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Graceful Passage Through Hopf Bifurcation

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Abstract: The concept of "graceful" transition through a Hopf bifurcation for a system of nonlinear ordinary differential equations (ODEs) is introduced. The key idea is to control the system such that its state space trajectory is close to the branch of equilibrium solutions or to the branch of periodic solutions associated with a Hopf bifurcation. This kind of evolution is called "graceful" and can be generated by formulating and solving optimization control problems.

Keywords: Bifurcation; control; optimization.

1. INTRODUCTION

The dynamics of nonlinear systems is rich in complicated phenomena that are usually not desired in engineering applications. Vibrations -periodic or not- have to be appropriately managed through control design. Common examples of practical applications include aircraft flutter control, surge and stall control of compressors, and control of power systems. One problem of particular interest is to adequately control the transition between qualitatively different operating conditions, because it is during these transitions that the most dangerous vibration related phenomena occur. In this paper we address one such situation and show how one can design control laws which guarantee the graceful transition of a nonlinear autonomous system of ODEs between an equilibrium operating condition and an oscillatory (periodic) one.

In particular, it is assumed that the equilibrium solution belongs to a branch of equilibria which loses its stability, as the control parameter is varied, through a Hopf bifurcation. Using feedback to stabilize a system with a Hopf bifurcation has been studied by several authors (Abed [1986], Berns [1998], Chen [1998], Yu [2004]). If the equilibrium state is linearly controllable, the bifurcation can be stabilized or shifted by linear feedback. However, nonlinear feedback is needed for systems with uncontrollable linearizations. Behtash and Sastry (see Bentah [1988]), Gu et al. (Gu [1999]), Hamzi et al. (Hamzi [2000]) studied bifurcations of uncontrollable systems. Yuen and Bau (see Yuen [2004]) demonstrated theoretically and experimentally the use of a nonlinear feedback controller in a thermal convection loop that renders a subcritical Hopf bifurcation supercritical.

Delayed feedback control (Just [1998], Pyragas [1998]) can be used to stabilize unstable periodic orbits. The advantage of the method is that neither the exact form of the periodic orbit nor knowledge of the system of equations is required. These algorithms are real-time implementable, as they only make use of a control signal obtained from the difference between the current state of the system and the state of the system delayed by one period of the unstable periodic orbit.

Our focus here is on controlling the transition of finite dimensional systems of ODEs through supercritical Hopf bifurcations. Briefly, this bifurcation leads to the emergence of a branch of asymptotically stable limit cycles when the branch of equilibria loses its stability. The main challenge, from a practical applications perspective, consists in controlling the evolution of the system between qualitatively different solutions of the system such that this evolution is "graceful". In this context, graceful evolution means that the trajectory of the system is close either to the equilibrium solution branch or to the periodic one.

The solution proposed herein, and illustrated via a simple example, is based on the formulation of optimization problems which guarantee "graceful" evolution as defined in the above and refined in the body of the paper.

2. PROBLEM FORMULATION

2.1 The System

Consider a nonlinear autonomous system of ODEs:

$$\dot{x} = \frac{dx}{dt} = f(x, u), x \in X \subset R^n, \lambda \in \Lambda \subset R, t \in T \subset R \quad (1)$$

where $f(x, \lambda)$ is a function of class C^k in $X \times \Lambda$, $k > 0$, x is the state vector, λ is the control, t is the time (independent variable), whereas X, Λ, and T are open sets.

2.2 Equilibrium and Periodic Solutions; Hopf Bifurcation

The equilibrium solutions of (1) are defined by

$$0 = f(x, \lambda). \quad (2)$$

Let (x_i, λ_i) be a solution pair of (2) for which the eigenvalues of

$$J_i = \frac{\partial f}{\partial x}(x_i, \lambda_i) \quad (3)$$

belong to the open left hand semi-plane of the complex plane, i.e. x_i is an exponentially stable equilibrium. Then, according to the implicit function theorem, there exists a branch of equilibrium solutions passing through (x_i, λ_i) and which can be expressed as an unique function, $x = g(\lambda)$, of class C^k on an open connected set (i.e. interval), $\Lambda_e \in \Lambda$, containing λ_i:

$$x = g(\lambda), x_i = g(\lambda_i), f(g(\lambda), \lambda) = 0, g : \Lambda_e \to X. \quad (4)$$

