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On The Optimal Spacecraft Fuel Consumption in Low Earth Orbit

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Abstract: The optimization problem for a 6DOF satellite equipped with constant specific impulse thrusters is addressed in terms of *fuel consumption*. The vehicle is assumed to move in Low Earth Orbit: the Keplerian motion around the Earth is perturbed both by the molecular air impingement (aerodynamic drag) and by the gravitational field disturbance known as J_2 effect. Further results are achieved by simplifying the environmental model and by assuming that the spacecraft is modeled as a point mass.

Keywords: Spacecraft fuel consumption, primer vector theory, calculus of variations, optimal control.

1. INTRODUCTION

The analytical study herein presented addresses the optimal path planning problem for 6DOF spacecraft moving in Low Earth Orbit (LEO) between fixed endpoints over an assigned time interval $[t_1, t_2] \subset \mathbb{R}$ in terms of *fuel consumption*.

The nonlinear methods available to tackle the trajectory optimization problem range from energy matching (Yang, Yang, Kapila, & de Queiroz 2000) to purely geometrical approaches (Faller, Bender, Hall, Hils, & Vincent, 1984), as well as from classical optimal control theory (Ross, 2006) to random search methods (Sarma, 1990) and stochastic optimization (Engelhardt, & Chien, 2000). The case of vehicles modelled as 6DOF rigid bodies in semi-realistic environmental conditions commonly need purely numerical approaches to be tackled. These methods, which are often reliable, are plagued by two main issues: singularities, which characterize the fuel optimization problem (Anselmo, & al., 2005), and difficulty to prove whether results satisfy necessary and sufficient conditions needed for the optimization of the assigned cost index (Sultan, Seereram, & Mehra, 2007). Fundamental analytical results for spacecraft modelled as 3DOF point masses have been achieved by the primer vector theory (PVT) (Lawden, 1963) under the assumption that environmental forces are function of the spacecraft position only. The present work extends the fundamental results of PVT to 6DOF by assuming that the external forces are function of the spacecraft position and velocity. Specifically, spacecraft are subject to Keplerian forces, *aerodynamic drag* (i.e. molecular air impingement), and the gravitational field perturbation known as J_2 *effect*.

The fuel consumption is herein optimized by applying Pontryagin's Principle (PP) in the formulation of Theorem 8 (Pontryagin, Boltyasnskii, Gamkrelidze, & Mishchenko, 1963) with assigned initial and final states over a fixed time interval. Originally PVT was developed by casting the optimization problem as a problem of Mayer (Bliss, 1968) but PP is more advantageous as it allows to formally assume that the *control vector* $\mathbf{u}(\cdot):[t_1, t_2] \subset \mathbb{R} \to \Gamma \subset \mathbb{R}^6$ is an integrable function and Γ a closed set. The vector $\mathbf{u}$ represents the translational and the rotational acceleration provided by the Attitude and Orbit Control Subsystem (AOCS).

To model the fuel consumption of spacecraft equipped with constant specific impulse (CSI) thrusters for translational and rotational control, the *fuel consumption* is modelled as

$$J_F[\mathbf{u}(\cdot)] := \int_{t_1}^{t_2} \|\mathbf{u}(t)\|_2 \, dt. \tag{1}$$

Theoretical results achieved herein are also verified by numerical simulations.

2. PHYSICAL BACKGROUND

The spacecraft herein discussed is schematized as 6DOF rigid body of constant mass m and matrix of inertia I_{in}. Fixed an inertial reference frame, define the spacecraft *position vector* $\mathbf{r}(\cdot):[t_1, t_2] \to \mathbb{R}^3$, the *velocity vector* $\mathbf{v}(t) := d\mathbf{r}(t)/dt$, and the *acceleration vector* $\mathbf{a}(t) := d\mathbf{v}(t)/dt$. Furthermore, let $\boldsymbol{\sigma}(\cdot):[t_1, t_2] \to \mathbb{R}^3$ define the spacecraft attitude in modified Rodrigues parameters and let $\boldsymbol{\omega}(\cdot)$ be the angular velocity vector in a principal body reference frame. The superposition principle holds and, under Keplerian hypothesis, the vehicle moves in a radial gravitational field generated by the Earth. Assuming that the inertial reference frame coincides with the Earth centric reference frame, the spacecraft experiences both a gravitational acceleration $\mathbf{a}_g(\mathbf{r}) := -\mu\mathbf{r}/\|\mathbf{r}\|_2^3$, where μ is the gravitational constant, and a perturbing acceleration $\mathbf{a}_{J2}(\mathbf{r}) := -\frac{\mu J_2 R^2}{2}\frac{\partial}{\partial \mathbf{r}}\left[\left(3\cos(\varphi(\mathbf{r}))-1\right)/\|\mathbf{r}\|_2^3\right]$ due to the zonal

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coefficient J_2, where $\varphi(\cdot):\mathbb{R}^3 \to (-\pi,\pi] \subset \mathbb{R}$ is the colatittude of its argument. Furthermore aerodynamic perturbing accelerations acting on the spacecraft are modelled as $\tilde{\mathbf{a}}_a(\mathbf{v}) := -\tilde{k}_D \|\mathbf{v}\|_2^2 \hat{\mathbf{v}}$, with $\tilde{k}_D = \dfrac{\rho S C_D}{2m}$, ρ the molecular air density, S the reference area, C_D the drag coefficient, and $\hat{\mathbf{v}}$ the velocity unit vector. Let $\mathbf{r}_{cp}$ be the position of the centre of pressure of the spacecraft in a body reference frame. Then $\mathbf{M}_a(\mathbf{v}) := m\tilde{k}_D \|\mathbf{v}\|_2^2 (\hat{\mathbf{v}} \wedge \mathbf{r}_{cp})$ is the aerodynamic disturbance moment (Larson, & Wertz, 2005). Here the notation $\wedge$ denotes the cross product between vectors.

