Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

A General Framework for the Analysis of the Quasi-Static Regime ⋆

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Abstract: Hysteresis is a nonlinear behavior encountered in a wide variety of processes in which the input-output dynamic relations between variables involve memory effects. In Ikhouane (2010) a framework aimed to characterize these systems is proposed. The systems that are considered are seen as operators that map an input signal and initial condition to an output signal, all of them belonging to some specified sets. A definition of the quasi-static regime is motivated and proposed. The aim of the present paper is to generalize the concepts introduced in Ikhouane (2010).

Keywords: Input/output, operator, quasi-static regime.

1. INTRODUCTION

Motivated by the mathematical characterization of hysteresis systems, Ikhouane (2010) proposed a framework for the analysis of the quasi-static regime. The systems that are considered are seen as operators that map an input signal and initial condition to an output signal, all of them belonging to some specified sets. The inputs and output are taken to be finite dimensional; however, if the system (or operator) has a state description, the state may be finite or infinite dimensional. The system (or operator) may be continuous or discontinuous, and if it is a hysteresis it may be rate-dependent or rate-independent. One of the main results of Ikhouane (2010) is that, when the input signal is such that its total variation is increasing, the existence, uniqueness and mathematical description of the quasi-static regime can be done in a very general framework in which the only assumption on the operator is causality. When the input signal can be constant on some interval (or intervals) so that its total variation is only nondecreasing, an additional assumption has to made on the operator in order to characterize its quasi-static regime. This condition expresses that, when the input stops and remains constant, the output stops and remains constant.

The objective of the present paper is to show that the quasi-static regime can be defined and analyzed even in the absence of the constant input constant output property. Within the same framework, the paper notes that, when the input is such that its total variation is increasing, the constant input constant output assumption is not needed to define the quasi-static regime. For this reason, if the input signal can be constant on some interval (or intervals) so that its total variation is only nondecreasing, a series of signals are constructed in such a way that their total variation is increasing, and they converge in some sense to the input signal. If the operator is such that a converging sequence of inputs leads to a converging sequence of outputs (in some precise sense), then the quasi-static regime can be defined using these sequences of inputs and outputs.

It is shown that this definition is consistent with the one given in Ikhouane (2010). Also, the relationship with this new definition of quasi-static operators and rate-independence is explored. A case-study is presented to illustrate the concepts that are introduced in the paper. Due to space limitation, the proofs have been eliminated. They are given in Ikhouane (2009).

2. BACKGROUND RESULTS

This section summarizes the results obtained in Ikhouane (2010).

2.1 Mathematical preliminaries

In this section, we present some mathematical tools that will be useful in the next sections.
The Lebesgue measure on $\mathbb{R}$ is denoted μ. We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Consider a function $f : I \subset \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^m$; we say that f is measurable when f is (M, B)-measurable where B is the class of Borel sets of $\mathbb{R}^m$ and M is the class of measurable sets of $\mathbb{R}_+$. For a measurable function $f : I \subset \mathbb{R}_+ \to \mathbb{R}^m$, $\|f\|_{\infty,I}$ denotes the essential supremum of the function $|f|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. When $I = \mathbb{R}_+$, it will be denoted simply $\|f\|_\infty$.
We consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u : \mathbb{R}_+ \to \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative $\dot{u}$ is defined a.e. and is equal a.e. to the weak derivative

⋆ Supported by grant DPI2008-06463-C02-01 of the Spanish Ministry of Science and Innovation.

of u. Moreover, we have $\|u\|_\infty < \infty$ and $\|\dot{u}\|_\infty < \infty$. Endowed with the norm $\|u\|_{1,\infty} = \max(\|u\|_\infty, \|\dot{u}\|_\infty)$, $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space Adams and Fournier (2003).

For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0,t]$.

$$\rho_u(t) = \int_0^t |\dot{u}(\tau)| \,\mathrm{d}\tau \in \mathbb{R}_+$$

The function $\rho_u(t)$ is well defined as $\dot{u} \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$. [1] It is nondecreasing and absolutely continuous. Denote $\rho_{u,\max} = \lim_{t\to\infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ is necessarily finite).
- $I_u = [0, \rho_{u,\max})$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ may be finite or infinite).

Now, given some value $\varrho \in I_u$, there exists at least some $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$ due to the continuity of ρ_u. The value t_ϱ may not be unique as the function ρ_u is not necessarily increasing.

Lemma 1. $u(t_\varrho)$ is independent of the particular choice of t_ϱ. It depends solely on ϱ.

Lemma 1 shows that we can define the following function

$$\begin{aligned} \psi_u : I_u &\to \mathbb{R}^n \\ \varrho &\to u(t_\varrho) \end{aligned}$$

The function ψ_u depends only on the function u, and we have $\mathrm{Dom}(\psi_u) = I_u$. Note that we have $\psi_u \circ \rho_u = u$.

