

# Suboptimal Switching State Feedback Control Consistency Analysis for Switched Linear Systems <sup>\*</sup>

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**Abstract:** This paper introduces the concept of consistency for continuous-time switched linear systems having the switching function as a primary control signal to be designed. A state feedback switching control strategy is strictly consistent whenever it improves performance compared to the ones of all isolated subsystems. It is proven that a min-type switching strategy is strictly consistent for the classes of  $H_2$  and  $H_\infty$  performance indexes. This property makes clear the importance of switching systems control design in both theoretical and practical application frameworks. Moreover, with this property it is not necessary to assume that all the subsystems are not stabilizable in order to make a switching strategy design problem well posed. The theory is illustrated by means of several academic examples.

Keywords: Switched linear systems, Optimality, Stabilization methods, State feedback.

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## 1. INTRODUCTION

Switched dynamical systems and, in particular, switched linear systems analysis and control design have been the concern of many researchers in the last decades since their introduction by Morse and collaborators in the early 70's, see the survey papers DeCarlo et al. [2000], Liberzon and Morse [1999], Lin and Antsaklis [2009], Shorten et al. [2007], the interesting and useful books Liberzon [2003], Schaft and Schumacher [2000], Sun and Ge [2005] and the references therein. For continuous-time switched linear systems stability analysis many results are available. They put in evidence the important fact that it is possible to orchestrate the subsystems through an adequate switching strategy in order to impose global stability. This property is valid even though all isolated subsystems are unstable, see Colaneri et al. [2008], Geromel et al. [2008], Geromel and Colaneri [2006], Liberzon and Morse [1999] and Zhao and Hill [2008]. For control design, several results are also available in two main different contexts. In the first one, controllers are designed in order to maintain global stability in the presence of switching viewed as unknown and arbitrary trajectories, see Hespanha and Morse [2002] for details. In the other framework the switching function is a control strategy that is used to improve performance, see for instance Deaecto and Geromel [2010], Geromel and Deaecto [2009], Ji et al. [2006], Ji et al. [2005], Savkin

<sup>\*</sup> This work was supported by "Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq" and "Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP", Brazil and by the ANR project ArHyCo, Programme "Systèmes Embarqués et Grandes Infrastructures" - ARPEGE, contract number ANR-2008 SEGI 004 01-30011459.

et al. [1996], Yan and Ozbay [2007], Zhai et al. [2005] and Zhai et al. [2001] among others. In this paper, the concept of consistency is introduced. Roughly speaking, a switching strategy is said to be strictly consistent if it improves performance when compared to all performances produced by the isolated subsystems. In other words, a strictly consistent switching strategy necessarily yields a performance gain. Clearly, an optimal switching strategy is consistent, but since it may be difficult to determine, we think that consistency is a valid certificate for suboptimal switching strategies quality. The consistency property clarifies the formulation of a switching strategy design problem in general in the sense that we do not need to assume that all the subsystems are not stabilizable to make the problem well posed. We show in this paper how to design a min-type switching strategy that is strictly consistent as far as  $H_2$  and  $H_\infty$  performance indexes are adopted. This is important in the general framework of switched linear systems and, in particular, when the isolated subsystems can be made asymptotically stable by means of an additional feedback control loop. The theory is illustrated by means of simple examples.

The notation is standard. For square matrices  $\text{Tr}(\cdot)$  denotes the trace function. For real matrices or vectors  $(\cdot)'$  indicates transpose. For symmetric matrices, the symbol  $(\bullet)$  denotes each of its symmetric blocks. The convex combination of matrices with the same dimension  $\{J_1, \dots, J_N\}$  is denoted by  $J_\lambda = \sum_{j=1}^N \lambda_j J_j$  where  $\lambda$  belongs to the unitary simplex  $\Lambda$  composed by all nonnegative vectors  $\lambda \in \mathbb{R}^N$  such that  $\sum_{j=1}^N \lambda_j = 1$ . The squared norm of a trajectory  $\xi(t)$  defined for all  $t \geq 0$ , denoted by  $\|\xi\|_2^2$  is

equal to  $\|\xi\|_2^2 = \int_0^\infty \xi(t)' \xi(t) dt$ . The set of all finite norm trajectories, such that  $\|\xi\|_2^2 < \infty$  is denoted by  $\mathcal{L}_2$ .