Let $\mathbf{x}_1 := [\mathbf{r}^T\ \mathbf{v}^T]^T$, $\mathbf{x}_2 := [\boldsymbol{\sigma}^T\ \boldsymbol{\omega}^T]^T$, and define $\mathbf{x} := [\mathbf{x}_1^T\ \mathbf{x}_2^T]^T$ as the *state vector*. The dynamic equations are therefore

$$\mathbf{x}' = \begin{bmatrix} \mathbf{v} \\ \mathbf{a}_g + \mathbf{a}_{J2} + \tilde{\mathbf{a}}_a \\ R_{rod}(\boldsymbol{\sigma})\boldsymbol{\omega} \\ -I_{in}^{-1}\omega^{\times} I_{in}\boldsymbol{\omega} + \boldsymbol{\Omega}_a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \mathbf{u} = \mathbf{f}_s(\mathbf{x}) + \mathbf{f}_u(\mathbf{u}) \tag{2}$$

where 0 and I are the zero and the identity matrices in $\mathbb{R}^3$ respectively, R_{rod} and $\omega^{\times}$ are matrices provided in Appendix A, and $\boldsymbol{\Omega}_a := I_{in}^{-1}\mathbf{M}_a$. The boundary conditions (BC) for (2) are $\mathbf{r}(t_i) = \mathbf{r}_i$, $\mathbf{v}(t_i) = \mathbf{v}_i$, $\boldsymbol{\sigma}(t_i) = \boldsymbol{\sigma}_i$, and $\boldsymbol{\omega}(t_i) = \boldsymbol{\omega}_i$, $i \in \{1,2\}$, where the vectors $\mathbf{r}_i$, $\mathbf{v}_i$, $\boldsymbol{\sigma}_i$, and $\boldsymbol{\omega}_i$ are given. Assume that $\mathbf{u}(\cdot) := [\mathbf{u}_1^T(\cdot)\ \mathbf{u}_2^T(\cdot)]^T$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the translational and the rotational control vectors, and $\mathbf{u}_i(\cdot):[t_1,t_2] \subset \mathbb{R} \to \Gamma_i \subset \mathbb{R}^3$, $i \in \{1,2\}$. Given the non-negative real constants ρ_1, ρ_2, ρ_3, ρ_4, we define the sets $\Gamma_1 := \{\mathbf{u}_1 : \rho_1 \le \|\mathbf{u}_1\|_2 \le \rho_2\} \cup \{\mathbf{0}_3\}$, $\Gamma_2 := \{\mathbf{u}_2 : \rho_3 \le \|\mathbf{u}_2\|_2 \le \rho_4\} \cup \{\mathbf{0}_3\}$, where $\mathbf{0}_3$ is the zero vector.

The controllability of the system is always assumed. Finally, to avoid singularities, impose that $\|\mathbf{v}\|_2 \ne 0$ and $\|\boldsymbol{\omega}\|_2 \ne 0$.

In the apex (') denotes the first derivative with respect to the independent variable, t.

3. FUEL CONSUMPTION OPTIMIZATION

Lemma 1: Optimal solutions $\mathbf{u}$ for (1) subject to (2) satisfy

$$\begin{cases} \begin{bmatrix} 0 & \mathcal{G}+\mathcal{J} & 0 & 0 \\ I & \mathcal{D} & 0 & \tilde{\mathcal{D}} \\ 0 & 0 & \Theta & 0 \\ 0 & 0 & R_{rod}^T & \mathcal{M} \end{bmatrix} \boldsymbol{\lambda}(t) = -\boldsymbol{\lambda}'(t) \\ \left[\boldsymbol{\lambda}_2^T(t)\ \boldsymbol{\lambda}_4^T(t)\right]^T = -\lambda_0 \mathbf{u}(t) / \|\mathbf{u}(t)\|_2 \end{cases} \tag{3}$$

with arbitrary BC on $\boldsymbol{\lambda}(t_i)$, $i \in \{1,2\}$. Furthermore the problem is normal, i.e. $\lambda_0 > 0$. The *costates* are $\boldsymbol{\lambda}(\cdot) := [\boldsymbol{\lambda}_1^T(\cdot)\boldsymbol{\lambda}_2^T(\cdot)\boldsymbol{\lambda}_3^T(\cdot)\boldsymbol{\lambda}_4^T(\cdot)]^T$ and λ_0.

The matrices Θ, $\mathcal{G}$, $\mathcal{J}$, $\mathcal{D}$, $\tilde{\mathcal{D}}$, $\mathcal{M}$, and their eigenpairs, if needed, are in Appendix A.