Lemma 2. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant. Then, $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$, $\|\psi_u\|_{\infty,I_u} = \|u\|_\infty$ and $\|\dot{\psi}_u\|_{\infty,I_u} = 1$. Also, $\mu\left[\left\{\varrho \in I_u / \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right] = 0$.

We consider the linear time scale change $s_\gamma(t) = \frac{t}{\gamma}$ for any $\gamma > 0$.

Lemma 3. For any $\gamma > 0$, we have $I_{u\circ s_\gamma} = I_u$ and $\psi_{u\circ s_\gamma} = \psi_u$.

Definition 1. Suppose that the function $w : \mathbb{R}_+ \to \mathbb{R}$ is continuous and periodic with the period $T > 0$. Furthermore we assume that there exists a scalar $0 < T^+ < T$ such that the function w is increasing and C^1 on the interval $(0, T^+)$, and decreasing and C^1 on the interval (T^+, T). We denote $w_{\min} = w(0)$ and $w_{\max} = w(T^+) > w_{\min}$ the minimal and maximal values of the function w respectively. Due to the particular shape of w, we call it wave-periodic.

Lemma 4. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a non-constant T-periodic input. Then, $I_u = \mathbb{R}_+$ and ψ_u is periodic of period $\rho_u(T) > 0$. Moreover, if u is wave-periodic as in Definition 1, then ψ_u is also wave-periodic and we have $\dot{\psi}_u(\varrho) = 1$ for $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for $\varrho \in (\rho_u(T^+), \rho_u(T))$.

2.2 Class of operators

Let Ξ be a set of initial conditions. Let $\mathcal{H}$ be an operator that maps the input function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ to an output in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ where m is a positive integer. So we have $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.

In this paper we consider only causal operators. This means that, for any function $y = \mathcal{H}(u, \xi^0)$, the value $y(t)$ may depend on the input values $u(\tau)$ for $\tau \le t$, but cannot depend on any value $u(\tau)$ for $\tau > t$. This is an intrinsic property of all physical systems so that it is a natural assumption for an operator that represents a physical system. Mathematically, this can be written as (Visintin, 1994, p.60): $\forall (u_1, \xi^0), (u_2, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, if $u_1 = u_2$ in $[0, \alpha]$, then $\mathcal{H}(u_1, \xi^0) = \mathcal{H}(u_2, \xi^0)$ in $[0, \alpha]$.

2.3 Constant input constant output operators

In this section, we consider that the operator $\mathcal{H}$ of Section 2.2 satisfies the following.

Assumption 1. Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$; if there exists a time instant $\theta \in \mathbb{R}_+$ such that u is constant in $[\theta, \infty)$, then the corresponding output $\mathcal{H}(u, \xi^0)$ is constant in $[\theta, \infty)$.

Note that, when the input u is constant on $\mathbb{R}_+$, the corresponding output is constant on $\mathbb{R}_+$ by Assumption 1 so that the graph output versus input $\{(u(t), y(t)), \forall t \ge 0\}$ is reduced to a single point which makes the analysis of the quasi-static regime irrelevant. For this reason, unless otherwise specified, we will consider inputs that are not constant on $\mathbb{R}_+$ which, due to the absolute continuity of u, implies that $\|\dot{u}\|_\infty \neq 0$ and $\overset{\circ}{I}_u \neq \emptyset$.

Some properties of the operator $\mathcal{H}$ Now, let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ and let $y = \mathcal{H}(u, \xi^0) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ where the causal operator $\mathcal{H}$ satisfies Assumption 1.

Lemma 5. For any $0 \le t_1 < t_2$, if u is constant on the interval $[t_1, t_2]$, then y is constant on $[t_1, t_2]$.

Lemma 6. Let $\varrho \in I_u$ be given and let $t_\varrho \in \mathbb{R}_+$ such that $\rho_u(t_\varrho) = \varrho$. Then, $y(t_\varrho)$ is independent of t_ϱ. It depends solely on ϱ.

Lemma 6 shows that we can define a function

$$\begin{aligned} \varphi_u : I_u &\to \mathbb{R}^m \\ \varrho &\to y(t_\varrho) \end{aligned}$$

The function φ_u depends only on the input u and the initial condition ξ^0, and we have $\mathrm{Dom}(\varphi_u) = I_u$. Note that we have $\varphi_u \circ \rho_u = y$.

Lemma 7. $\varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ and $\|\varphi_u\|_{\infty,I_u} \le \|y\|_\infty$. If y is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty,I_u} = \|y\|_\infty$.

Characterization of the quasi-static regime The objective of this section is to present a criterion for the existence of a quasi-static regime. To this end, we denote G_u the graph output versus input defined as $G_u = \{(u(t), y(t)), \forall t \in \mathbb{R}_+\} \subset \mathbb{R}^n \times \mathbb{R}^m$. Define the function

$$\begin{aligned} \phi_u : I_u &\to \mathbb{R}^n \times \mathbb{R}^m \\ \varrho &\to (\psi_u(\varrho), \varphi_u(\varrho)) \end{aligned}$$

[1] $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of locally measurable functions $\mathbb{R}_+ \to \mathbb{R}^n$.