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following switched linear system

$$\dot{x}(t) = A_\sigma x(t) + Hw(t) \quad (1)$$

$$z(t) = E_\sigma x(t) + Gw(t) \quad (2)$$

evolving from zero initial condition  $x(0) = 0$ . The vectors  $x \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^m$ , and  $z \in \mathbb{R}^r$  are the state, the exogenous disturbance and the controlled output, respectively. The switching function denoted by  $\sigma(t)$  selects at each time instant  $t \geq 0$  a subsystem among those belonging to the set  $\mathbb{K} = \{1, \dots, N\}$ . The state space realization of each subsystem is defined by matrices  $(A_i, H, E_i, G)$  of appropriate dimensions for each  $i \in \mathbb{K}$ . Matrices  $H$  and  $G$  are supposed to be switching independent but the possibility to relax this assumption will be discussed in details afterwards. For a given switching trajectory  $\sigma(t), \forall t \geq 0$ , two different performance indexes can be associated to the switched linear system (1)-(2) depending on the class of external perturbation  $w$  considered. With a little abuse they are denominated  $H_2$  and  $H_\infty$  performance indexes, respectively, and are defined as follows:

- **$H_2$  performance:** For strictly proper subsystems ( $G = 0$ ), the controlled output  $z(t)$  associated to impulsive disturbances of the form  $w(t) = e_k \delta(t)$  where  $e_k \in \mathbb{R}^m$  is the  $k^{th}$  column of the identity matrix provides the index

$$J_2(\sigma) = \sum_{k=1}^m \|z_k\|_2^2 \quad (3)$$

- **$H_\infty$  performance:** The controlled output  $z(t)$  associated to arbitrary square integrable disturbances  $w \in \mathcal{L}_2$  provides the index

$$J_\infty(\sigma) = \sup_{w \neq 0 \in \mathcal{L}_2} \frac{\|z\|_2^2}{\|w\|_2^2} \quad (4)$$

The rationale behind these definitions is that whenever the switching rule is kept constant, that is,  $\sigma(t) = i \in \mathbb{K}$  for all  $t \geq 0$ , and matrices  $A_i, i \in \mathbb{K}$  are Hurwitz then the indexes (3) and (4) equal the standard  $H_2$  and  $H_\infty$  squared norms of the  $i$ -th subsystem transfer function from the input  $w$  to the controlled output  $z$ , respectively.

Let us now define the set  $\mathcal{S}$  that contains all state feedback switching functions of the form  $\sigma(t) = g(x(t))$  for some  $g : \mathbb{R}^n \rightarrow \mathbb{K}$  ensuring global asymptotic stability of the origin and the set  $\mathcal{C}$  that contains only the  $N$  time constant policies  $\sigma(t) = i \in \mathbb{K}$  for all  $t \geq 0$ . Clearly,  $\mathcal{C}$  is a subset of  $\mathcal{S}$  if and only if all subsystems matrices are Hurwitz. To ease the notation, in the sequel,  $\alpha$  denotes  $\{2, \infty\}$ .

*Definition 1.* A particular switching strategy  $\sigma_\alpha$  is consistent with respect to the index  $J_\alpha$  if it belongs to  $\mathcal{S}$  and guarantees that  $J_\alpha(\sigma_\alpha) \leq J_\alpha(\sigma), \forall \sigma \in \mathcal{C}$ . If this inequality is strict the strategy is said strictly consistent.

A switching strategy is consistent whenever it imposes to the switched linear system a performance that is not worse than the performance produced by each isolated subsystem. In other words, a strictly consistent switching

strategy improves performance. Clearly, it is seen that the optimal state feedback switching strategy provided by

$$\sigma_\alpha^* = \arg \inf_{\sigma \in \mathcal{S}} J_\alpha(\sigma) \quad (5)$$

is consistent. Indeed, if  $\sigma \in \mathcal{C}$  also belongs to  $\mathcal{S}$  then by definition  $J_\alpha(\sigma_\alpha^*) \leq J_\alpha(\sigma)$ . On the contrary, if  $\sigma \in \mathcal{C}$  but it does not belong to  $\mathcal{S}$  then the same inequality holds because  $J_\alpha(\sigma)$  is unbounded. Since the computation of the optimal state feedback strategy (5) is a very difficult task, suboptimal solutions are of great interest but only if they are strictly consistent. Otherwise, such a switching strategy does not improve performance when compared to some strategy of  $\mathcal{C}$ .