Proof: The Hamiltonian function is

$$H(\mathbf{x},\mathbf{u}) = \lambda_0 \|\mathbf{u}\|_2 + \boldsymbol{\lambda}^T \mathbf{x}'. \tag{4}$$

By imposing that $\partial H / \partial \mathbf{x} = -\boldsymbol{\lambda}'^T(t)$ and that $\partial H / \partial \mathbf{u} = \mathbf{0}_6^T$, (3) is achieved. Assume $\lambda_0 = 0$, then from (3) it follows that $\boldsymbol{\lambda} \equiv \mathbf{0}_6$, which is in contradiction with PP, hence the problem is *normal* and therefore from PP it follows that $\lambda_0 > 0$ □.

With negligible lack of rigour, because of the normality of the problem, it is hereafter assumed that $\lambda_0 = 1$.

The invertible symmetric matrix $\mathcal{G}$ accounts for the gravitational acceleration, $\mathcal{J}$ for the gravitational perturbing acceleration, $\mathcal{D}$, symmetric negative definite, and $\tilde{\mathcal{D}}$ for the aerodynamic perturbing effects, and finally $\mathcal{M}$ for the attitude dynamics. It is worth to stress that, if I_i and ω_i are the i-th moment of inertia in the principal body reference frame and the i-th component of the angular velocity vector respectively, $\mathcal{M}$ is invertible for $\omega_i \ne \omega_j$ and $I_i \ne I_j$, $(i,j) \in \{1,2,3\}^2$, $i \ne j$.

Lemma 2: Candidate optimal $\mathbf{u}$'s for (1) subject to (2) are:

Table 1. $\|\mathbf{u}_1\|_2$ and $\|\mathbf{u}_2\|_2$ as functions of $\|\boldsymbol{\lambda}_2\|_2$ and $\|\boldsymbol{\lambda}_4\|_2$

	$\|\boldsymbol{\lambda}_2\|_2$	$\|\boldsymbol{\lambda}_4\|_2$
> 1	$\|\mathbf{u}_1\|_2 = \rho_2$	$\|\mathbf{u}_2\|_2 = \rho_4$
$= 1$	Any $\|\mathbf{u}_1\|_2$ so that $\|\mathbf{u}_1\|_2 \boldsymbol{\lambda}_2 \in \Gamma_1$	Any $\|\mathbf{u}_2\|_2$ so that $\|\mathbf{u}_2\|_2 \boldsymbol{\lambda}_4 \in \Gamma_2$
< 1	$\|\mathbf{u}_1\|_2 = 0$	$\|\mathbf{u}_2\|_2 = 0$

Proof: According to PP, one needs to minimize (4) with respect to $\mathbf{u}$. Note that $H = \|\mathbf{u}\|_2 + \boldsymbol{\lambda}_2^T\mathbf{u}_1 + \boldsymbol{\lambda}_4^T\mathbf{u}_2 + \boldsymbol{\lambda}^T\mathbf{f}_s(\mathbf{x})$. The minimum for $\boldsymbol{\lambda}_2^T\mathbf{u}_1$ is $-\|\boldsymbol{\lambda}_2\|_2\|\mathbf{u}_1\|_2$ and is achieved when the two vectors are collinear. Similarly $\min_{\mathbf{u}_2} \boldsymbol{\lambda}_4^T\mathbf{u}_2 = -\|\boldsymbol{\lambda}_4\|_2\|\mathbf{u}_2\|_2$. Because $\|\mathbf{u}\|_2 \le \|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2$, one can write $H_{\min} \le (1-\|\boldsymbol{\lambda}_2\|_2)\|\mathbf{u}_1\|_2 + (1-\|\boldsymbol{\lambda}_4\|_2)\|\mathbf{u}_2\|_2 + \boldsymbol{\lambda}^T\mathbf{f}_s(\mathbf{x})$. Thus, if $\|\boldsymbol{\lambda}_2\|_2 > 1$, then $H_{\min}$ could be minimized for $\|\mathbf{u}_1\|_2 = \rho_2$; if $\|\boldsymbol{\lambda}_2\|_2 < 1$, then $H_{\min}$ could be minimized for $\|\mathbf{u}_1\|_2 = 0$;

finally if $\|\boldsymbol{\lambda}_2\|_2 = 1$, then $\|\mathbf{u}_1\|_2$ can assume any value such that $\mathbf{u}_1 \in \Gamma_1$. Table 1 is completed by applying the same reasoning to $\|\boldsymbol{\lambda}_4\|_2$ □.