Then, we have $\text{Dom}(\phi_u) = I_u$ and $\text{Range}(\phi_u) = \{(\psi_u(\varrho), \varphi_u(\varrho)), \forall \varrho \in I_u\} \subset \mathbb{R}^n \times \mathbb{R}^m$.

Lemma 8. $\text{Range}(\phi_u) = G_u$.

Using Lemma 3 it follows that

$$G_{u\circ s_\gamma} = \text{Range}\left(\{(\psi_u(\varrho), \varphi_{u\circ s_\gamma}(\varrho)), \varrho \in I_u\}\right) \qquad (1)$$

which motivates the following definition.

Definition 2. Let $\mathcal{H}$ be an operator as in Section 2.2 that satisfies Assumption 1. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ be given. The operator $\mathcal{H}$ is said to have a quasi-static regime with respect to the input u and initial condition ξ^0 if and only if the series of functions $\{\varphi_{u\circ s_\gamma}\}_{\gamma>0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$.

Note that $\text{Dom}(\varphi_{u\circ s_\gamma}) = I_u$ for all $\gamma > 0$ by Lemma 3. Definition 2 implies that there exists a function $\varphi_u^\star \in L^\infty(I_u, \mathbb{R}^m)$ such that $\lim_{\gamma\to\infty} \|\varphi_{u\circ s_\gamma} - \varphi_u^\star\|_{\infty, I_u} = 0$. Define the function

$$\begin{aligned} \phi_u^\star : I_u &\to \mathbb{R}^n \times \mathbb{R}^m \\ \varrho &\to (\psi_u(\varrho), \varphi_u^\star(\varrho)) \end{aligned}$$

Then, we have $\text{Dom}(\phi_u^\star) = I_u$ and $\text{Range}(\phi_u^\star) = \{(\psi_u(\varrho), \varphi_u^\star(\varrho)), \forall \varrho \in I_u\} \subset \mathbb{R}^n \times \mathbb{R}^m$. The function $\phi_u^\star$ describes completely the quasi-static behavior of the operator $\mathcal{H}$ with respect to (u, ξ^0). This function is unique due to the uniqueness of the limit $\varphi_u^\star$ in $L^\infty(I_u, \mathbb{R}^m)$.
Now, for any two nonempty compact sets S_1 and S_2 in $\mathbb{R}^{n+m}$, define the Hausdorff metric

$$d(S_1, S_2) = \max\left\{ \sup_{\eta_1\in S_1}\left(\inf_{\eta_2\in S_2} |\eta_1 - \eta_2|\right), \sup_{\eta_2\in S_2}\left(\inf_{\eta_1\in S_1} |\eta_1 - \eta_2|\right)\right\}$$

Then, the collection of all nonempty compact subsets of $\mathbb{R}^{n+m}$ is a complete metric space with respect to the Hausdorff metric d (Edgar, 1990, p.67).

Lemma 9. Suppose that the operator $\mathcal{H}$ has a quasi-static regime with respect to (u, ξ^0). Then, the series of sets $\{\text{Closure}(G_{u\circ s_\gamma})\}_{\gamma>0}$ converges to $\text{Closure}(\text{Range}(\phi_u^\star))$ with respect to the metric d.

2.4 The case of periodic inputs

In this section we consider input functions $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ that are T-periodic. In this case, Lemmas 2 and 4 show that the function $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is $\rho_u(T)$-periodic. The quasi-static regime is characterized by the function $\phi^\star = (\psi_u, \varphi_u^\star)$. Define the functions $\varphi_{u,k}^\star \in L^\infty([0, \rho_u(T)], \mathbb{R}^m)$, $k \in \mathbb{N}$ by the relation $\varphi_{u,k}^\star(\varrho) = \varphi_u^\star(k\rho_u(T) + \varrho)$, $\forall \varrho \in [0, \rho_u(T)]$.

Definition 3. The quasi-static regime of the operator $\mathcal{H}$ defined by the function $\phi^\star$ is said to have a steady-state if and only if the series of functions $\{\varphi_{u,k}^\star\}_{k\in\mathbb{N}}$ converges with respect to the norm $\|\cdot\|_{\infty,[0,\rho_u(T)]}$ in $L^\infty([0, \rho_u(T)], \mathbb{R}^m)$.

