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Optimal Tracking Design of an MIMO Linear System with Quantization Effects ⋆

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Abstract: This paper studies optimal design for a linear time-invariant (LTI) MIMO discrete-time networked feedback system in tracking a step signal. It is assumed that the outputs of the controller are quantized by logarithm quantization laws, respectively, and then transmitted through a communication network to the remote plant in the feedback system, whereas the quantization errors in all quantized signals are modeled as a product of a white noise with zero mean and the source signal respectively, the variances of the white noises are determined by the accuracies of the quantization laws. The tracking performance of the system we interested in is defined as the averaged energy of the error between the output of the plant and the reference input. Three problems are studied for the system: 1) For a set of given logarithm laws, how to design an optimal stabilizing controller for the closed-loop system in mean-square stability sense? 2) What is a minimal communication load to stabilize the networked feedback system in terms of the characteristics of the logarithm quantization laws? 3) For a set of given logarithm laws, how to design an optimal controller to achieve minimal tracking cost? We find that the problems 1 and 3 have a unique solution, respectively, and obtain an analytic solution for problem 2 when the plant is a minimum phase system.

1. INTRODUCTION

The network has been widely used in applications over the last two decades. Due to the fact that networked control systems include some unique features which are likely to degrade performance of these systems, the design of networked control systems must incorporate the inherent constraints brought about by the use of communication channels. Because of this reason, there has been growing attention devoted to the studies of networked control problems (for example see [5], [7], [11] and the references therein). In these studies, an overriding theme has been the modeling and use of communication links or networks for control, and accordingly, whether and how networked control systems can be stabilized via feedback over the duly modeled communication links. While significant understanding has been achieved on stabilization problems subject to quantization effects ([4], [5] and [7]), time delays [12], bandwidth constraints [2] and bit rate limitations [11] of communication channels, seldom have performance aspects of networked feedback systems been addressed, which the present paper seeks to investigate.

Our particular goal dwells on the question as to how quantization errors may adversely affect performance of a networked feedback system. In the system, logarithm quantization laws which can be used to model floating point arithmetic numerical calculations are adopted and the quantization errors are modeled as a product of the source signal and a white noise (see [14] and [6]). Under this model, the quantized control system in our research is actually a linear system with multiplicative noises. Such system which also has rich applications in many engineering problems has been studied for long time (see for example [13] and [10] etc.). [13] formulated the mean-square (or second-order stochastic) stability problem for both discrete-time and continuous-time linear time-invariant (LTI) SISO feedback systems with a multiplicative noise and presented a sufficient and necessary condition of the stability for the systems. [10] presented a small gain theorem for mean-square stability of a discrete-time LTI MIMO system. It was shown by [10] that the optimal $\mathcal{H}_2$ design for the system via output feedback is related to the solution of a matrix equation with high nonlinearity. In their work, a locally optimal output feedback design approach was presented.

In this paper, the optimal tracking problem is studied for the system via output feedback. The tracking performance is measured by the averaged energy of the error between the system's output and a given reference input. Our formulation postulates a step signal as the reference input, and that the control signals are quantized and then transmitted to the plant via a communication network. To achieve asymptotical tracking, the internal mode principle

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is used in the control design. And it is shown that the averaged tracking error energy of the system is bounded if and only if the closed-loop system is mean-square stable. For a minimum phase plant, we find that the optimal robust stabilization design is a generalized eigenvalue problem which has a unique solution (see for example [1]). On the other hand, we investigate minimum communication load in stabilizing the feedback system with quantization effect. A channel capacity is defined in terms of the signal-to-noise ratios of the quantization laws. An explicit expression is presented for the minimum capacity which is need in stabilizing a minimum phase MIMO system via output feedback. This problem was also studied in [8] and [15] for MIMO linear discrete time systems via state feedback with the sector bound model of logarithm quantization laws, and packet loss in communication channels, respectively.

The remainder of this paper is organized as follows. We proceed in Section 2 to formulate the optimal tracking problem for networked control system under quantization constraint. A stochastic model is presented for the quantization laws in the system. In Section 3, the averaged tracking performance is derived in terms of the model of quantization law and two parameter control scheme (i.e.,Youla parametrization) for the feedback system. The stabilization and tracking problem are studied for SISO systems. In Section 4, for given quantization laws, the largest stable radius and minimum channel capacity for stabilizing the networked feedback system are studied, respectively. We then proceed in Section 5 to investigate the minimum tracking performance in terms of the characteristics of the quantization laws. Section 6 concludes the paper.