Assume that the stability of this branch is lost through a Hopf bifurcation, which means that, as λ varies it reaches a point, λ_H, at which a pair of eigenvalues of the Jacobian of $f(x, \lambda)$ evaluated along this branch,

$$J(\lambda) = \frac{\partial f}{\partial x}(g(\lambda), \lambda), \quad (5)$$

crosses the imaginary axis in a transversal manner. If the pair of eigenvalues of interest is $\gamma(\lambda) \pm j\omega(\lambda)$, this means:

$$\omega(\lambda_H) \neq 0, \frac{\partial \gamma}{\partial \lambda}(\lambda_H) \neq 0. \quad (6)$$

It is also assumed that the Hopf bifurcation is supercritical, which means that the bifurcation is one sided, i.e., in the neighborhood of the Hopf bifurcation point, the branch of periodic solutions exists only for $\lambda > \lambda_H$. The supercritical Hopf bifurcation leads to the emergence of a branch of asymptotically stable periodic solutions, while the stability of the equilibrium branch is lost.

2.3 Graceful Evolution

In this paper we are interested in controlling the transition between an equilibrium x_i which belongs to the equilibrium branch and a periodic solution, $x_f(s)$,

$$x_f(s) = x_f(s + \tau), \quad (7)$$

which belongs to the branch of periodic solutions (here s is just a notation for the parameter used to parameterize the periodic solution while $\tau > 0$ is the period of $x_f(s)$). Let $\lambda_f > \lambda_H$ be the value of the control for which solution $x_f(s)$ is obtained. Additionally, it is required that the transition is conducted in a "graceful" manner, which is characterized as follows. Firstly, as long as the control parameter is in the region corresponding to the branch of stable equilibrium solutions the system's trajectory is sufficiently close to this branch, such that if the control is frozen the system settles down to the corresponding asymptotically stable equilibrium. Secondly, when the control parameter passes into the region of the branch of asymptotically stable periodic solutions (i.e. $\lambda > \lambda_H$) the system's trajectory is sufficiently close to this branch such that, if the control is frozen the system's trajectory settles down to the corresponding asymptotically stable periodic solution. In other words, the state of the system belongs either to the basin of attraction of an asymptotically stable equilibrium or to the one of an asymptotically periodic solution, depending on the current value of the control. (The idea can be slightly enlarged by not requiring that x_i is an equilibrium but an initial condition close to the branch of equilibrium solutions).

There are several practical advantages associated with graceful evolution of which the most important is that this is a fault tollerant controlled transition. If failure in the actuating mechanisms occur (e.g. due to power loss) this will not lead to catastrophic behavior: in the region of the asymptotically stable equilibrium branch the system will settle down to a stable equilibrium, whereas in the region of the asymptotically stable periodic solutions branch it will settle down to a stable limit cycle. Thus, in any case, a predicted, stable behaviour is achieved. Other practical advantages are described in detail in Sultan [2007] where controlled transition (though not graceful) through a Hopf bifurcation has been numerically explored.

3. OPTIMIZATION FOR CONTROL DESIGN

To guarantee graceful evolution, for each of the two regions of interest (i.e. of the branch of equilibria and of the branch of periodic solutions, respectively) optimization control problems can be formulated. Controling the system such that its state space trajectory is arbitrarily close to a branch of equilibrium solutions has been previously addressed so in the following we shall only summarize one possible procedure based on the equilibrium branch parameterization.

3.1 Control in the Equilibrium Solutions Branch Region

Let $\lambda_e(s)$, $x_e(s) = g(\lambda_e(s))$ be a parameterization of the equilibrium branch and let the controls vary along this curve i.e. $\lambda(t) = \lambda_e(t)$, $t \in T_e$, where T_e is a time interval which is used to parameterize the curve. Let t_i denote the starting point of this interval. The length of T_e depends on several factors, for example the length of the segment of the curve used in this control method and the speed with which the control varies along this curve, as it will be discussed later. Clearly we should have $\lambda(t_i) = \lambda_e(t_i) = \lambda_i$. Let the corresponding solution to the initial value problem be labeled $x_d(t)$ and called the deployment path, i.e.,