Let $L^\infty([0, \rho_u(T)], \mathbb{R}^m) \ni \varphi_u^\diamond = \lim_{k\to\infty} \varphi_{u,k}^\star$, and let $\psi_u|_{[0,\rho_u(T)]}$ be the restriction of the $\rho_u(T)$-periodic function ψ_u to the interval $[0, \rho_u(T)]$. The steady-state is characterized completely by the function $\phi^\diamond = (\psi_u|_{[0,\rho_u(T)]}, \varphi_u^\diamond)$ defined on the interval $[0, \rho_u(T)]$. The range of the function $\phi^\diamond$, that is the set $\{(\psi_u(\varrho), \varphi_u^\diamond(\varrho)), \varrho \in [0, \rho_u(T)]\}$ is called the steady-state graph, and it satisfies the following.

Lemma 10. Suppose that the quasi-static regime of the operator $\mathcal{H}$ has a steady-state as in Definition 3. Then, given $z \in \text{Range}(\phi^\diamond)$, there exists an increasing divergent sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty,\gamma\to\infty} |(u\circ s_\gamma(\gamma t_i), [\mathcal{H}(u\circ s_\gamma, \xi^0)](\gamma t_i)) - z| = 0$.

3. RELAXATION OF THE CONSTANT INPUT CONSTANT OUTPUT ASSUMPTION

We consider a causal operator $\mathcal{H}$ as in Section 2.2 for which the constant input constant output assumption (that is Assumption 1) may fail. If the input u is such that ρ_u is increasing, then the function φ_u can be constructed exactly as in Section 2.3. In this case, all the results of Section 2.3 hold. In particular, Definition 2 of the quasi-static regime needs no change and all the results that are related to this definition hold. This means that, if the input u is such that ρ_u is increasing, the analysis of the quasi-static regime can be done as in Section 2.3 under a very general framework in which the only assumption on the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is that it is causal.
However, when the input has some interval (or intervals) in which it is constant, the function φ_u can no longer be constructed. Indeed, take $t_1 < t_2$ such that u is constant in $[t_1, t_2]$. Then, for all $t_1 \le t \le t_2$, we have $\rho_u(t) = \text{constant} = \varrho$ so that $\varphi_u(\varrho)$ should correspond to a single value while $y(t)$ needs not to be constant on $[t_1, t_2]$ so that we cannot have $\varphi_u(\varrho) = y(t)$. To solve this issue, one may propose that φ_u is a set-valued map. However, as shown in Ikhouane (2010), the use of sets to characterize the quasi-static regime is not appropriate as they do not incorporate information on the orientation of the trajectories.
The objective of the subsequent analysis is to resolve this issue.

3.1 Approximation of the input u by the functions u_γ

Note first that if the input u is constant on $\mathbb{R}_+$, then $u \circ s_\gamma = u$ for all $\gamma > 0$ which means that the quasi-static regime is characterized by the set $\{(u, [\mathcal{H}(u, \xi^0)](t)), t \in \mathbb{R}_+\}$. Thus in the rest of the paper, we assume that u is not constant on $\mathbb{R}_+$, but is constant at least on some interval not reduced to a single point. Let $t \in \mathbb{R}_+$ and define $J_t = \{\tau \in \mathbb{R}_+ / u(\nu) = u(t), \forall t \le \nu \le \tau \text{ and } \forall \tau \le \nu \le t\}$. Note that all sets J_t are of the form $[t_1, t_2]$ or $[t_1, \infty)$ and, if there exist J_t and $J_{t'}$ that are not reduced to a single point, then either $J_t = J_{t'}$ or $J_t \cap J_{t'} = \emptyset$. Let J be the union of all intervals J_t that are not reduced to a single point, then $J \cap [0, n]$ is a measurable set for all $n \in \mathbb{N}$ as its interior measure is equal to its exterior measure, both being finite. This implies that J is a measurable set so that its characteristic function χ_J is measurable (Rudin, 1987, p. 11). Now, for every $\gamma > 0$ define the function u_γ as follows. Let $t \in \mathbb{R}_+$, then

$$u_\gamma(t) = u(0) + \int_0^t \dot{u}_\gamma(\tau)\mathrm{d}\tau \tag{2}$$

$$\dot{u}_\gamma(\tau) = \dot{u}(\tau) + \chi_J \frac{c}{\gamma(1+\tau^2)} \text{ where } c \in \mathbb{R}^n \text{ verifies } |c| = 1. \tag{3}$$

It is clear that $\dot{u}_\gamma$ is a measurable function so that u_γ is absolutely continuous.

Lemma 11. We have the following properties.