The notation used throughout this paper is fairly standard. For any complex number z, we denote its complex conjugate by $\bar{z}$. For any vector u, we denote its transpose by u^T and conjugate transpose by u^*. For any matrix A, the transpose and conjugate transpose are denoted by A^T and A^*, respectively. For any real rational function $f(z)$, $z \in \mathbb{C}$, define $f^{\sim}(z) = f(1/z)$. Denote the expectation operator with respect to the random variable α by $\underset{\alpha}{E}\{\cdot\}$. Let the open unit disc be denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the closed unit disc by $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, the unit circle by $\partial\mathbb{D}$, and the complements of $\mathbb{D}$ and $\bar{\mathbb{D}}$ by $\mathbb{D}^c$ and $\bar{\mathbb{D}}^c$, respectively. The space $\mathcal{L}_\infty$ is a Banach space and defined by

$$\mathcal{L}_\infty := \left\{ f : f(k),\ k = 0, 1, 2, \cdots,\ \sup_k |f(k)| < \infty \right\}.$$

The space $\mathcal{L}_2$ is a Hilbert space and defined by

$$\mathcal{L}_2 := \left\{ f : f(z) \text{ measurable in } \partial\mathbb{D}, \quad \|f\|_2 := \left(\frac{1}{2\pi} \int_\pi^\pi \|f(e^{j\theta})\|^2 d\theta \right)^{1/2} < \infty \right\}.$$

For the space $\mathcal{L}_2$, the inner product is defined as

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(e^{j\theta}) g(e^{j\theta}) d\theta.$$

It is well-known that $\mathcal{L}_2$ admits an orthogonal decomposition into the subspaces

$$\mathcal{H}_2 := \left\{ f : f(z) \text{ analytic in } \bar{\mathbb{D}}^c, \quad \|f\|_2 := \left(\sup_{r>1} \frac{1}{2\pi} \int_\pi^\pi \|f(re^{j\theta})\|^2 d\theta \right)^{1/2} < \infty \right\},$$

and

$$\mathcal{H}_2^\perp := \left\{ f : f(z) \text{ analytic in } \mathbb{D}, \quad \|f\|_2 := \left(\sup_{r<1} \frac{1}{2\pi} \int_\pi^\pi \|f(re^{j\theta})\|^2 d\theta \right)^{1/2} < \infty \right\}.$$

Thus, for any $f \in \mathcal{H}_2^\perp$ and $g \in \mathcal{H}_2$, $\langle f, g \rangle = 0$. Similarly, define

$$\mathcal{H}_\infty := \{ f : f(z) \text{ bounded and analytic in } \mathbb{D}^c \}.$$

A subset of $\mathcal{H}_\infty$, denoted by $\mathcal{RH}_\infty$, is the set of all proper stable rational transfer functions in the discrete-time sense. Note that we have used the same notation $\|\cdot\|_2$ to denote the corresponding norm for spaces $\mathcal{L}_2$, $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$.

2. PROBLEM STATEMENTS

In this section, we formulate the tracking problem for a networked linear feedback system which is depicted in Fig. 1. The plant in the system is an MIMO system whose control signals are sent from a remote site through network channels. The network effect is modeled as a set of parallel quantizers, i.e.,

$$Q = \operatorname{diag}\{Q_1, \cdots, Q_m\}$$

with noise free channels NC. The control signals are quantized and transmitted to the plant through communication channels independently. The quantization law adopted is logarithm quantization (see for example [7]).