In this paper, our purpose is to provide suboptimal switching strategies that are strictly consistent with respect to  $J_\alpha$  for both  $\alpha \in \{2, \infty\}$ . They are formally expressed as  $\sigma_{so}(t) = g(x(t))$  where

$$g(x) = \arg \min_{i \in \mathbb{K}} x' P_i x \quad (6)$$

with  $P_i \in \mathbb{R}^{n \times n}$  being positive definite matrices for all  $i \in \mathbb{K}$  that satisfy some conditions to be presented in the sequel. To this end, we need to introduce the following matrix sets, see Deaecto and Geromel [2010] and Geromel and Colaneri [2006]. The first, denoted  $\mathcal{X}_2$ , is composed by positive definite matrices  $P_i \in \mathbb{R}^{n \times n}, i \in \mathbb{K}$  and a Metzler matrix  $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$  satisfying the so called Lyapunov-Metzler inequalities

$$\begin{bmatrix} A_i' P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j & \bullet \\ E_i & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \quad (7)$$

whereas the second one denoted  $\mathcal{X}_\infty$  is composed by positive definite matrices  $P_i \in \mathbb{R}^{n \times n}, i \in \mathbb{K}$ , a Metzler matrix  $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$  and a positive scalar  $\rho \in \mathbb{R}$  satisfying the so called Riccati-Metzler inequalities

$$\begin{bmatrix} A_i' P_i + P_i A_i + \sum_{j \in \mathbb{K}} \pi_{ji} P_j & \bullet & \bullet \\ H' P_i & -\rho I & \bullet \\ E_i & G & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \quad (8)$$

The elements of the Metzler matrices under consideration are such that  $\pi_{ij} \geq 0, \forall i \neq j \in \mathbb{K} \times \mathbb{K}$  and

$$\sum_{i \in \mathbb{K}} \pi_{ij} = 0, \quad \forall j \in \mathbb{K} \quad (9)$$

which implies that  $\pi_{ii} \leq 0, \forall i \in \mathbb{K}$ . The following are Metzler matrices of this class: the null matrix  $\Pi = 0$  and  $\Pi = \Theta^j$  with null elements except  $\theta_{ii} = -\beta$  and  $\theta_{ji} = \beta$  for all  $i = 1, \dots, N, i \neq j$ , for some  $j \in \mathbb{K}$  and  $\beta \geq 0$ . Moreover, the feasibility of the sets  $\mathcal{X}_\alpha$  does not require all matrices  $\{A_1, \dots, A_N\}$  be Hurwitz. Indeed, in Geromel and Colaneri [2006] it is proven that the inequalities (7) admit a solution with respect to the matrix variables  $P_i > 0, i \in \mathbb{K}$  and  $\Pi$  satisfying (9) whenever there exists  $\lambda \in \Lambda$  such that  $A_\lambda$  is Hurwitz. In other words, the convex hull  $\text{co}\{A_1, \dots, A_N\}$  must contain a Hurwitz matrix. Clearly the same result holds for the Riccati-Metzler inequalities (8) because they reduce to the previous ones for  $\rho \rightarrow +\infty$ . Finally, it is simple to verify that for  $\Pi$  fixed, both are expressed by LMIs with respect to the remaining variables.

### 3. MAIN RESULTS

This section is devoted to present conditions to assure that the switching strategy based on the state feedback function (6) is strictly consistent. The next theorem provides conditions for the case of  $H_2$  performance, that is  $\alpha = 2$ .

*Theorem 2.* Assume the set  $\mathcal{X}_2$  is not empty. The optimal solution of

$$J_2^{so} = \inf_{\{P_1, \dots, P_N, \Pi\} \in \mathcal{X}_2} \min_{i \in \mathbb{K}} \text{Tr}(H' P_i H) \quad (10)$$

provides positive definite matrices  $P_i, i \in \mathbb{K}$  such that the state feedback switching function  $\sigma_{so}(t) = g(x(t))$  is strictly consistent.