(1) $\forall \gamma > 0$, $u_\gamma \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$.
(2) $\|u - u_\gamma\|_\infty \leq \frac{\pi}{2\gamma}$ so that $\lim_{\gamma\to\infty} \|u - u_\gamma\|_\infty = 0$.
(3) $\|\dot{u} - \dot{u}_\gamma\|_\infty \leq \frac{1}{\gamma}$ so that $\lim_{\gamma\to\infty} \|\dot{u} - \dot{u}_\gamma\|_\infty = 0$.
(4) $\|\rho_u - \rho_{u_\gamma}\|_\infty \leq \frac{\pi}{2\gamma}$ so that $\lim_{\gamma\to\infty} \|\rho_u - \rho_{u_\gamma}\|_\infty = 0$.
(5) $\forall \gamma > 0$, $I_u \subset I_{u_\gamma}$. If $I_u \neq \mathbb{R}_+$ then $I_{u_\gamma} \neq \mathbb{R}_+$ and $\lim_{\gamma\to\infty} \mu(I_{u_\gamma} \backslash I_u) = 0$.
(6) $\forall \gamma > 0$, ρ_{u_γ} is increasing.
(7) $\lim_{\gamma\to\infty} \|\psi_u - \psi_{u_\gamma}|_{I_u}\|_{\infty, I_u} = 0$.

3.2 Characterization of the quasi-static regime

Definition 4. Given a function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, we say the the series $\{\wp(u,\gamma)\}_{\gamma>0}$ of functions $\wp(u,\gamma) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is an approximation of u if and only if the following holds.

$$\text{If } \rho_u \text{ is increasing }, \wp(u,\gamma) = u, \forall \gamma > 0. \tag{4}$$

$$\lim_{\gamma\to\infty} \|u - \wp(u,\gamma)\|_\infty = 0. \tag{5}$$

$$\lim_{\gamma\to\infty} \|\rho_u - \rho_{\wp(u,\gamma)}\|_\infty = 0. \tag{6}$$

$$\forall \gamma > 0, I_u \subset I_{\wp(u,\gamma)}. \text{ If } I_u \neq \mathbb{R}_+ \text{ then } I_{\wp(u,\gamma)} \neq \mathbb{R}_+ \text{ and}$$
$$\lim_{\gamma\to\infty} \mu(I_{\wp(u,\gamma)} \backslash I_u) = 0. \tag{7}$$

$$\forall \gamma > 0, \rho_{\wp(u,\gamma)} \text{ is increasing.} \tag{8}$$

$$\exists k > 0 / \forall \gamma > 0, \left\| \frac{\mathrm{d}\wp(u,\gamma)(t)}{\mathrm{d}t} \right\|_\infty \leq k. \tag{9}$$

An example of such approximation is the series $\{u_\gamma\}_{\gamma>0}$ of Section 3.1.

Lemma 12. $\lim_{\gamma\to\infty} \|\psi_u - \psi_{\wp(u,\gamma)}|_{I_u}\|_{\infty, I_u} = 0$

Since $\rho_{\wp(u,\gamma)}$ is increasing, the function $\varphi_{\wp(u,\gamma)}$ can be constructed as in Section 2.3 without the need of Assumption 1, and we have $\varphi_{\wp(u,\gamma)} = \mathcal{H}(\wp(u,\gamma), \xi^0) \circ \rho^{-1}_{\wp(u,\gamma)}$.

Assumption 2. The operator $\mathcal{H}$ verifies the following. $\forall \epsilon > 0$, there exists $\delta > 0$ such that, $\forall u, v \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ we have $\|v - u\|_\infty < \delta \Rightarrow \|\mathcal{H}(v, \xi^0) - \mathcal{H}(u, \xi^0)\|_\infty < \epsilon$.

By Definition 4, the series of functions $\{\wp(u,\gamma)\}_{\gamma>0}$ converges with respect to the norm $\|\cdot\|_\infty$ to the function u, thus by Assumption 2 the series of functions $\{\mathcal{H}(\wp(u,\gamma), \xi^0)\}_{\gamma>0}$ converges in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ to $\mathcal{H}(u, \xi^0)$. This fact motivates the following definition.

Definition 5. Let $\mathcal{H}$ be an operator as in Section 2.2 that verifies Assumption 2. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ be given, and let series $\{\wp(u,\gamma)\}_{\gamma>0}$ be an approximation of the function u as in Definition 4. The operator $\mathcal{H}$ is said to have a quasi-static regime with respect to the input u and initial condition ξ^0 if and only if the series of functions $\{\varphi_{\wp(u,\gamma_1)\circ s_{\gamma_2}}|_{I_u}\}_{\gamma_1>0,\gamma_2>0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$ when $\gamma_1 \to \infty$ and $\gamma_2 \to \infty$. Similar to Section 2.3, we denote $\varphi^\star_u \in L^\infty(I_u, \mathbb{R}^m)$ the function that verifies $\lim_{(\gamma_1,\gamma_2)\to(\infty,\infty)} \|\varphi_{\wp(u,\gamma_1)\circ s_{\gamma_2}}|_{I_u} - \varphi^\star_u\|_{\infty, I_u} = 0$. [2] The quasi-static function $\phi^\star$ is defined as before, that is $\phi^\star = (\psi_u, \varphi^\star_u)$.

Note that if ρ_u is increasing, $\wp(u,\gamma) = u$, $\forall \gamma > 0$ so that Definition 5 reduces to Definition 2. The objective of the following analysis is to check that Definition 5 is independent of the particular choice of the series $\{\wp(u,\gamma)\}_{\gamma>0}$.