The reference signal r of the system is a step signal, i.e.,

$$r(k) = \begin{cases} r_0, & \text{for} \quad k = 0, 1, 2, 3, \cdots, \\ 0, & \text{for} \quad k < 0 \end{cases} \tag{1}$$

where r_0 is a real constant vector. To achieve asymptotical tracking, a set of integrators are used to cope with the tracking error which is caused by the quantizer Q. The

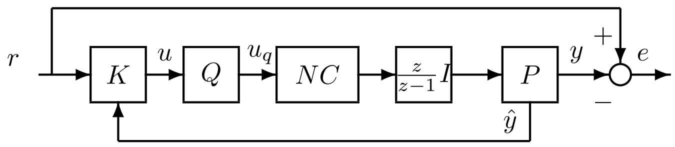

Fig. 1. Structure of closed-loop networked control system

signals $y(k)$ and $\hat{y}(k)$ are the tracking output and the measurement, respectively. The tracking error is $e(k) = r(k) - y(k)$. The plant transfer matrix function P is partitioned compatibly as follows:

$$P(z) = \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} \tag{2}$$

where the outputs of $G(z)$ and $H(z)$ are $y(k)$ and $\hat{y}(k)$, respectively.

Compared with the systems in conventional tracking problem, the system shown in Fig. 1 includes a new component quantizer Q which yields a quantization error Δu,

$$\Delta u(t) = \begin{bmatrix} \Delta u_1(t) \\ \vdots \\ \Delta u_m(t) \end{bmatrix} = \begin{bmatrix} u_1(t) - Q_1(u_1(t)) \\ \vdots \\ u_m(t) - Q_m(u_m(t)) \end{bmatrix}. \quad (3)$$

It is shown in [5] and [7] that the quantization error in every channel is a product of its source signal and relative quantization error, i.e.,

$$\Delta u_i(t) = \omega_i u_i(t), \quad i = 1, \cdots, m.$$

The relative quantization errors ω_i, $i = 1, \cdots, m$ satisfy that

$$|\omega_i(k)| \le \delta_i < 1 \quad (4)$$

where δ_i is a measure to the accuracy of the quantization law Q_i. Due to high nonlinearities which are involved in the quantization error ω_i, precisely modeling this error is a hard task. It is turned out in [6] that when the quantization density is not too small (i.e., the quantization law has middle or high resolution approximately), the quantization error and source signal are uncorrelated,

$$E\{u_i(k)\Delta u_i(k)\} = 0, \quad i = 1, \cdots, m. \quad (5)$$

Moreover, it is shown by numerical simulations that the signal-to-noise for logarithm quantization law is given by

$$\frac{E\{u_i^2(k)\}}{E\{\Delta u_i^2(k)\}} \approx \frac{\delta_i^2}{3}, \quad i = 1, \cdots, m. \quad (6)$$

Similar results are presented in [14] with complete analysis for floating point quantization which can be considered as a practical model of the logarithm quantization law. In this work, a multiplicative noise model is used to describe the statistical features (5) and (6) of the quantization law.

Denote the whole of relative quantization error ω caused by the quantizer Q by:

$$\omega = \operatorname{diag}\{\omega_1, \cdots, \omega_m\}.$$

We assume that three assumptions hold for relative quantization errors $\omega(k)$, $k = 0, 1, \cdots, \infty$.

Assumption 1. The relative quantization error $\omega_i(k)$ is a random variable which has uniform distribution over $[-\delta_i, \delta_i]$. For any $k_1,\ k_2 \in \{0, 1, 2, \cdots\}$, it holds that

$$E[\omega_i(k_1)\omega_i(k_2)] = \begin{cases} \sigma_i^2, & k_1 = k_2 \\ 0, & k_1 \ne k_2 \end{cases}$$

where $\sigma_i^2 = \frac{\delta_i^2}{3}$.

Assumption 2. The relative quantization error $\omega_i(k)$ is independent of the source signal $u_i(k)$ of the quantizer Q_i, i.e., $E\{\omega_i(k)u_i(k)\} = 0, \quad k > 0$.

Assumption 3. For any $i, j \in \{1, \cdots, m\}$, $i \ne j$, it holds that $E\{\omega_i(k_1)\omega_j(k_2)\} = 0, \quad k > 0, \quad \forall k_1, k_2 > 0$.