*Proof:* Consider the switched system (1)-(2) with zero input  $w(t) = 0$  and arbitrary initial condition  $x(0) = x_0$ . Since  $\mathcal{X}_2$  is not empty take any  $\{P_1, \dots, P_N, \Pi\} \in \mathcal{X}_2$  and adopt the Lyapunov function  $v(x) = \min_{i \in \mathbb{K}} x' P_i x$ . Following the same reasoning as in Geromel and Colaneri [2006] it is seen that  $\|z\|_2^2 < v(x_0) = \min_{i \in \mathbb{K}} x_0' P_i x_0$ . Hence, applying this to the switched linear system (1)-(2) with successive inputs  $w(t) = e_k \delta(t)$  from zero initial condition, using (3) we get

$$\begin{aligned} J_2(\sigma_{so}) &< \sum_{k=1}^m \min_{i \in \mathbb{K}} (He_k)' P_i (He_k) \\ &< \min_{i \in \mathbb{K}} \text{Tr}(H' P_i H) \end{aligned} \quad (11)$$

which enables us to say that the optimal solution of problem (10) provides the most favorable upper bound to the  $H_2$  performance index, that is  $J_2(\sigma_{so}) < J_2^{so}$ . Consider now that there exists at least one  $\sigma \in \mathcal{C}$  that also belongs to  $\mathcal{S}$  because otherwise the claim follows immediately from the fact that  $J_2(\sigma)$  is unbounded for all  $\sigma \in \mathcal{C}$ . Consequently, we proceed by assuming that matrix  $A_\ell$  is Hurwitz for some  $\ell \in \mathbb{K}$  in which case it is well known, see Boyd et al. [1994], that the equality

$$\begin{aligned} &\|E_\ell(sI - A_\ell)^{-1} H\|_2^2 = \\ &= \inf_{Q_\ell > 0} \{ \text{Tr}(H' Q_\ell H) : A_\ell' Q_\ell + Q_\ell A_\ell + E_\ell' E_\ell < 0 \} \end{aligned} \quad (12)$$

holds. Moreover, taking the Metzler matrix  $\Pi = \Theta^\ell$  the constraints that define the set  $\mathcal{X}_2$  are rewritten as

$$\begin{bmatrix} A_i' P_i + P_i A_i + \beta(P_\ell - P_i) & \bullet \\ E_i & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \quad (13)$$

making simple to verify that for  $\beta > 0$  large enough, matrices  $P_\ell = Q_\ell$  and  $P_i > Q_\ell$  arbitrary but fixed for  $i = 1, \dots, N, i \neq \ell$  are such that  $\{P_1, \dots, P_N, \Pi\} \in \mathcal{X}_2$ , yielding

$$\begin{aligned} J_2^{so} &= \inf_{\{P_1, \dots, P_N, \Pi\} \in \mathcal{X}_2} \min_{i \in \mathbb{K}} \text{Tr}(H' P_i H) \\ &\leq \text{Tr}(H' Q_\ell H) \\ &\leq \|E_\ell(sI - A_\ell)^{-1} H\|_2^2 \end{aligned} \quad (14)$$

Since this is true for every index  $\ell \in \mathbb{K}$  such that  $A_\ell$  is Hurwitz, the conclusion is that  $J_2(\sigma_{so}) < J_2^{so} \leq J_2(\sigma)$  for all  $\sigma \in \mathcal{C}$  and the proof is completed.  $\square$

From the proof of Theorem 2 the following remarks are important:

*Remark 3.* It is clear that  $J_2^{so}$  is only an upper bound to the true value of the cost  $J_2(\sigma_{so})$ . Hence, in general, we have  $J_2(\sigma_{so}) \ll J_2^{so}$ .

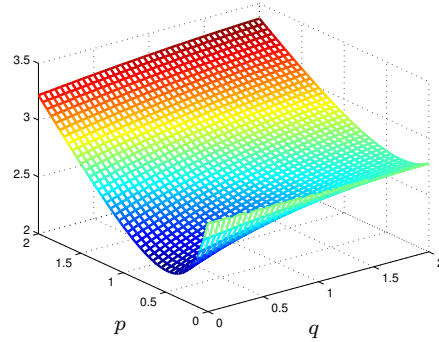


Fig. 1.  $J_2^{so}$  cost determination

*Remark 4.* Theoretically, it may occur that  $J_2^{so} = J_2(\sigma)$  for some  $\sigma \in \mathcal{C}$  and  $J_2(\sigma_{so})$  be arbitrarily close to it. In this situation, switching does not improve performance.