It is worth to note that Lemma 2 is proven without accounting for any of the conditions of Lemma 1. Both these lemmae lead to the following relevant result:

Theorem 1: Candidate optimal $\mathbf{u}$ for (1) subject to (2) are such that

Table 2. $\|\mathbf{u}_1\|_2$ and $\|\mathbf{u}_2\|_2$ as functions of $\|\boldsymbol{\lambda}_2\|_2$

If	*Then*
$\|\boldsymbol{\lambda}_2\|_2 = 1$	Any $\|\mathbf{u}_1\|_2$ so that $\|\mathbf{u}_1\|_2 \boldsymbol{\lambda}_2 \in \Gamma_1 ; \|\mathbf{u}_2\|_2 = 0$
$\|\boldsymbol{\lambda}_2\|_2 = 0$	Any $\|\mathbf{u}_2\|_2$ so that $\|\mathbf{u}_2\|_2 \boldsymbol{\lambda}_4 \in \Gamma_2 ; \|\mathbf{u}_1\|_2 = 0$
$0 < \|\boldsymbol{\lambda}_2\|_2 < 1$	$\|\mathbf{u}_1\|_2 = \|\mathbf{u}_2\|_2 = 0$

Proof: According to PP both $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_4$ are real vector functions. Hence from Lemma 1 it follows that $\|\boldsymbol{\lambda}_2\|_2 \in [0,1]$ and $\|\boldsymbol{\lambda}_4\|_2^2 = 1 - \|\boldsymbol{\lambda}_2\|_2^2$. As a consequence, the first row of Table 1 does not hold and Table 2 is obtained.

From the second of (3) it follows that $\mathbf{u}_1 = -\boldsymbol{\lambda}_2 \|\mathbf{u}\|_2$ and $\mathbf{u}_2 = -\boldsymbol{\lambda}_4 \|\mathbf{u}\|_2$. By substituting these values in (4), the Hamiltonian evaluated along a candidate optimal trajectory is $H = \boldsymbol{\lambda}^T \mathbf{f}_s(\mathbf{x})$. It is relevant to notice that this value is independent of $\mathbf{u}$. By substituting in (4) the control laws $\mathbf{u}_1$ and $\mathbf{u}_2$ given in Table 2, it yields that $H = \boldsymbol{\lambda}^T \mathbf{f}_s(\mathbf{x})$. Hence the control law given in Table 2 is candidate optimal according to PP □.

Remark 1: The vectors $-\boldsymbol{\lambda}_2$ and $-\boldsymbol{\lambda}_4$ extend the concept of primer vector and therefore they are herein defined as the translational and the rotational primer vectors respectively. As a matter of fact, if the spacecraft is modelled as a point mass with 3DOF, then $-\boldsymbol{\lambda}_2$ reduces to the primer vector introduced in (Lawden, 1963). Moreover, according to PP, $\boldsymbol{\lambda}$ is continuous and from (3) it follows that also $\boldsymbol{\lambda}'$ is continuous, hence $\boldsymbol{\lambda} \in C^1(t_1, t_2)$.

It is relevant to notice that one major advantage of applying the control law of Table 2 is its practicality: most thrusters onboard some spacecraft provide "switching controls", which provide constant control forces over finite time intervals.

Theorem 2:: If $\|\mathbf{u}_1\|_2 = \|\mathbf{u}_2\|_2 = 0$, then there exist a constant $c \in \mathbb{R}$ such that $\boldsymbol{\lambda}^T \mathbf{x}' = c$ on (t_1, t_2).

Proof: According to the Weierstrass – Erdmann condition, on an optimal trajectory it holds that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, for $t \in (t_1, t_2)$. It is herein assumed that H does not explicitly depend on t, hence $\frac{\partial H}{\partial t} = 0$, and there exists a first integral $H = const$ □.

Theorem 3: Assume that the spacecraft is equipped with impulsive thrusters, then translational impulses $\mathbf{u}_1$ occur in the direction of $-\boldsymbol{\lambda}_2$ when $\|\boldsymbol{\lambda}_2\|_2 = 1$. Similarly, rotational impulses $\mathbf{u}_2$ occur in the direction of $-\boldsymbol{\lambda}_4$ when $\|\boldsymbol{\lambda}_2\|_2 = 0$.

Proof: This is a direct consequence of Theorem 2 □.

Theorem 4: Assume that the spacecraft is equipped with impulsive thrusters. If the spacecraft moves along an optimal trajectory, there exists a constant $\alpha \in \mathbb{R}$ such that $\alpha \boldsymbol{\lambda}_1^T \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3^T \boldsymbol{\lambda}_4 = 0$ for all $t \in (t_1, t_2)$.

Proof: It holds that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$, for $t \in (t_1, t_2)$. Integrating both sides of this equality over the infinitesimal duration of an impulsive thrust, since the right hand side remains finite throughout this time interval, one obtains that $H^+ - H^- = 0$ on an optimal trajectory. The superscripts (+) and (–) indicate some values immediately before and after an impulse. Thus H is continuous throughout the impulse. When the thrusters are not active, it holds that $H = \boldsymbol{\lambda}_1^T \mathbf{v} + \boldsymbol{\lambda}_2^T \mathbf{a} + \boldsymbol{\lambda}_3^T \boldsymbol{\sigma}' + \boldsymbol{\lambda}_4^T \boldsymbol{\omega}'$. According to PP $\boldsymbol{\lambda}$ is continuous on (t_1, t_2) and, by the environmental conditions of par. 2, both $\mathbf{a}$ and $\boldsymbol{\omega}'$. Hence $\boldsymbol{\lambda}_1^T \mathbf{v} + \boldsymbol{\lambda}_3^T \boldsymbol{\sigma}'$ must be continuous across the impulse, i.e. $\boldsymbol{\lambda}_1^T (\mathbf{v}^+ - \mathbf{v}^-) + \boldsymbol{\lambda}_3^T (\boldsymbol{\sigma}'^+ - \boldsymbol{\sigma}'^-) = 0$. Since the direction of an impulse is collinear with the corresponding primer vector (Theorem 3), i.e. $\mathbf{v}^+ - \mathbf{v}^-$ is parallel to $\boldsymbol{\lambda}_2$ and $\boldsymbol{\sigma}'^+ - \boldsymbol{\sigma}'^-$ is parallel to $\boldsymbol{\lambda}_4$, then there exist two constants $(\alpha_1, \alpha_2) \in \mathbb{R}_{++}^2$ so that $\alpha_1 \boldsymbol{\lambda}_1^T \boldsymbol{\lambda}_2 + \alpha_2 \boldsymbol{\lambda}_3^T \boldsymbol{\lambda}_4 = 0$ □.