What we are interested in is: Beyond the limitation on the tracking performance from the plants in the traditional feedback systems (see for example [3]), what are constraints from quantization error ω on the tracking performance. In our study, the tracking performance is measured by an averaged tracking error energy regarding to the relative quantization error ω, i.e.,

$$J = \mathop{E}_{\omega}\left\{\sum_{k=0}^{\infty}[r(k) - y(k)]^2\right\}. \quad (7)$$

Denote the set of the controllers by $\mathcal{K}$, with which the averaged tracking error energy of the closed-loop system regrading to the relative quantization error ω is bounded. The best tracking performance is

$$J_{opt} = \inf_{K \in \mathcal{K}}\left\{\mathop{E}_{\omega}\left\{\sum_{k=0}^{\infty}[r(k) - y(k)]^2\right\}\right\}. \quad (8)$$

It will be shown in this work that the cost function J given in (7) is a function of variances $\{\sigma_1, \cdots, \sigma_m\}$ of the relative quantization errors $\omega_1, \cdots, \omega_m$. Denote Ω be a set of variances $\{\sigma_1, \cdots, \sigma_m\}$ such that there exists at least one controller K to result a bounded tracking cost, i.e., $K \in \mathcal{K}$. To study the minimum performance J_{opt}, we consider the mean-square stability for the system.

Definition 1. Under Assumptions 1-3, the linear system in Fig. 1 is said to be mean-square stable if every input sequence $r(k)$ with bounded second-order stochastic, i.e., $E\{r^2(k)\} \in \mathcal{L}_\infty$, generates error and output sequences $\{e(k)\}$, $\{y(k)\}$, with bounded second-order stochastic, i.e., $E\{e^2(k)\}, E\{y^2(k)\} \in \mathcal{L}_\infty$.

The problems under studied in this work are:

P1: *How to design controller K such that the resultant closed-loop system to achieve optimal robust stability in mean-square sense for given $\{\sigma_1, \cdots, \sigma_m\}$?*

Notice the fact that the variance σ_i is a characteristic of the quantizer Q_i for its accuracy and the communication load associated with the i-th channel. To measure this communication load, we borrow the concept of the channel capacity from communication theory, which is defined by the signal to noise ratio of the i-th channel $\frac{1}{\sigma_i^2}$ as below:

$$C_i = \frac{1}{2}\log\left(1 + \frac{1}{\sigma_i^2}\right).$$

The total capacity of m channels is defined as:

$$C_p = \sum_{i=1}^{m}\frac{1}{2}\log\left(1 + \frac{1}{\sigma_i^2}\right).$$

The problem which we are interested in is:

P2: *What is the minimum value of C_p for all possible $\{\sigma_1, \cdots, \sigma_m\} \in \Omega$?*

The third problem which we are interested in is:

P3: *For given $\{\sigma_1, \cdots, \sigma_m\} \in \Omega$, how to design an optimal control K such that the cost function J achieves minimum?*

3. TRACKING PERFORMANCE UNDER QUANTIZATION EFFECT

For the tracking problem to be meaningful, we make the following assumption throughout this paper.

Assumption 4. $P(z), G(z)$ and $H(z)$ have the same unstable poles.

This assumption means that the plant is stablilizable via output feedback of the measurement $\hat{y}$, a premise to achieve the tracking, while the measurement channel dose not introduce more unstable poles.

It is noticeable that when the quantization effect is void, the tracking problem in Fig. 1 is degraded to conventional

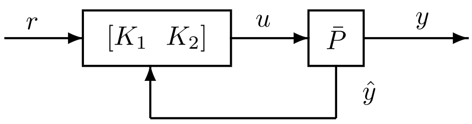

Fig. 2. Traditional tracking problem

tracking problem shown in Fig. 2, in which two parameter control scheme is adopted, i.e.,

$$u(z) = K_1(z)r(z) + K_2(z)\hat{y}(z)$$

and the plant $\bar{P}$ is defined as

$$\bar{P}(z) = \frac{z}{z-1}P(z).$$

Subsequently, let

$$\bar{G}(z) = \frac{z}{z-1}G(z), \quad \bar{H}(z) = \frac{z}{z-1}H(z).$$

To design a two parameter control scheme for the system, let right and left coprime factorization of the plant $\bar{H}$ be LM^{-1} and $\tilde{M}^{-1}\tilde{L}$. And let the right and left coprime factorization of the plant $\bar{G}$ be NM^{-1} and $\tilde{M}^{-1}\tilde{N}$. The factors L, M, N, $\tilde{L}$, $\tilde{M}$, $\tilde{N}$ are from $\mathcal{R}H_\infty$, and the factors L, M, $\tilde{L}$, $\tilde{M}$ satisfy Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{L} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ L & X \end{bmatrix} = I \tag{9}$$

where X, Y, $\tilde{X}$, $\tilde{Y} \in \mathcal{R}H_\infty$.