Solving problem (10) is not a simple task because, for each  $i \in \mathbb{K}$ , we have to handle nonconvex constraints. Indeed, the Lyapunov-Metzler inequalities are BMIs whose global solution determination requires the use of powerful numerical methods based on polynomial optimization. These aspects are illustrated by means of the next simple example where the switched linear system (1)-(2) state space realization is given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad (15)$$

and  $E_1 = [1 \ 0]$ ,  $E_2 = [0 \ 1]$ . Since matrices  $A_1$  and  $A_2$  are Hurwitz, it is immediate to verify that the best performance that we can obtain with a constant strategy is by adopting  $\sigma(t) = 1, \forall t \geq 0$  with the associated cost

$$J_2(\sigma) = \min_{\ell \in \{1,2\}} \|E_\ell(sI - A_\ell)^{-1} H\|_2^2 = 2.7778 \quad (16)$$

In this particular case of only two subsystems, the Metzler matrix  $\Pi \in \mathbb{R}^{2 \times 2}$ , written as

$$\Pi = \begin{bmatrix} -p & q \\ p & -q \end{bmatrix} \quad (17)$$

enables us to solve problem (10) by gridding the objective function for all  $(p, q)$  in the box  $[0, 2] \times [0, 2]$  as indicated in Figure 1. Notice that the global optimal belongs to this box which provides the optimal Metzler matrix defined by the pair  $(p, q) = (0.45, 0.00)$  and  $J_2^{so} = 2.1929$ . Finally, the time simulation of the closed-loop system allows the determination of the true cost  $J_2(\sigma_{so}) = 1.6357$ . This confirms that the  $\sigma_{so}$  strategy is strictly consistent and presents a cost 40% smaller than the best cost produced by any constant strategy  $\sigma \in \mathcal{C}$ . This simple example shows that, even in the case of all subsystems are asymptotically stable, we may have an important performance improvement whenever switching is allowed.

We now move our attention to the case of  $H_\infty$  performance. The situation is more demanding since only one scalar variable  $\rho$  is present in all Riccati-Metzler inequalities that define the set  $\mathcal{X}_\infty$ .

*Theorem 5.* Assume the set  $\mathcal{X}_\infty$  is not empty. The optimal solution of

$$J_\infty^{so} = \inf_{\{P_1, \dots, P_N, \Pi, \rho\} \in \mathcal{X}_\infty} \rho \quad (18)$$

provides positive definite matrices  $P_i, i \in \mathbb{K}$  such that the state feedback switching function  $\sigma_{so}(t) = g(x(t))$  is strictly consistent.

*Proof:* Consider the switched linear system (1)-(2) with zero initial condition  $x(0) = 0$ . Since  $\mathcal{X}_\infty$  is not empty take any  $\{P_1, \dots, P_N, \Pi, \rho\} \in \mathcal{X}_\infty$  and adopt the Lyapunov function  $v(x) = \min_{i \in \mathbb{K}} x' P_i x$ . Following the same reasoning as in Deaecto and Geromel [2010] it is possible to conclude that  $\|z\|_2^2 - \rho \|w\|_2^2 < 0$  for all  $w \in \mathcal{L}_2$ . Hence, we get

$$J_\infty(\sigma_{so}) < \rho \quad (19)$$

which enables us to say that the optimal solution of problem (18) provides the most favorable upper bound to the  $H_\infty$  performance index, that is  $J_\infty(\sigma_{so}) < J_\infty^{so}$ . As before, consider that there exists at least one  $\sigma \in \mathcal{C}$  that also belongs to  $\mathcal{S}$  because otherwise the claim follows immediately from the fact that  $J_\infty(\sigma)$  is unbounded for all  $\sigma \in \mathcal{C}$ . Consequently, we proceed by assuming that matrix  $A_\ell$  is Hurwitz for some  $\ell \in \mathbb{K}$  in which case it is well known, see Boyd et al. [1994], that the following equality

$$\begin{aligned} & \|E_\ell(sI - A_\ell)^{-1}H + G\|_\infty^2 = \\ & = \inf_{Q_\ell > 0, \rho_\ell} \left\{ \rho_\ell : \begin{bmatrix} A'_\ell Q_\ell + Q_\ell A_\ell & \bullet & \bullet \\ H' Q_\ell & -\rho_\ell I & \bullet \\ E_\ell & G & -I \end{bmatrix} < 0 \right\} \quad (20) \end{aligned}$$

holds. On the other hand, with the Metzler matrix  $\Pi = \Theta^\ell$  the constraints that define the set  $\mathcal{X}_\infty$  are rewritten as