By specifying the environmental model according to par. 2, further results can be achieved.

Corollary 1: Assume $\mathbf{a}_{J2} \equiv \mathbf{0}_3$, $\mathbf{r}_{cp} = \mathbf{0}_3$, $\mathbf{u}_2 \equiv \mathbf{0}_3$, and $\boldsymbol{\omega}(t_1) = \mathbf{0}_3$, then along an optimal trajectory

$$\boldsymbol{\lambda}_2^T \boldsymbol{\lambda}_1 > 0 \text{ and } \left\|\boldsymbol{\lambda}_1'\right\|_2 \le 2 / \|\mathbf{r}\|_2^3. \tag{5}$$

Proof: Since $\mathbf{u}_1 / \|\mathbf{u}_1\|_2 = -\boldsymbol{\lambda}_2$, from Lemma 1 it follows that $\|\boldsymbol{\lambda}_2\|_2 = 1$. As $\left(\boldsymbol{\lambda}_2^T \boldsymbol{\lambda}_2\right)' = 0$, $\boldsymbol{\lambda}_2^T \mathcal{D} \boldsymbol{\lambda}_2 = -\boldsymbol{\lambda}_2^T \boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2^T \boldsymbol{\lambda}_2' = 0$. As $\mathcal{D}$ is negative definite, then $\boldsymbol{\lambda}_2^T \boldsymbol{\lambda}_1 > 0$. Furthermore, under these assumptions $J \equiv 0$. Taking the Euclidean norm of $\boldsymbol{\lambda}_1'$, from the definition of matrix induced norm it follows that $\left\|\boldsymbol{\lambda}_1'\right\|_2 \le \varsigma_{\max}\left(\mathcal{G}\right)$, where $\varsigma_{\max}\left(\cdot\right)$ denotes the largest singular value of a matrix. As the singular values of $\mathcal{G}$ are equal to $2/\left\|\mathbf{r}\right\|_2^3$, the proof is concluded □.

Corollary 2: In addition to the hypothesis of Corollary 1, assume that $\tilde{k}_D = 0$ and $\boldsymbol{\lambda}_2 \in C^2\left(t_1,t_2\right)$. Then along an optimal trajectory it holds that

$$\left|l_3\right| \le \sqrt{\frac{2+\left\|\mathbf{r}\right\|_2^2\left\|\boldsymbol{\lambda}_1\right\|_2^4}{3}} \text{ and } \left\|\boldsymbol{\lambda}_1\right\|_2^2 \le \frac{1}{\left\|\mathbf{r}\right\|_2} \tag{6}$$

with $l_i\left(\cdot\right):\left[t_1,t_2\right]\to\mathbb{R}$, $i\in\left\{1,2,3\right\}$, the *i*-th component of the $\boldsymbol{\lambda}_2$ in a principal reference frame in the space of $\boldsymbol{\lambda}_2$.

Proof: Under these assumptions $\boldsymbol{\sigma}\left(t\right) = const$ and then from Lemma 1 it follows that

$$\begin{cases}\boldsymbol{\lambda}_2'' = \mathcal{G}\boldsymbol{\lambda}_2,\ \boldsymbol{\lambda}_2' = -\boldsymbol{\lambda}_1 \\ \mathbf{u}_1/\left\|\mathbf{u}_1\right\|_2 = -\boldsymbol{\lambda}_2,\ \boldsymbol{\lambda}\left(t_1\right),\ \boldsymbol{\lambda}\left(t_2\right) \text{ arbitrary.}\end{cases} \tag{7}$$

Since $\boldsymbol{\lambda}_2^T\boldsymbol{\lambda}_2 = 1 \quad \forall t\in\left[t_1,t_2\right]$, then $\frac{d^2}{dt^2}\boldsymbol{\lambda}_2^T\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_2^T\boldsymbol{\lambda}_2'' + {}$ $+\boldsymbol{\lambda}_2'^T\boldsymbol{\lambda}_2' = 0$ and from (7) we have

$$\left\{\boldsymbol{\lambda}_2^T\mathcal{G}\boldsymbol{\lambda}_2 = -\boldsymbol{\lambda}_1^T\boldsymbol{\lambda}_1;\ \boldsymbol{\lambda}_2^T\boldsymbol{\lambda}_2 = 1.\right. \tag{8}$$

Diagonalizing $\mathcal{G}$ and accounting for its eigenpairs, $l_2^2 = \frac{2+\left\|\mathbf{r}\right\|_2^2\left\|\boldsymbol{\lambda}_1\right\|_2^4}{3} - l_3^2$, $l_1^2 = \frac{1-\left\|\mathbf{r}\right\|_2^2\left\|\boldsymbol{\lambda}_1\right\|_2^4}{3}$ □.