All possible controllers for the system are given

$$[K_1, K_2] = (\tilde{X} - R_2\tilde{L})^{-1}\begin{bmatrix} R_1 & \tilde{Y} - R_2\tilde{M} \end{bmatrix} \tag{10}$$

where R_1, $R_2 \in \mathcal{R}H_\infty$ are parameters to be designed.

Apply the two parameter control scheme (10) to the networked system. The system is restructured to that in Fig. 3 where $C_0 = (\tilde{X} - R_2\tilde{L})^{-1}$. Let $T = (Y - MR_2)\tilde{L}$

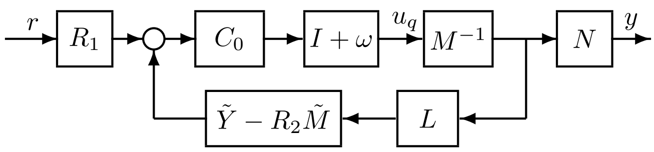

Fig. 3. Closed-loop system after loop transformations

and denote the ij-th entry of T by T_{ij}. The necessary and sufficient condition of mean-square stability for the system is as follows:

Lemma 1. (see [10], [13]) The system in Fig. 3 is mean-square stable if and only if

$$\rho\left(\begin{bmatrix} \sigma_1^2\|T_{11}\|_2^2 & \cdots & \sigma_1^2\|T_{1m}\|_2^2 \\ \vdots & \ddots & \vdots \\ \sigma_m^2\|T_{m1}\|_2^2 & \cdots & \sigma_m^2\|T_{mm}\|_2^2 \end{bmatrix}\right) < 1 \tag{11}$$

where $\rho(\cdot)$ is the spectral radius of the matrix.

Now, the tracking performance is considered for the system. It follows from the structure of the system that the tracking error $e(k)$ of the system is given by

$$e(k) = \begin{bmatrix} e_1(k) \\ \vdots \\ e_m(k) \end{bmatrix} = (I - NR_1)r(k) - N(\tilde{X} - R_2\tilde{L})\Delta u(k). \tag{12}$$

Let

$$e_0(k) = (I - NR_1)r(k) = [e_{01}(k), \cdots, e_{0m}(k)]^T$$

and

$$Q_1 = N(\tilde{X} - R_2\tilde{L}).$$

Denote the impulse response of ij-th entry in Q_1 by $\{q_{ij}(k),\ k = 0, 1, 2, \cdots, \infty\}$. Then we have

$$e_i(k) = e_{0i}(k) + \sum_{\tau=0}^{k}\sum_{j=1}^{m} q_{ij}(k-\tau)\Delta u_j(\tau). \tag{13}$$

From Assumptions 1-3, the variance of the tracking error $e_i(k)$ is given as below:

$$E_\omega\left\{e_i^2(k)\right\} = e_{0i}^2(k) + \sum_{\tau=0}^{k}\sum_{j=1}^{m}\sigma_j^2 q_{ij}^2(k-\tau)E_\omega\left\{u_j^2(\tau)\right\}.$$

With straightforwardly manipulations, we obtain

$$J = \sum_{i=1}^{m}\sum_{k=0}^{\infty} e_{0i}^2(k) + \sum_{j=1}^{m}\sigma_j^2\left[\sum_{i=1}^{m}\sum_{k=0}^{\infty} q_{ij}^2(k)\right]\sum_{\tau=0}^{\infty}E_\omega\left\{u_j^2(\tau)\right\}. \tag{14}$$

Denote j-th column by $[N(\tilde{X} - R_2\tilde{L})]_j$. It holds that

$$\|[N(\tilde{X} - R_2\tilde{L})]_j\|_2^2 = \sum_{i=1}^{m}\sum_{k=0}^{\infty} q_{ij}^2(k). \tag{15}$$

Notice that

$$\sum_{i=1}^{m}\sum_{k=0}^{\infty} e_{0i}^2(k) = \|r(z) - NR_1r(z)\|_2^2. \tag{16}$$