Lemma 13. Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ be given and let $\{\wp_1(u,\gamma)\}_{\gamma>0}$ and $\{\wp_2(u,\gamma)\}_{\gamma>0}$ be approximations of the function u. Assume that the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ verifies Assumption 2. If $\exists \varphi^\star_u \in L^\infty(I_u, \mathbb{R}^m)$ such that $\lim_{(\gamma,\gamma')\to(\infty,\infty)} \|\varphi_{\wp_1(u,\gamma)\circ s_{\gamma'}}|_{I_u} - \varphi^\star_u\|_{\infty, I_u} = 0$ then $\lim_{(\gamma,\gamma')\to(\infty,\infty)} \|\varphi_{\wp_2(u,\gamma)\circ s_{\gamma'}}|_{I_u} - \varphi^\star_u\|_{\infty, I_u} = 0$.

We need to check that, if the operator $\mathcal{H}$ verifies both Assumptions 1 and 2, Definitions 2 and 5 coincide.

Lemma 14. If the operator $\mathcal{H}$ verifies both Assumptions 1 and 2, then $\forall (u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, the functions $\mathcal{H}(u, \xi^0)$ and φ_u are uniformly continuous. Moreover, if the operator $\mathcal{H}$ has a quasi-static regime with respect to (u, ξ^0) as in Definition 2 or as in Definition 5, then the corresponding function $\varphi^\star_u$ is uniformly continuous.

Lemma 15. Assume that the operator $\mathcal{H}$ verifies both Assumptions 1 and 2. Then Definitions 2 and 5 of the quasi-static regime are equivalent. That is, the operator $\mathcal{H}$ has a quasi-static regime with respect to $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ as in Definition 2 if and only if it has a quasi-static regime with respect to (u, ξ^0) as in Definition 5. Moreover, the function $\phi^\star_u$ obtained using Definition 2 is equal to the one obtained using Definition 5.

The following result characterizes the definition of quasi-static regime in terms of rate-independence property.

Lemma 16. Let $\mathcal{H}$ be an operator as in Section 2.2 that verifies Assumption 2, and has a quasi-static regime with respect to $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ as in Definition 5. Then, $\forall \epsilon > 0$, $\exists \gamma_\epsilon > 0$ such that $\forall \gamma_1 \geq 1$ and $\forall \gamma_2 \geq \gamma_\epsilon$ we have $\left\|\mathcal{H}(u \circ s_{\gamma_2} \circ s_{\gamma_1}, \xi^0) - \mathcal{H}(u \circ s_{\gamma_2}, \xi^0) \circ s_{\gamma_1}\right\|_\infty < \epsilon$.

Rate independence means that $\mathcal{H}(u \circ f, \xi^0) = \mathcal{H}(u, \xi^0) \circ f$ for all nondecreasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$. Lemma 16 is an almost rate-independence property in which f is replaced by a linear function s_{γ_1} and u by $u \circ s_{\gamma_2}$ for large values of γ_2. That is the operator $\mathcal{H}$ is almost rate-independent with respect to linear scale time-change at low frequencies. This latter fact is closer to experimental observations in the case of hysteresis systems than the property of rate-independence.

We end this paragraph by this property of the quasi-static graph.

Lemma 17. The function $\varphi^\star_u$ is continuous on $I_u \backslash \{0\}$.

[2] That is $\forall \epsilon > 0$, $\exists \gamma_1, \gamma_2 > 0$ such that $\forall \gamma_3 \geq \gamma_1$ and $\forall \gamma_4 \geq \gamma_2$ we have $\left\|\varphi^\star_u - \varphi_{\wp(u,\gamma_3)\circ s_{\gamma_4}}|_{I_u}\right\| < \epsilon$.

3.3 Case-study: Application to Wiener systems

Example 1. Consider the so-called Wiener system with input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, output $y \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and initial condition $x(0) \in \mathbb{R}^p$:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{10}$$
$$y(t) = f\left(Cx(t) + Du(t)\right) \tag{11}$$
$$x(0) = \xi^0 \tag{12}$$

where the matrix $A \in \mathbb{R}^{p\times p}$ is Hurwitz, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{m\times p}$, $D \in \mathbb{R}^{m\times n}$, $t \in \mathbb{R}_+$, and the function $f : \mathbb{R}^m \to \mathbb{R}^m$ is k-Lipschitz, $k > 0$. [3] A special subclass of Wiener systems are linear time-invariant systems obtained when the function f is the identity ($f(\alpha) = \alpha$). In this case, the static gain of the linear system (10)-(12) is given by the matrix $G = -CA^{-1}B + D$ (P. Albertos, 2004, p.60) (A is is invertible as it is Hurwitz). Let $\mathcal{H}$ be the operator that that maps (u, ξ^0) to y; note that the operator $\mathcal{H}$ is causal but, in general, it does not verify Assumption 1.