Having l_1, l_2, and l_3, it is possible to deduce the vector $\boldsymbol{\lambda}_2$, hence $\mathbf{u}_1/\left\|\mathbf{u}_1\right\|_2$ from (7). This result also applies in the context of the classical PVT.

Remark 4: For spacecraft one can assume that $\left\|\mathbf{r}\right\|_2^2 \gg 1$. Consequently $l_1 \approx 0$, $l_2 \approx 0$, and $\left|l_3\left(t\right)\right| \approx 1$ and therefore the direction of $\mathbf{u}_1$ is then given by

$$\boldsymbol{\lambda}_2 = l_3\frac{r_3}{\left\|\mathbf{r}\right\|_2}\left[r_1\ -r_1\ r_2\right]^T \tag{9}$$

where r_i, $i\in\left\{1,2,3\right\}$, is the *i*-th component of $\mathbf{r}$.

4. ANALYSIS ON $\partial\Gamma_1$ AND $\partial\Gamma_2$

Lemma 3: Assume that $\mathbf{a} = \mathbf{u}_1 + \mathbf{a}_g + \mathbf{a}_{J2}$, $\tilde{k}_D \equiv 0$, and that $\mathbf{u}\in C^2\left(t_1,t_2\right)$, then any candidate optimal solution $\mathbf{u}$ for (1) subject to (2) verifies the following relations:

$$\begin{cases}\left\|\left(\mathcal{G}+\mathcal{J}\right)^{-1}\boldsymbol{\Lambda}\right\|_2 \le \left\|\mathbf{u}\right\|_2^4, \\ \left\|\mathcal{M}^{-1}\left[\left\|\mathbf{u}\right\|_2\mathbf{u}_2' + \left(\mathbf{u}'^T\mathbf{u}\right)\mathbf{u}_2 + \left\|\mathbf{u}\right\|_2^2 R_{rod}^T\boldsymbol{\lambda}_3\right]\right\|_2 \le \left\|\mathbf{u}\right\|_2^2\end{cases} \tag{10}$$

with the vector $\boldsymbol{\Lambda} := \left(\mathbf{u}'^T\mathbf{u}\right)\left[\left(\left\|\mathbf{u}\right\|_2^3 + \left\|\mathbf{u}\right\|_2^2 - 2\left\|\mathbf{u}\right\|_2\right)\mathbf{u}_2' - 2\mathbf{u}_1\right] + {}$ $+\left\|\mathbf{u}\right\|_2^2\left[\left\|\mathbf{u}\right\|_2\mathbf{u}_1'' + \left(\mathbf{u}''^T\mathbf{u} + \mathbf{u}'^T\mathbf{u}'\right)\mathbf{u}_1\right]$.

Proof (brief): Eq. (10) is achieved from (3) by noticing that $\mathcal{D}$=$\tilde{\mathcal{D}}$=0, that $\tilde{\mathbf{a}}_a \equiv \mathbf{0}_3$, and by exploiting $\boldsymbol{\lambda}_i'$, $i\in\left\{2,4\right\}$, as function of $\mathbf{u}$ □.

Lemma 3 can be specialized for $\mathbf{u}_1\in\partial\Gamma_1$ and $\mathbf{u}_2\in\partial\Gamma_2$:

Theorem 5: On an optimal trajectory it holds that

$$\begin{cases}\left\|\left(\mathcal{G}+\mathcal{J}\right)^{-1}\boldsymbol{\lambda}_2''\right\|_2 \le 1 \\ \left\|\mathcal{M}^{-1}\left(R_{rod}^T\boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_4'\right)\right\|_2 \le 1\end{cases} \tag{11}$$

Proof: From Theorem 1 it follows that, if $0 < \left\|\boldsymbol{\lambda}_2\right\|_2 < 1$, then (10) become two identities. Both $\boldsymbol{\Lambda}$ and the second of (10) can be specialized for $\left\|\boldsymbol{\lambda}_2\right\|_2 = 1$ and $\left\|\boldsymbol{\lambda}_2\right\|_2 = 0$. By noticing that, if $\left\|\mathbf{u}_i\right\|_2 = \rho_{2\cdot i}$, $i\in\left\{1,2\right\}$, then $\mathbf{u}_i$ lays upon a sphere or radius $\rho_{2\cdot i}$ centred in $\mathbf{0}_3$, (11) can be achieved □.