Substituting (15) and (16) into (14) leads to

$$J = \|r(z) - NR_1r(z)\|_2^2 + \sum_{j=1}^{m}\sigma_j^2\|[N(\tilde{X} - R_2\tilde{L})]_j\|_2^2\sum_{\tau=0}^{\infty}E_\omega\left\{u_j^2(\tau)\right\}. \tag{17}$$

Notice that the control signal is given by

$$u = MR_1r + (Y - MR_2)\tilde{L}\Delta u. \tag{18}$$

Applying the same argument as that used in deriving the equation (17) yields that

$$\begin{bmatrix} \underset{\omega}{E}\|u_1\|_2^2 \\ \vdots \\ \underset{\omega}{E}\|u_m\|_2^2 \end{bmatrix} = \begin{bmatrix} \|M_1R_1r\|_2^2 \\ \vdots \\ \|M_mR_1r\|_2^2 \end{bmatrix} + W^T\begin{bmatrix} \underset{\omega}{E}\|u_1\|_2^2 \\ \vdots \\ \underset{\omega}{E}\|u_m\|_2^2 \end{bmatrix} \tag{19}$$

where

$$W = \begin{bmatrix} \sigma_1^2\|[(Y - MR_2)\tilde{L}]_{11}\|_2^2 & \cdots & \sigma_1^2\|[(Y - MR_2)\tilde{L}]_{m1}\|_2^2 \\ \vdots & \ddots & \vdots \\ \sigma_m^2\|[(Y - MR_2)\tilde{L}]_{1m}\|_2^2 & \cdots & \sigma_m^2\|[(Y - MR_2)\tilde{L}]_{mm}\|_2^2 \end{bmatrix} \tag{20}$$

and $[(Y - MR_2)\tilde{L}]_{ij}$ is (i, j)-th entry of the matrix, and M_i, $i = 1, \cdots, m$ are i-th rows of M, respectively.

Substituting (19) into (17) leads to

$$J = \|(1 - NR_1)r(z)\|_2^2 + \sum_{i=1}^{m}\alpha_i\left\|M_iR_1r(z)\right\|_2^2 \tag{21}$$

and

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = (I - W)^{-1} V, \quad V = \begin{bmatrix} \sigma_1^2 \|[N(\tilde{X} - R_2 \tilde{L})]_1\|_2^2 \\ \vdots \\ \sigma_m^2 \|[N(\tilde{X} - R_2 \tilde{L})]_m\|_2^2 \end{bmatrix}. \tag{22}$$

Remark 1. The feedback system asymptotically track a step reference if and only if $\rho(W) < 1$. Hence, from Lemma 1, asymptotical tracking is achievable for the system in tracking a step reference if and only if the feedback system is mean-square stable. In this case, the internal principle plays a key role. Without the integrators in the system, the tracking performance function J may not be bounded, i.e., mean-square stabiliy of the system does not imply that asymptotical tracking is achievable for the system.

If the plant H in the system is an SISO plant, the tracking performance (21) is written as

$$J = \|(1 - NR_1) r(z)\|_2^2 + \alpha \|MR_1 r(z)\|_2^2 \tag{23}$$

and

$$\alpha = \frac{\sigma^2 \|N(\tilde{X} - R_2 \tilde{L})\|_2^2}{1 - \sigma^2 \|(Y - MR_2)\tilde{L}\|_2^2}.$$

Theorem 1. If H is an SISO plant, the optimal tracking is a quasi-convex problem. If the plant H is a minimum phase system with relative degree one, the resultant closed-loop is second-order stochastic stable if and only if

$$\log\left(1 + \frac{1}{\sigma^2}\right) > \frac{1}{2} \sum_1^n \log|\lambda_i|.$$

Due to space limit, the proof is omitted. In the remainder part of this work, we will study the results for the MIMO system.

4. LARGEST STABLE RADIUS AND MINIMUM CHANNEL CAPACITY

In this section, the optimal design of robust stabilization is studied for the networked system shown in Fig. 3.