$$\begin{bmatrix} A'_i P_i + P_i A_i + \beta(P_\ell - P_i) & \bullet & \bullet \\ H' P_i & -\rho I & \bullet \\ E_i & G & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \quad (21)$$

and we verify that for  $\beta > 0$  large enough, matrices  $P_\ell = Q_\ell$ ,  $P_i > Q_\ell$  arbitrary but fixed for  $i = 1, \dots, N$ ,  $i \neq \ell$  and  $\rho = \rho_\ell$  the inequalities (21) reduce to the one in (20) for  $i = \ell$  and to  $\rho_\ell > G'G$  for  $i \neq \ell \in \mathbb{K}$  that are redundant. Consequently, since  $\{P_1, \dots, P_N, \Pi, \rho\} \in \mathcal{X}_\infty$ , we get

$$\begin{aligned} J_\infty^{so} & = \inf_{\{P_1, \dots, P_N, \Pi, \rho\} \in \mathcal{X}_\infty} \rho \\ & \leq \rho_\ell \\ & \leq \|E_\ell(sI - A_\ell)^{-1}H + G\|_\infty^2 \quad (22) \end{aligned}$$

Since this is true for every index  $\ell \in \mathbb{K}$  such that  $A_\ell$  is Hurwitz, the conclusion is that  $J_\infty(\sigma_{so}) < J_\infty^{so} \leq J_\infty(\sigma)$  for all  $\sigma \in \mathcal{C}$  and the proof is completed.  $\square$

The same remarks valid for the  $H_2$  performance index are also valid for  $H_\infty$  performance. The Riccati-Metzler inequalities are difficult to handle due to the nonconvexity inherited by the product of variables. This fact certainly makes more demanding the solution of problem (18). However, following the proofs of Theorem 2 and Theorem 5 an important difference between them is apparent. Suppose all isolated subsystems are asymptotically stable, choose  $\Pi = 0$  which decouples the inequalities that define the set  $\mathcal{X}_2$  and determine the associated cost (which clearly may not be the optimal one for problem (10))

$$\begin{aligned} J_2^{so} & = \min_{i \in \mathbb{K}} \inf_{P_i > 0} \{ \text{Tr}(H' P_i H) : A'_i P_i + P_i A_i + E'_i E_i < 0 \} \\ & = \min_{i \in \mathbb{K}} \|E_i(sI - A_i)^{-1}H\|_2^2 \quad (23) \end{aligned}$$

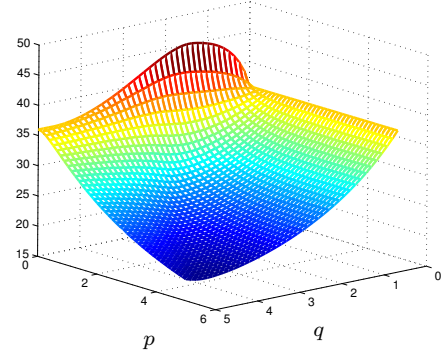


Fig. 2.  $J_\infty^{so}$  cost determination

where it is to be noticed that  $P_i > 0$  is arbitrarily close to the observability gramian of the  $i$ -th subsystem. Since these matrices together with  $\Pi = 0$  are feasible, the cost of the switching strategy  $\sigma_{so}$  satisfies (11) and so, it is strictly consistent. Unfortunately, the same reasoning is not valid for  $H_\infty$  performance. The main reason is that for  $\Pi = 0$  the inequalities defining  $\mathcal{X}_\infty$  are not decoupled and the associated cost

$$\begin{aligned} J_\infty^{so} & = \inf_{P_i > 0, \rho} \left\{ \rho : \begin{bmatrix} A'_i P_i + P_i A_i & \bullet & \bullet \\ H' P_i & -\rho I & \bullet \\ E_i & G & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \right\} \\ & = \max_{i \in \mathbb{K}} \|E_i(sI - A_i)^{-1}H + G\|_\infty^2 \quad (24) \end{aligned}$$

can not be used to establish consistency. Consider now the following illustrative example where the subsystems are defined as before and  $G = 1$ . Both isolated subsystems are asymptotically stable and present approximately the same  $H_\infty$  norm. Moreover, with the constant strategy  $\sigma(t) = 2, \forall t \geq 0$  we get the associated cost

$$J_\infty(\sigma) = \min_{\ell \in \{1,2\}} \|E_\ell(sI - A_\ell)^{-1}H + G\|_\infty^2 = 35.9356 \quad (25)$$