$$\dot{x}_d = f(x_d, \lambda_e(t)), \quad x_d(t_i) = x_i. \quad (8)$$

An optimization problem to guarantee that $x_d(t)$ is close to the equilibrium branch can be formulated as

$$\min_{\lambda_e(t)} I \quad \text{subject to (8)} \quad \text{and}$$
$$\| x_d(t) - x_e(t) \| \leq \eta, \forall t \in T_e \quad (9)$$

where I is a performance index to be minimized (e.g. deployment time, which is the length of T_e, energy, etc.), $\|\|$ represents the Euclidean norm, and $\eta > 0$ a prescribed bound. Remark that now the system in (8) is non-autonomous because $\lambda_e(t)$ is time varying. Additional inequality constraints, such as quasi-stationarity constraints, i.e. $\| \dot{x}_d(t) \| \leq \delta$, where δ is a small positive scalar, or collision avoidance constraints can be considered, or the index I can be ignored altogether to simplify the problem (see Sultan [2007] for a detailed discussion of this problem). Khalil's results using Lyapunov functions on slowly varying systems (see Khalil [2002]) easily lead to a set of conditions that guarantee *arbitrarily* small η, of which the most important conditions are that $f(x, \lambda)$ is sufficiently smooth, the equilibrium branch is exponentially stable uniformly in λ for the frozen system, and the controls variation is sufficiently smooth and slow (this was the idea pursued, for example, in Sultan [2007]). In Sultan [2008] conditions for *arbitrarily* small η as well as *arbitrarily* small δ were sought after, avoiding Lyapunov functions in order to reduce conservatism. Using only basic concepts from topology and mathematical analysis less stringent conditions were proved (e.g. the controls can be piecewise constant and not smooth, exponential stability can be replaced by asymptotical stability, etc.). In these proofs enforcing the controls to vary only along the equilibrium curve is crucial whereas the results from slowly varying systems theory (Khalil [2002]) are not limited to this kind of parameterization.

3.2 Control in the Periodic Solutions Branch Region

In the region of periodic solutions branch the problem is much more complex. Let λ_a denote a value of the control in this region, $\lambda_H < \lambda_a < \lambda_f$. For graceful evolution the control law must be computed such that for any λ_a there exists a point on the corresponding periodic solution which is sufficiently close to the system's trajectory. In a controlled transition λ varies in time, i.e. $\lambda = \lambda(t)$ and when t reaches t_f then $\lambda(t)$ reaches λ_f and the control is fixed at λ_f. If the corresponding point on the system's trajectory belongs to the basin of attraction of the corresponding periodic solution, $x_f(s)$, the system's trajectory will settle down, asymptotically in time to $x_f(s)$ so the desired, final periodic solution will be achieved.

This is definitely a much more difficult problem than maintaining the system's trajectory close to a branch of equilibrium solutions and needs to be theoretically explored in future research to derive conditions for the system's trajectory to be sufficiently/arbitrarily close to the periodic solutions branch.

In the following, to illustrate the concept and clarify ideas, we shall use a simple example of a system exhibiting a supercritical Hopf bifurcation at $\lambda_H = 0$, namely

$$\dot{x_1} = x_1\lambda - x_2 - x_1^3 - x_1x_2^2 \tag{10}$$

$$\dot{x_2} = x_1 + x_2\lambda - x_2x_1^2 - x_2^3. \tag{11}$$

Clearly $x_1 = x_2 = 0$ is a branch of exponentially stable equilibria for $\lambda < 0$. At $\lambda = \lambda_H = 0$ this branch losses its stability in a transverse manner and a branch of asymptotically stable periodic solutions emerges for $\lambda > \lambda_H$. For the analysis of these periodic solutions transformation to polar coordinates is recommended, yielding (after obvious simplification through $R \neq 0$),

$$\dot{R} = R(\lambda - R^2) \tag{12}$$

$$\dot{\varphi} = 1 \tag{13}$$

which will be used to illustrate our methodology. Note that, for fixed λ, the initial condition problem (i.e. given R_0 for $t = t_0$) for (12) can be solved yielding

$$R^2 = \lambda \frac{1}{1 - \frac{R_0^2 - \lambda}{R_0^2} e^{-2\lambda(t-t_0)}}. \tag{14}$$