5. NUMERICAL SIMULATIONS

Consider a spacecraft in a geocentric equatorial reference frame that needs to perform in 40 *min* a manoeuvre from $\left[6900\ 0\ -450\right]^T$ *km* to $\left[-6500\ 150\ 660\right]^T$ *km*. Impose that the initial and the final velocities are such that the spacecraft leaves from the perigee of an orbit of eccentricity 0.6 and reaches at the apogee an orbit of eccentricity 0.8 whose major axes coincide. The vector $\mathbf{u}_1$ at the initial and final positions is chosen parallel to the velocity vector. The spacecraft is modelled as a parallelepiped of 100 *kg* whose volume is $2\times2\times2\ m^3$. It is assumed that the vector normal to one of the surfaces of the spacecraft has initially a radial direction with respect to the Earth and that at the end of the manoeuvre it is rotated of $3/2\pi$ around the intermediate inertial axis with a rest to rest manoeuvre. It is imposed that $\mathbf{r}_{cp} = 10^{-2}\left[1\,1\,1\right]^T m$ in the principal body reference frame. Finally it is assumed that $\rho_2 = \rho_4 = 15\frac{m}{s^2}$, that the system is controllable, and that impulsive thrusters are mounted on the vehicle. By applying Theorem 1 the optimal trajectory shown in Fig. 1 is achieved. The Earth is scaled for clarity. The plots of $\left\|\boldsymbol{\lambda}_2\right\|$ and $\left\|\boldsymbol{\lambda}_4\right\|$ are in Fig 2.

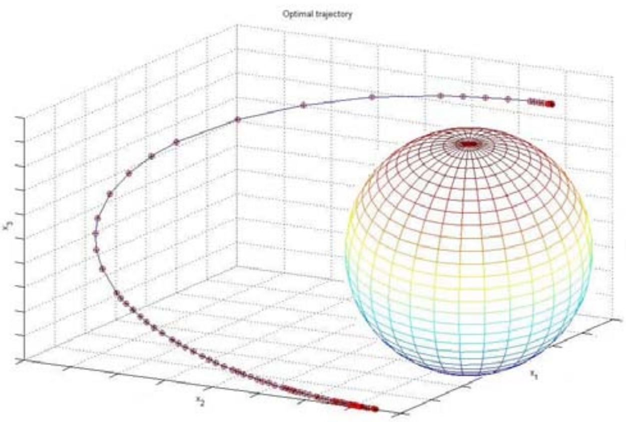

Fig. 1. Optimal trajectory for the given mission scenario.

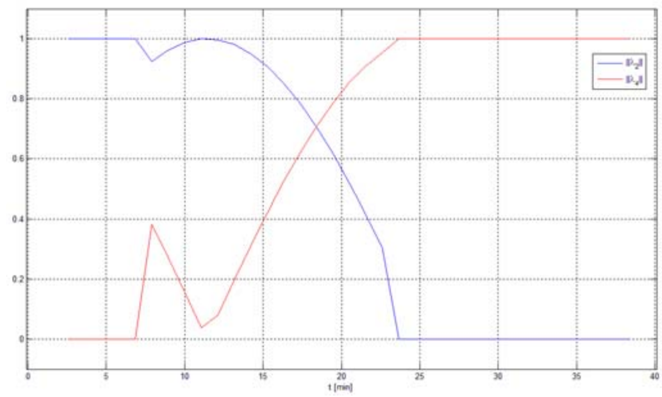

Fig. 2. Plots of $\|\boldsymbol{\lambda}_2\|$ and $\|\boldsymbol{\lambda}_4\|$.

6. CONCLUSIONS

The *fuel consumption* problem for a 6DOF rigid spacecraft moving between fixed endpoints over an assigned time interval has been herein tackled. A realistic model of the Low Earth Orbit space environment has been considered, as well as more schematic ones. This study led to several necessary conditions to identify the optimal control vector and to verify results achieved by numerical integration.

Some results of Lawden's primer vector theory have been extended to 6DOF vehicles subject to external forces depending on the spacecraft position and velocity. Pontryagin's principle has been employed as it allows formally discussing integrable control vector functions and arbitrarily constrained control sets. Numerical simulations presented verify the analytical results proven.

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Appendix A. MATRICES

Let the subscript (i, j) determine the element on the *i*-th row and *j*-th column of a matrix and let the single subscript *k* be the *k*-th component of a vector, with $(i, j, k) \in \{1,2,3\}^3$.

$$R_{rod}\left(\boldsymbol{\sigma}(t)\right) = \frac{1}{4}\left[\left(1-\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)I + 2\sigma^{\times} + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]$$

$$\sigma^{\times}(t) = \begin{bmatrix} 0 & -\sigma_3(t) & \sigma_2(t) \\ \sigma_3(t) & 0 & -\sigma_1(t) \\ -\sigma_2(t) & \sigma_1(t) & 0 \end{bmatrix}.$$

$$\Theta\left(\boldsymbol{\sigma}(t)\right) = \frac{1}{2}\boldsymbol{\sigma}^T(t)\boldsymbol{\omega}(t) I +$$
$$+\frac{1}{2}\begin{bmatrix} 0 & -\omega_1\sigma_2 + \omega_2\sigma_1 + \omega_3 & -\omega_1\sigma_3 - \omega_3\sigma_1 + \omega_2 \\ \omega_1\sigma_2 - \omega_2\sigma_1 - \omega_3 & 0 & \omega_3\sigma_2 - \omega_2\sigma_3 + \omega_1 \\ \omega_1\sigma_3 + \omega_3\sigma_1 - \omega_2 & -\omega_3\sigma_2 + \omega_2\sigma_3 - \omega_1 & 0 \end{bmatrix}$$

