m)+\alpha\qquad(37)$$

where α is given by (34).

7. A USEFUL RESULT

Theorem 8. Assume $\tilde{D}(t)$, $\tilde{L}(t)$ and $\tilde{R}(t)$ are continuous in t. Then the worst case performance problem (21), (33) and (35) has a finite infimum for any $m\in\mathbb{R}^{(n+n_c)}$ if and only if the RDE (28)-(29) has a solution $Z(.)$ on $[0,t_f]$.

8. EQUIVALENCE BETWEEN THE DETERMINISTIC WORST CASE PERFORMANCE PROBLEM AND THE STOCHASTIC WORST CASE PERFORMANCE PROBLEM

Theorem 9. For γ sufficiently large, the RDE (28)-(29) has a solution on $[0,t_f]$.

As a consequence of Theorem 9, the following set is nonempty:

$$\begin{aligned}\hat{\Gamma}=\{&\tilde{\gamma}>0|\text{the RDE (28)-(29) has a solution on }[0,t_f]\\&\forall\gamma\geq\tilde{\gamma}\}.\end{aligned}$$

Define $\hat{\gamma}$ as $\hat{\gamma}=\inf\left\{\gamma:\gamma\in\hat{\Gamma}\right\}$.

Theorem 10. For $\gamma = \hat{\gamma}$, the RDE (28)-(29) has a finite escape time.

Theorem 11. If the RDE (28)-(29) admits a solution defined over $[0, t_f]$ then $\inf_{\beta_w \in \mathcal{W}} \check{J}_\gamma(\mathcal{K}, \beta_w) > -\infty$ and $\inf_{\beta_w \in \mathcal{W}} \tilde{J}_\gamma(\mathcal{K}, \beta_w) > -\infty$.

9. EQUIVALENCE BETWEEN THE QUANTUM WORST CASE PERFORMANCE PROBLEM AND THE DETERMINISTIC WORST CASE PERFORMANCE PROBLEM

Let $J_\gamma(\mathcal{K}, \beta_w) = J_\gamma(\mathcal{K}, \hat{\beta}_w) - \gamma^2 \check{x}_0^T Q_0 \check{x}_0$.

Theorem 12. (a) In the deterministic case, $\tilde{J}_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$ if and only only if $\gamma > \hat{\gamma}$.

(b) In the stochastic case, $\check{J}_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\gamma > \hat{\gamma}$.

(c) $\check{J}_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\tilde{J}_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$.

(d) In the quantum case, $J_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$ if and only if $\tilde{J}_\gamma(\mathcal{K}, \beta_w)$ has a finite infimum in β_w for all $m \in \mathbb{R}^{(n+n_c)}$.

10. SOLUTION TO THE FINITE HORIZON H^∞ CONTROL PROBLEM FOR SAMPLED-DATA MEASUREMENTS SYSTEMS

In order to solve the finite horizon quantum H^∞ problem for sampled-data measurements systems, we now introduce the GRDE (Generalized Riccati Differential Equation) in $\Sigma(t)$:

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A(t)^T + \gamma^{-2}\Sigma(t)Q(t)\Sigma(t) + D(t)D(t)^T \quad (38)$$

$$\Sigma(t_k^+) = \Sigma(t_k^-)[I + C(t_k)^T \tilde{N}(t_k)^{-1} C(t_k)\Sigma(t_k^-]^{-1} \quad (39)$$

where $\tilde{N}(t) = N(t)N(t)^T$ and $\Sigma(0) = Q_0^{-1}$. Here, Q_0 is a positive definite matrix. We also introduce the GRDE in $Z(t)$:

$$\begin{aligned}&\dot{Z}(t) + Z(t)A(t) + A(t)^T Z(t) - \\ &Z(t)\left(B(t)B(t)^T - \gamma^{-2}D(t)D(t)^T\right)Z(t) + Q(t) = 0, \\ &Z(t_f) = Q_f.\end{aligned} \quad (40)$$

In addition, we introduce the following condition

$$\forall t \in [0, t_f], \quad \rho(\Sigma(t)Z(t)) < \gamma^2 \quad (41)$$

where $\rho(.)$ denotes the spectral radius.

Let

$$\begin{aligned}\dot{\check{\xi}}(t) &= \left(A + \gamma^{-2}\Sigma(t)Q(t)\right)\check{\xi}(t) + B(t)\hat{\beta}_{u_1}(t), \quad (42) \\ \check{\xi}(0) &= 0; \\ \check{\xi}(t_k^+) &= \check{\xi}(t_k^-) + \Sigma(t_k^+)C(t_k)^T \tilde{N}(t_k)^{-1} \\ &\quad (y(t_k) - C(t_k)\check{\xi}(t_k^-)) \quad (43)\end{aligned}$$

where

$$\hat{\beta}_{u_1}(t) = -B(t^T)Z(t)(I - \gamma^{-2}\Sigma(t)Z(t))^{-1}\check{\xi}(t). \quad (44)$$

To keep matters simple, we will assume in the subsequent development that $D(t)N(t)^T = 0$.

Theorem 13. Consider the disturbance attenuation problem $\mathcal{P}_\gamma$ with continuous imperfect system variable measurement $y(t)$ as given in (4), and let the corresponding optimum attenuation level be γ_q^*.

(i) For a given $\gamma > 0$, if the Riccati differential equations (38)- (39) and (40) have solutions over $[0, t_f]$, and if the condition (41) is satisfied, then necessarily $\gamma \geq \gamma_q^*$.

(ii) For each such γ, there exists an optimal controller, given by (42) (43) and (44).

(iii) If either (38)-(39) or (40) has a conjugate point in $[0, t_f]$, or if (41) fails to hold, then $\gamma \leq \gamma_q^*$; i.e., for any smaller γ (and possibly for the one considered) the supremum in problem $\mathcal{P}_\gamma$ is infinite for any admissible controller.

11. CONCLUSION

This paper shows that solving the finite horizon H^∞ control problem for sampled-data measurements systems is equivalent to solving a corresponding deterministic continuous-time problem with imperfect state measurements. From this, the solution to the finite horizon quantum H^∞ control problem for sampled-data measurements systems can be obtained in terms of a pair of GRDEs.

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