From now on the periodic solutions (obtained for $\lambda > \lambda_H = 0$) are of interest. The amplitude of the limit cycle generated through the Hopf bifurcation scales as $\sqrt{\lambda}$. Since the equations are uncoupled we will focus our attention on the $\dot{R} = R(\lambda - R^2)$ equation with initial condition

$$R(0) = R_0 << 1. \tag{15}$$

A control law $\lambda(t)$ is sought such that the evolution in this region is graceful. Let $R(t)$ be the solution to this initial condition problem (note that now, unlike for (14), λ is not a fixed number but a time varying function). For graceful evolution we want to find $\lambda(t)$ to solve

$$\min_{\lambda(t)} \sup_{t \in [0\ t_f]} \| R(t) - \sqrt{\lambda(t)} \|$$

subject to

$$\lambda(0) = 0, \quad \lambda(t_f) = \lambda_f = \sqrt{A}. \tag{16}$$

where A is the amplitude of the desired, final periodic solution (corresponding to $\lambda(t_f)$).

Introducing the nondimensional scales

$$\tilde{t} = t/t_f, \quad r = R/A \tag{17}$$

and the auxiliary function $\kappa\left(\tilde{t}\right) = \frac{\sqrt{\lambda(\tilde{t})}}{A}$, the above problem is rewritten as

$$\min_{\kappa(\tilde{t})} \sup_{\tilde{t} \in [0\ 1]} \left\| r\left(\tilde{t}\right) - \kappa\left(\tilde{t}\right) \right\| \tag{18}$$

subject to

$$r' = A^2 t_f r \left(\kappa^2 - r^2\right), r\left(0\right) = \frac{R_0}{A} \tag{19}$$

$$\kappa\left(0\right) = 0, \quad \kappa\left(1\right) = 1, \tag{20}$$

Here $'$ denotes differentiation with respect to $\tilde{t}$.

In the following we consider $A = 1$, so $\lambda_f = \sqrt{A} = 1$. The simplest function for $\kappa\left(\tilde{t}\right)$ which verifies the boundary conditions and one might be tempted to try out is the linear one,

$$\kappa\left(\tilde{t}\right) = \tilde{t}. \tag{21}$$

Fig. 1 shows the resulting "Error Function", defined as $\kappa\left(\tilde{t}\right) - r(\tilde{t})$, for $r_0 = 0.01$ and $t_f = 50, 150, 250, 350, 450, 550,$ in terms of the "Normalized Time", $\tilde{t}$. Fig. 2 shows the system's trajectory together with the Hopf surface (i.e. the surface generated by the limit cycles in the x, y, λ) space for $t_f = 50$. It is clear that the evolution is not graceful, especially for small t_f and small λ, when the error is large and the trajectory is rather far from the Hopf surface (2). Lengthening the time of deployment, t_f (i.e. using slower time varying control), somewhat alleviates the large error but not as dramatically as one would be tempted to believe. This was also remarked in Sultan [2007] using a different system exhibiting Hopf bifurcation. It has actually been ascertained previously Holden [1993] that the transition from oscillatory states to steady states through a supercritical Hopf bifurcation (i.e. the reverse of the process considered here) is also delayed even if the rate of change of the control is extremelly small. In order to further reduce the error a more complex control law is designed next.