$$\mathcal{M}\left(\boldsymbol{\omega}(t)\right) = \begin{bmatrix} 0 & -\omega_Y(t)\tilde{I}_2 & -\omega_P(t)\tilde{I}_3 \\ -\omega_Y(t)\tilde{I}_1 & 0 & -\omega_R(t)\tilde{I}_3 \\ -\omega_P(t)\tilde{I}_1 & -\omega_R(t)\tilde{I}_2 & 0 \end{bmatrix}$$

with $\tilde{I}_1 = (I_P - I_Y)/I_R$, $\tilde{I}_2 = (I_R - I_Y)/I_P$, $\tilde{I}_3 = (I_P - I_R)/I_Y$, $I_{R/P/Y}$ and $\omega_{R/P/Y}$ the moments of inertia and the angular velocity unit vectors respectively along the first, second, and third axis of the principal body reference frame.

$$\omega^{\times}(t) = \begin{bmatrix} 0 & -\omega_Y(t) & \omega_P(t) \\ \omega_Y(t) & 0 & -\omega_R(t) \\ -\omega_P(t) & \omega_R(t) & 0 \end{bmatrix}.$$

$$\mathcal{D}_{(i,i)}(\mathbf{v}(t)) = -\tilde{k}_D \left(v_i^2(t) + \|\mathbf{v}(t)\|_2^2\right) \Big/ \|\mathbf{v}(t)\|_2$$

$$\mathcal{D}_{(i,j)}(\mathbf{v}(t)) = -\tilde{k}_D v_i(t) v_j(t) \Big/ \|\mathbf{v}(t)\|_2$$

The eigenvalues of $\mathcal{D}$ are $-\|\mathbf{v}\|_2 [1 \;\; 1 \;\; 2]^T$. The eigenvectors $[-v_2/v_1 \;\; 1 \;\; 0]^T$, $[-v_3/v_1 \;\; 0 \;\; 1]^T$, and $\mathbf{v}$ respectively.

$$\mathcal{G}(\mathbf{r}(t)) = \frac{3\mu}{\|\mathbf{r}(t)\|_2^5} \mathbf{r}(t)\mathbf{r}^T(t) - \frac{\mu}{\|\mathbf{r}(t)\|_2^3} I$$

Eigenvectors of $\mathcal{G}$ are $\mathbf{r}$, $[1 \;\; 0 \;\; -r_1/r_3]^T$, $[0 \;\; 1 \;\; -r_2/r_3]^T$ with eigenvalues $\mu(2,-1,-1) \Big/ \|\mathbf{r}\|_2^3$.

$$\tilde{\mathcal{D}}_{(i,i)}(\mathbf{v}(t)) = -\mathcal{D}_{(i,i)}(\mathbf{v}(t)) \left(\mathbf{v}(t) \wedge \mathbf{r}_{cp}\right)_i^{'}$$

$$\tilde{\mathcal{D}}_{(i,j)}(\mathbf{v}(t)) = -\mathcal{D}_{(i,j)}(\mathbf{v}(t)) \left(\mathbf{v}(t) \wedge \mathbf{r}_{cp}\right)_i^{'}.$$

$$J_{(i,i)}(\mathbf{r}(t)) = -\frac{3\mu J_2 R^2}{2} \frac{(4r_3(t) - 1)\left(\|\mathbf{r}(t)\|_2^2 - 6r_i^2(t)\right)}{\|\mathbf{r}(t)\|_2^8}, \; i \in \{1,2\}$$

$$J_{(3,3)}(\mathbf{r}(t)) = -\frac{3\mu J_2 R^2}{2\|\mathbf{r}(t)\|_2^9} \Big[12 r_3^3(t) \|\mathbf{r}(t)\|_2 - 3r_3^2(t) \|\mathbf{r}(t)\|_2^2 +$$

$$-r_3^4(t) - 12 r_3(t) \|\mathbf{r}(t)\|_2 \left(r_1^2(t) + r_2^2(t)\right) + r_2^2(t)\left(2r_1^2(t) + r_2^2(t)\right)\Big]$$

$$J_{(i,j)}(\mathbf{r}(t)) = -\frac{9\mu J_2 R^2 r_i(t) r_j(t)}{\|\mathbf{r}(t)\|_2^8} (4r_3(t) - 1), \; (i,j) \in \{1,2\}^2$$

$$J_{(3,j)}(\mathbf{r}(t)) = \frac{3\mu J_2 R^2 r_j(t)}{\|\mathbf{r}(t)\|_2^8} \left(2\|\mathbf{r}(t)\|_2^2 + 12 r_3^2(t) + 3r_3(t)\right), \; j \in \{1,2\}$$

$$J_{(i,3)}(\mathbf{r}(t)) = -\frac{3\mu J_2 R^2 r_j(t)}{\|\mathbf{r}(t)\|_2^8} \Big[20 r_3^2(t) - 5r_3(t) \|\mathbf{r}(t)\|_2^2 +$$

$$-4\|\mathbf{r}(t)\|_2 \left(\|\mathbf{r}(t)\|_2^2 - r_3^2(t)\right)\Big], \; i \in \{1,2\}.$$