Notice the fact that the the closed-loop system in Fig. 3 is mean-square stable if and only if the inequality (11) holds or $\rho(W) < 1$. For any given $\sigma_1, \cdots, \sigma_m$, a minimum spectral radius of the matrix W yields a good robustness in second-order stochastic stability to the resultant closed-loop system. The robust stabilization problem P1 presented in Section 2 is formulated as below: For any given $\{\sigma_1, \cdots, \sigma_m\} \in \Omega$, find an optimal controller K to minimize the radius of the matrix W, i,e,:

$$K = \arg \inf_{K \in \mathcal{K}} \rho(W(\sigma_1, \cdots, \sigma_m)). \tag{24}$$

It is well-known (see [9]) that for positive matrix W (all entries of W are positive), $\inf_{R_2} \rho(W) < 1$ if and only if it holds that

$$\inf_{\Gamma} \inf_{R_2} \|\Gamma W \Gamma^{-1}\|_\infty < 1 \tag{25}$$

where $\|\cdot\|_\infty$ is a matrix norm (see [9]) and Γ is any positive diagonal matrix, $\Gamma = \text{diag}\{\gamma_1, \cdots, \gamma_m\}$, $\gamma_i > 0$, $i = 1, \cdots, m$.

Selecting an appropriate R_2 leads to

$$\begin{aligned} &\inf_{\Gamma} \inf_{R_2} \|\Gamma W \Gamma^{-1}\|_\infty \\ &= \max\left\{\sigma_i^2 \|e_i - M_{\Gamma in} M_{\Gamma in}^{-1}(\infty) e_i\|_2^2, i = 1, \cdots, m\right\} \end{aligned} \tag{26}$$

where $M_{\Gamma in}$ is an inner of ΓM.

Denote the realization of $M_{\Gamma in}$ by :

$$M_{\Gamma in} = \left[\begin{array}{c|c} A_{\Gamma in} & B_{\Gamma in} \\ \hline C_{\Gamma in} & D_{\Gamma in} \end{array}\right].$$

It holds that

$$A_{\Gamma in}^* X A_{\Gamma in} - X + C_{\Gamma in}^* C_{\Gamma in} = 0 \tag{27}$$

and

$$\|M_{\Gamma in} M_{\Gamma in}^{-1}(\infty) e_i - e_i\|_2^2 = e_i^T D_{\Gamma in}^{*-1} B_{\Gamma in}^* X B_{\Gamma in} D_{\Gamma in}^{-1} e_i. \tag{28}$$

So the minimization problem for the spectral radius of W is to find minimum $\gamma_0 > 0$ such that, for $i = 1, \cdots, m$,

$$\sigma_i^2 e_i^T D_{\Gamma in}^{*-1} B_{\Gamma in}^* X B_{\Gamma in} D_{\Gamma in}^{-1} e_i < \gamma_0 \tag{29}$$

or

$$\sigma_i^2 e_i^T D_{in}^{*-1} B_{in}^* X B_{in} D_{in}^{-1} e_i < \gamma_0 e_i^T \Gamma^2 e_i. \tag{30}$$

This minimization problem is a generalized eigenvalue problem. It has a unique solution (see for example [1]).

Theorem 2. For any given $\{\sigma_1, \cdots, \sigma_m\} \in \Omega$, if the plant zH is invertible in $\mathcal{R}H_\infty$, the robust stabilization design problem is a quasi-convex problem.

Now, the problem P2 in Section 2 is studied. To do this, we need following lemma.

Lemma 2. For a given plant $\bar{H}$, there exist coprime matrices L and $M \in \mathcal{R}H_\infty$ such that

$$\bar{H} = LM^{-1}$$

and the matrix M is upper triangular, with each diagonal element m_{ii}, $i = 1, \cdots, m$ given by

$$m_{ii} = \frac{\prod_1^{k_i}(z - \lambda_j)}{\prod_1^{k_i}(1 - \lambda_j z)} \tag{31}$$

where $\lambda_{i1}, \cdots, \lambda_{ik_i}$, $i = 1, \cdots, m$ are all unstable poles of $\bar{H}$.

The proof of this lemma is omitted due to space limit.