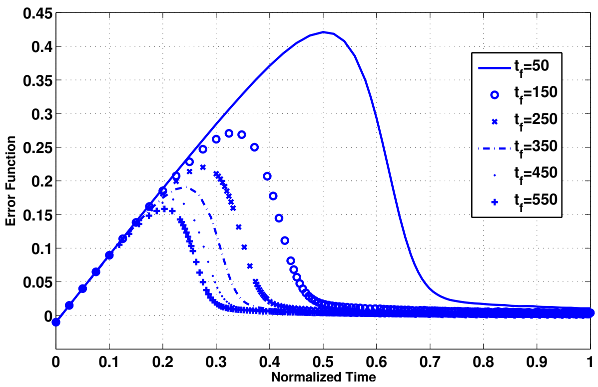

Fig. 1. Error function time variation for linear control

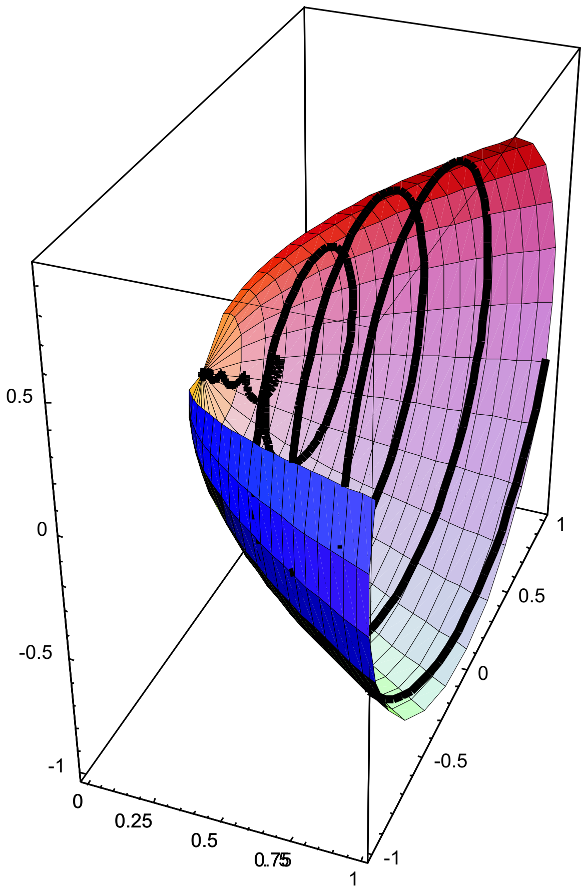

Fig. 2. Pictorial representation of the Hopf surface and system's trajectory for $t_f = 50$ and linear control

4. POLYNOMIAL APPROXIMATION OF THE CONTROLS

The simplest class of functions to satisfy the boundary conditions for $\kappa\left(\tilde{t}\right)$ are polynomials. Admissible polynomials $\kappa\left(\tilde{t}\right)$ consist of two parts, the base polynomial and the perturbing polynomial:

$$\kappa\left(\tilde{t}\right) = \kappa_0(\tilde{t}) + \Delta\kappa(\tilde{t}). \tag{22}$$

The base polynomial $\kappa_0(\tilde{t})$ satisfies the boundary conditions, while the perturbing polynomial $\Delta\kappa(\tilde{t})$ and its derivatives up to the required order vanish on the boundaries (higher order smoothness might be required to reduce mode excitation).

The perturbing polynomial is selected as a weighted sum of polynomials whose values and necessary derivatives vanish at both ends of the time interval and therefore do not affect the boundary conditions. These are the so-called *trim-constrained splines*. The perturbing polynomial $\Delta\kappa(\tilde{t})$ is herein written as a product of two parts: one which constrains the desired boundary conditions of $\Delta\kappa(\tilde{t})$ to be zero, and a weighted sum of basis polynomials

$$\Delta\kappa(\tilde{t}) = Q(\tilde{t}) \sum_{i=1}^{N} a_i \psi_i(\tilde{t}). \tag{23}$$

Here $Q(\tilde{t})$ is a polynomial, $\{\psi_i(\tilde{t})\}$ is a set of basis polynomials, and a_i are the weighting coefficients.

To ensure that the boundary conditions of $\kappa_0(\tilde{t})$ are not changed with the addition of $\Delta\kappa(\tilde{t})$, $Q(\tilde{t})$ is selected as

$$Q(\tilde{t}) = \tilde{t}^{\alpha}(1-\tilde{t})^{\beta}, \tag{24}$$

where $\alpha > 0$, $\beta > 0$ are natural numbers. Since the first $\alpha - 1$ and $\beta - 1$ derivatives of $Q(\tilde{t})$ are zero at the initial and final times (0 and 1) the same will be true for any polynomial resulting from the multiplication of $Q(\tilde{t})$ and another polynomial.