Adopting the Metzler matrix as in (17), Figure 2 shows the objective function of problem (18) inside the box  $[0, 5] \times [0, 5]$ . Searching a solution in this box we found  $(p, q) = (5.00, 4.50)$  and the corresponding cost  $J_\infty^{so} = 18.0677$ . This represents almost 50% reduction when compared to (25). From Theorem 5, the switching strategy  $\sigma_{so}$  is strictly consistent and its cost is even smaller than  $J_\infty^{so}$ . Even though we can not guarantee that the solution found inside that box is optimal, it improves the performance.

As a final remark we want to analyze the possibility to treat more general models than (1)-(2). Initially, using again the results of Deaecto and Geromel [2010], it can be verified that there is no difficulty to replace the full column rank input matrix  $H \in \mathbb{R}^{n \times m}$  by the  $\sigma$ -dependent matrix  $H_\sigma$ . Indeed, doing this, the result of Theorem 2 remains valid because for any  $Q_\ell > 0$ , it is always possible to define  $P_i > Q_\ell$  such that  $H'_i P_i H_i > H'_\ell Q_\ell H_\ell$ , for all  $i = 1, \dots, N$ ,  $i \neq \ell$ . The proof of Theorem 5 remains unchanged. Unfortunately, the same does not hold for matrix  $G$ . Following the proof of Theorem 5 it is clear that if we replace  $G$  by the  $\sigma$ -dependent matrix  $G_\sigma$  then even for  $\beta$  large enough, the constraints  $\rho_\ell > G'_i G_i$  for all  $i = 1, \dots, N$ ,  $i \neq \ell$  may not be redundant to the constraint

(20). Hence, in this case, the proof as it stands is not valid anymore. This point is left for future research.

#### 4. STATE FEEDBACK

In this section, our goal is to apply the concept of consistency to cope with state feedback control design problems. Consider a switched linear system of the form

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t) + Hw(t) \quad (26)$$

$$z(t) = E_\sigma x(t) + F_\sigma u(t) + Gw(t) \quad (27)$$

evolving from zero initial condition  $x(0) = 0$  where, besides vectors  $x$ ,  $w$  and  $z$  defined before,  $u \in \mathbb{R}^p$  is the control input. For  $u(t) = K_\sigma x(t)$  with  $K_\sigma \in \{K_1, \dots, K_N\}$  to be determined, the closed-loop system is given by

$$\dot{x}(t) = (A_\sigma + B_\sigma K_\sigma)x(t) + Hw(t) \quad (28)$$

$$z(t) = (E_\sigma + F_\sigma K_\sigma)x(t) + Gw(t) \quad (29)$$

We need to determine a set of matrix gains  $\{K_1, \dots, K_N\}$  such that the switching strategy  $\sigma_{so}(t) = g(x(t))$  is strictly consistent with respect to  $J_\alpha$  for each  $\alpha \in \{2, \infty\}$ . As a first step it is important to generalize the Lyapunov and Riccati-Metzler inequalities introduced before. Hence, for the closed-loop system (28)-(29), the set  $\mathcal{Y}_2$  is composed by all positive definite matrices  $S_i \in \mathbb{R}^{n \times n}$ , matrices  $Y_i \in \mathbb{R}^{p \times n}$ ,  $T_{ij} \in \mathbb{R}^{n \times n}$  and a Metzler matrix  $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$  satisfying for all  $i \in \mathbb{K}$  the Lyapunov-Metzler inequalities

$$\begin{bmatrix} \mathcal{V}_i + \sum_{j \neq i \in \mathbb{K}} \pi_{ji} T_{ij} & \bullet \\ E_i S_i + F_i Y_i & -I \end{bmatrix} < 0 \quad (30)$$

whereas the set  $\mathcal{Y}_\infty$  is composed by all positive definite matrices  $S_i \in \mathbb{R}^{n \times n}$ , matrices  $Y_i \in \mathbb{R}^{p \times n}$ ,  $T_{ij} \in \mathbb{R}^{n \times n}$ , a Metzler matrix  $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$  and a scalar  $\rho \in \mathbb{R}$  satisfying for all  $i \in \mathbb{K}$  the Riccati-Metzler inequalities

$$\begin{bmatrix} \mathcal{V}_i + \sum_{j \neq i \in \mathbb{K}} \pi_{ji} T_{ij} & \bullet & \bullet \\ H^T & -\rho I & \bullet \\ E_i S_i + F_i Y_i & G & -I \end{bmatrix} < 0 \quad (31)$$