Lemma 18. The operator $\mathcal{H}$ verifies Assumption 2. Furthermore, if $\xi^0 = -A^{-1}Bu(0)$, then $\mathcal{H}$ has a quasi-static regime with respect to (u, ξ^0) as in Definition 5 and we have $\phi^\star = (\psi_u, f(G\psi_u))$.

Proof. Due to the linearity of the differential equation (10), its solution x exists over $\mathbb{R}_+$. Let $\epsilon > 0$ be given, and let $u, v \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$. Then, we have

$$\dot{x}_1(t) = Ax_1(t) + Bu(t) \tag{13}$$
$$w_1(t) = Cx_1(t) + Du(t) \tag{14}$$
$$\left[\mathcal{H}\left(u, \xi^0\right)\right](t) = f(w_1(t)) \tag{15}$$
$$x_1(0) = \xi^0 \tag{16}$$

and

$$\dot{x}_2(t) = Ax_2(t) + Bv(t) \tag{17}$$
$$w_2(t) = Cx_2(t) + Dv(t) \tag{18}$$
$$\left[\mathcal{H}\left(v, \xi^0\right)\right](t) = f(w_2(t)) \tag{19}$$
$$x_2(0) = \xi^0 \tag{20}$$

Define $x_3 = x_2 - x_1$, then we have

$$\dot{x}_3(t) = Ax_3(t) + B\left(v(t) - u(t)\right) \tag{21}$$
$$w_2(t) - w_1(t) = Cx_3(t) + D\left(v(t) - u(t)\right) \tag{22}$$
$$\left[\mathcal{H}\left(v, \xi^0\right)\right](t) - \left[\mathcal{H}\left(u, \xi^0\right)\right](t) = f(w_2(t)) - f(w_1(t)) \tag{23}$$
$$x_3(0) = \mathbf{0} \tag{24}$$

Due to (Khalil, 2000, Corollary 5.2, p.205), it follows that

$$\left\|w_2 - w_1\right\|_\infty \le k' \left\|v - u\right\|_\infty \tag{25}$$

where the constant $k' > 0$ depends on matrices A, B, C, D. Since the function f is k-Lipschitz, it follows that

$$\left\|\mathcal{H}\left(v, \xi^0\right) - \mathcal{H}\left(u, \xi^0\right)\right\|_\infty \le k \left\|w_2 - w_1\right\|_\infty \tag{26}$$

Thus, to have $\left\|\mathcal{H}\left(v, \xi^0\right) - \mathcal{H}\left(u, \xi^0\right)\right\|_\infty < \epsilon$ in Assumption 2, it suffices to take $\delta = \frac{\epsilon}{kk'}$, which proves the first assertion of the lemma.

Now, consider Equations (10)-(12) in which u is replaced by the function $u_\gamma \circ s_\gamma$ where $\gamma \ge 1$, that is

$$\dot{x}_\gamma(t) = Ax_\gamma(t) + Bu_\gamma \circ s_\gamma(t) \tag{27}$$
$$y_\gamma(t) = f\left(Cx_\gamma(t) + Du_\gamma \circ s_\gamma(t)\right) \tag{28}$$
$$x_\gamma(0) = \xi^0 \tag{29}$$

The exact solution of (27)-(29) is given by

$$y_\gamma(t) = f\left(C\left(e^{tA}\xi^0 + \int_0^t e^{(t-\tau)A} Bu_\gamma \circ s_\gamma(\tau)\mathrm{d}\tau\right) + Du_\gamma \circ s_\gamma(t)\right) \tag{30}$$

Integrating by parts we get

$$\int_0^t e^{(t-\tau)A} Bu_\gamma \circ s_\gamma(\tau)\mathrm{d}\tau = A^{-1}\left(e^{tA}Bu_\gamma \circ s_\gamma(0) - Bu_\gamma \circ s_\gamma(t)\right) + A^{-1}\int_0^t e^{(t-\tau)A} B \frac{\mathrm{d}u_\gamma \circ s_\gamma}{\mathrm{d}t}(\tau)\,\mathrm{d}\tau \tag{31}$$

By Equation (2) we have $u_\gamma \circ s_\gamma(0) = u(0)$. If $\xi^0 = -A^{-1}Bu(0)$ then Equation (31) becomes

$$\int_0^t e^{(t-\tau)A} Bu_\gamma \circ s_\gamma(\tau)\mathrm{d}\tau = -e^{tA}\xi^0 - A^{-1}Bu_\gamma \circ s_\gamma(t) + \frac{A^{-1}}{\gamma}\int_0^t e^{(t-\tau)A} B\dot{u}_\gamma \circ s_\gamma(\tau)\,\mathrm{d}\tau \tag{32}$$

as A^{-1} and e^{tA} commute. Combining Equations (32) and (30) it follows that

$$y_\gamma(t) - f\left(Gu_\gamma \circ s_\gamma(t)\right) = f\left(Gu_\gamma \circ s_\gamma(t) + \frac{CA^{-1}}{\gamma} \times \int_0^t e^{(t-\tau)A} B\dot{u}_\gamma \circ s_\gamma(\tau)\,\mathrm{d}\tau\right) - f\left(Gu_\gamma \circ s_\gamma(t)\right) \tag{33}$$