Theorem 3. If the plant zH is invertible in $\mathcal{R}H_\infty$ and $\lambda_1, \cdots, \lambda_k$ are unstable poles of the plant, the closed-loop system is stabilizible, i.e.,

$$\{\sigma_1, \cdots, \sigma_m\} \in \Omega$$

if and only if the total channel capacity C_p satisfies:

$$C_p = \sum_{i=1}^m \frac{1}{2} \log\left(1 + \frac{1}{\sigma_i^2}\right) \geq \sum_{j=1}^k \log|\lambda_j|. \tag{32}$$

The proof of this theorem is omitted. The key of this proof is that by selecting an appropriate Γ and using Lemma 2, the matrix $\Gamma W \Gamma^{-1}$ approaches a diagonal matrix. This leads to (32).

In the robust stabilization problem, our goal is to minimize the spectral radius of the matrix W for given $\sigma_1, \cdots, \sigma_m$ and in minimal capacity problem, our goal is to minimize the total capacity C_p. For an MIMO system, these problems have no a common solution in general. However, for a SISO systems, these two different problems are solved by minimizing the function $\|(Y - MR_2)\tilde{L}\|_2^2$. In general, if the directions of all poles of the plant are parallel or orthogonal mutually, these two problems share a common solution.

5. OPTIMAL TRACKING FOR MINIMUM PHASE SYSTEMS

Now, the optimal design which minimize the cost function J in (21) is studied. Here, we only consider the case in which $\tilde{L}$ is invertible in $\mathcal{R}H_\infty$. Selecting $R_2 = \hat{R}_2\tilde{L}^{-1}$ leads to

$$(Y - MR_2)\tilde{L} = Y\tilde{L} - M\hat{R}_2$$

and

$$N(\tilde{X} - R_2\tilde{L}) = N(\tilde{X} - \hat{R}_2).$$

Denote the i-th column of the matrix $\hat{R}_2$ by $\hat{R}_{2i}$.

Lemma 3. For a given i, $1 \le i \le m$, if the all columns $\hat{R}_{2j}$, $j \ne i$, of $\hat{R}_2$ are given, then the weights $\alpha_1, \cdots, \alpha_m$ in (22) for the cost function J in (21) are quasi-convex in $\hat{R}_{2i}$ and have a common minimizer $\hat{R}^*_{2i}$.

Proof: Denote the i-th row of matrix W in (20) and i-th entry of vector V in (22) by w_i and v_i, respectively. It is clear that w_i and v_i are functions of $\hat{R}_{2i}$ only. Let W_i and V_i be W and V with the i-th row zeroed out, respectively. It is well-known that

$$(I - W)^{-1} = (I - W_i)^{-1} + \frac{1}{1 - w_i(I - W_i)^{-1}e_i}(I - W_i)^{-1}e_i w_i (I - W_i)^{-1}.$$

For any nonzero $\varphi = [\varphi_1, \cdots, \varphi_m]$ with $\varphi_i \ge 0$, $i = 1, \cdots, m$, consider the cost function β,

$$\beta = \varphi(I - W)^{-1}V.$$

It can be verified that

$$\beta = \varphi(I - W_i)^{-1}V_i + \varphi(I - W_i)^{-1}e_i \frac{w_i(I - W_i)^{-1}V_i + v_i}{1 - w_i(I - W_i)^{-1}e_i}.$$

Notice the facts that $(I - W_i)^{-1}e_i$ is independent of $\hat{R}_{2i}$, its entries are all non-negative. The minimizer of β is determined by $\dfrac{w_i(I - W_i)^{-1}V_i + v_i}{1 - w_i(I - W_i)^{-1}e_i}$ only. Since w_i and v_i are convex in $\hat{R}_{2i}$. The function β is quasi-convex in $\hat{R}_{2i}$ and its minimizer $\hat{R}^*_{2i}$ is unique and independent of the choice of φ. Hence, all α_i will have the same unique minimizer.

The results above lead to the following main result:

Theorem 4. The cost function J in (7) is quasi-convex in $\hat{R}_2$.

The proof is omitted due to space limit.

6. CONCLUSION

In this paper, the optimal tracking problem of a networked control system with an MIMO linear discrete-time plant in tracking a step signal has been discussed. Quantization is the only constraint from network under consideration. The linear feedback system with quantization is modeled a stochastic system with multiplicative noises. We found that for minimum phase plants the optimal design for mean square stabilization and optimal tracking design problems have unique optimal solutions, respectively. The minimum communication capacity, which is related to the accuracies of quantization laws, for the stabilization problem is presented.

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