With this form for $Q(\tilde{t})$, the set of functions $\{\psi_i(\tilde{t})\}$ can be chosen to be any polynomial basis set. The combination of each $\psi_i(\tilde{t})$ with $Q(\tilde{t})$ forms a new set of trim-constrained splines, $\{\phi_i(\tilde{t})\}$,

$$\phi_i(\tilde{t}) = Q(\tilde{t})\psi_i(\tilde{t}). \tag{25}$$

For the boundary conditions in (20) the simplest base polynomial is, clearly, $\kappa_0(\tilde{t}) = \tilde{t}$ and the corresponding simplest $Q(\tilde{t})$ is

$$Q(\tilde{t}) = \tilde{t}(1-\tilde{t}). \tag{26}$$

Of course if higher order derivatives are desired zero at the end points, higher order polynomials should be considered.

Then the function

$$\kappa(\tilde{t}) = \tilde{t} + \sum_{i=1}^{N} a_i \phi_i(\tilde{t}) \tag{27}$$

with $\phi_i(\tilde{t})$ given by (25) matches the specified boundary conditions. The next step is to determine the coefficients a_i.

Once the number of basis functions N has been selected the previous infinite dimensional, functional optimization problem (18)-(20), reduces to a finite dimensional, parameter optimization problem over a_i for which many reliable solution algorithms exist. While t_f can represent an optimization parameter also, here it has been set to $t_f = 550$. In the following we provide the results for a simple example in which $\{\psi_i(\tilde{t})\} = \{1, \tilde{t}, \tilde{t}^2, ...\}$, $N = 3$, and with the additional constraints that $a_i \in [-2, 2]$. The minimum error is achieved by the choice $a_0 = -0.4$, $a_1 = a_2 = -1.6$ and it is substantially reduced compared to the linear control case (see Fig. (3) which shows the variation of the error with λ). Fig. (4) shows the corresponding polar radius, R, for this optimal trajectory and the radius of the limit cycles generated via the Hopf bifurcation as a function of λ. Clearly this optimal trajectory is very close to the branch of periodic solutions of interest.

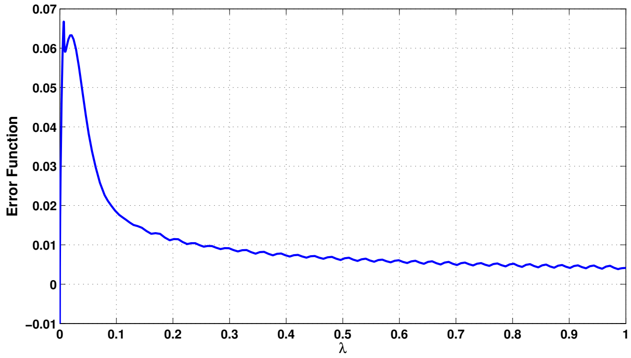

Fig. 3. Error function variation with λ for optimal control

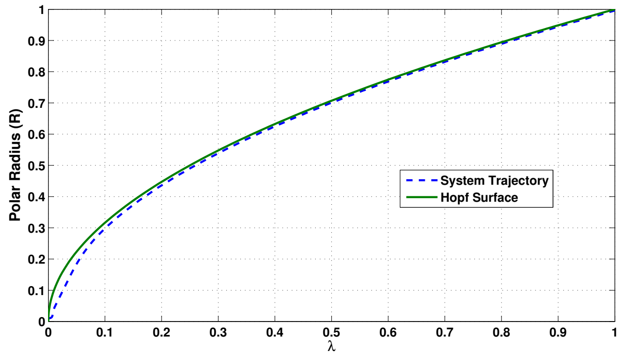

Fig. 4. Hopf surface and optimal trajectory polar radius variation with λ

CONCLUSIONS

The main motivation for conducting the passage of a nonlinear system of ODEs through a Hopf bifurcation in a graceful manner is the fault tolerance of the transition process with respect to unexpected events (e.g. power loss) which might result in freezing controls. If such a situation occurs, due to the close proximity of the system's trajectory to asymptotically stable equilibria or periodic solutions, the system settles down to predictable, asymptotically stable behaviour (equilibrium or limit cycle depending on the frozen control value). The idea is further explored and illustrated, with a focus on the branch of periodic solutions, using a very simple system exhibiting Hopf bifurcation. An optimization control problem is formulated and solved to ensure proximity of the system's trajectory to the branch of periodic solutions.

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