On the other hand, by (Loan, 1977, Eq(2.8)) there exist $\beta > 0$ and $r \in \mathbb{N}\backslash\{0\}$ that depend solely on A such that $\left\|e^{\delta A}\right\| \le \beta e^{\alpha\delta}\delta^r, \forall\delta \ge 0$ where $\|X\|$ is the largest singular

[3] That is $\forall\alpha, \beta \in \mathbb{R}^m, |f(\alpha) - f(\beta)| \le k|\alpha - \beta|$

value of the matrix X, and α is the largest real part of the eigenvalues of A. Note that $\alpha < 0$ as A is Hurwitz. By Lemma 11 Property 3 we have $|\dot{u}_\gamma \circ s_\gamma(\tau)| \leq \|\dot{u}\|_\infty + 1$ for almost all $\tau \in [0, \infty)$ as $\gamma \geq 1$. Thus, Equation (33) along with the k-Lipschitz property of f lead to

$$|y_\gamma(t) - f(Gu_\gamma \circ s_\gamma(t))| \leq \frac{kk_1}{\gamma} \int_0^t e^{(t-\tau)\alpha} (t-\tau)^r \, \mathrm{d}\tau,$$

$$\forall t \geq 0, \forall \gamma \geq 1 \tag{34}$$

for some constant $k_1 \geq 0$ that depends on A, B, C and $\|\dot{u}\|_\infty$. Using the change of variable $t - \tau = \lambda$ it follows that

$$|y_\gamma(t) - f(Gu_\gamma \circ s_\gamma(t))| \leq \frac{kk_1}{\gamma} \int_0^t e^{\lambda\alpha} \lambda^r \mathrm{d}\lambda$$

$$\leq \frac{kk_1}{\gamma} \int_0^\infty e^{\alpha\lambda} \lambda^r \mathrm{d}\lambda, \forall t \geq 0, \forall \gamma \geq 1 \tag{35}$$

Since $\int_0^\infty e^{\alpha\lambda}\lambda^r \mathrm{d}\lambda$ is finite, it follows that

$$|y_\gamma(t) - f(Gu_\gamma \circ s_\gamma(t))| \leq \frac{k_2}{\gamma}, \forall t \geq 0, \forall \gamma \geq 1 \tag{36}$$

for some constant $k_2 \geq 0$ that depends on A, B, C and $\|\dot{u}\|_\infty$. Now, we have $\psi_{u_\gamma \circ s_\gamma} \circ \rho_{u_\gamma \circ s_\gamma} = u_\gamma \circ s_\gamma$, and by Lemma 3 we have $\psi_{u_\gamma \circ s_\gamma} = \psi_{u_\gamma}$. On the other hand, we have $\varphi_{u_\gamma \circ s_\gamma} \circ \rho_{u_\gamma \circ s_\gamma} = y_\gamma$ so that Equation (36) can be written as

$$\left|\varphi_{u_\gamma \circ s_\gamma}(\varrho) - f\left(G\psi_{u_\gamma}(\varrho)\right)\right| \leq \frac{k_2}{\gamma}, \forall \varrho \in I_{u_\gamma \circ s_\gamma}, \forall \gamma \geq 1 \tag{37}$$

By Lemma 3 we have $I_{u_\gamma \circ s_\gamma} = I_{u_\gamma}$, and by Lemma 11 Property 5 we get $I_u \subset I_{u_\gamma}$ so that Equation (37) leads to

$$\left\|\varphi_{u_\gamma \circ s_\gamma}|_{I_u} - f\left(G\psi_{u_\gamma}|_{I_u}\right)\right\|_{\infty, I_u} \leq \frac{k_2}{\gamma}, \forall \gamma \geq 1 \tag{38}$$

The second assertion of the lemma follows from Lemma 11 Property 7 and the fact that f is Lipschitz. □

4. CONCLUSION

The objective of the paper was to provide a mathematical framework as general as possible for the characterization of the quasi-static regime of operators mapping an input and initial condition to an output, all of them belonging to some specified sets. When the total variation of the input is increasing, the quasi-static regime can be characterized under the only assumption the the operator is causal. If the input presents an interval (or intervals) in which it is constant, then two classes of operators are considered: those that verify a constant input constant output assumption (Section 2.3), and those that verify a smoothness assumption (Section 3). In both cases, the quasi-static regime can been defined appropriately. The case-studies of the semi-linear Duhem model and Winer model illustrate the concepts introduced in the paper.

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