In both inequalities (30) and (31), for each  $i \in \mathbb{K}$ , the matrix  $\mathcal{V}_i$  denotes  $\mathcal{V}_i = A_i S_i + S_i A_i' + B_i Y_i + Y_i' B_i'$  and in both sets the matrix variables are coupled by the LMIs

$$\begin{bmatrix} T_{ij} + S_i & \bullet \\ S_i & S_j \end{bmatrix} > 0, \quad i \neq j \in \mathbb{K} \times \mathbb{K} \quad (32)$$

From the previous results of Geromel and Deaecto [2007], Deaecto and Geromel [2010], it is known that any feasible solution of  $\mathcal{Y}_2$  or  $\mathcal{Y}_\infty$  allows us to synthesize the switching strategy  $\sigma_{so}(t) = g(x(t))$  with  $P_i = S_i^{-1}$  and the control input  $u(t)$  with  $K_i = Y_i S_i^{-1}$  for all  $i \in \mathbb{K}$ . In fact, applying the Schur Complement to (32) with respect to the last row and column, we obtain  $T_{ij} > S_i S_j^{-1} S_i - S_i$  which multiplied both sides by  $\pi_{ji}$ ,  $j \neq i \in \mathbb{K}$  provides

$$\begin{aligned} \sum_{j \neq i \in \mathbb{K}} \pi_{ji} T_{ij} &> \sum_{j \neq i \in \mathbb{K}} \pi_{ji} (S_i S_j^{-1} S_i - S_i) \\ &> \sum_{j \in \mathbb{K}} \pi_{ji} S_i S_j^{-1} S_i \end{aligned} \quad (33)$$

since  $\sum_{j \neq i \in \mathbb{K}} \pi_{ji} = -\pi_{ii}$  for all  $i \in \mathbb{K}$ . The inequality (33) puts in evidence that conditions (30) and (31) multiplied both sides by the nonsingular matrices  $\text{diag}\{S_i^{-1}, I\}$  and  $\text{diag}\{S_i^{-1}, I, I\}$ , respectively, are equivalent to (7) and (8) with the replacements  $(A_i + B_i K_i, E_i + F_i K_i) \rightarrow (A_i, E_i)$  for all  $i \in \mathbb{K}$ . Hence, the results of Theorem 2 and Theorem 5 remain valid for the sets  $\mathcal{Y}_2$  and  $\mathcal{Y}_\infty$ , respectively. We conclude that we can use Theorem 2 to determine the gains  $\{K_1, \dots, K_N\}$  and a strictly consistent switching strategy  $\sigma_{so} = g(x)$  by solving

$$J_2^{so} = \inf_{\{S_i, Y_i, W_i, T_{ij}\} \in \mathcal{Y}_2} \min_{i \in \mathbb{K}} \left\{ \text{Tr}(W_i) : \begin{bmatrix} W_i & \bullet \\ H & S_i \end{bmatrix} > 0 \right\} \quad (34)$$

Adopting the same reasoning for the  $H_\infty$  case, the gains  $\{K_1, \dots, K_N\}$  and a strictly consistent switched strategy  $\sigma_{so} = g(x)$  are obtained from Theorem 5. More specifically, from the optimal solution of

$$J_\infty^{so} = \inf_{\{S_i, Y_i, T_{ij}, \rho\} \in \mathcal{Y}_\infty} \rho \quad (35)$$

Some aspects of these results are illustrated by means of the following two simple examples. They are related to the  $H_2$  case because for  $H_\infty$ , the determination of the true value of the index  $J_\infty(\sigma_{so})$  is extremely difficult. Consider the system (26)-(27) with matrices

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

and  $H = [0 \ 10]'$ . As the pairs  $(A_1, B_1)$  and  $(A_2, B_2)$  are controllable the best performance taking into account the individual subsystems (that is with  $\sigma \in \mathcal{C}$ ) is

$$J_2(\sigma) = \min_{K_\ell, \ell \in \{1, 2\}} \|\tilde{E}_\ell(sI - \tilde{A}_\ell)^{-1} H\|_2^2 = 11.4745 \quad (36)$$

where  $\tilde{E}_\ell = (E_\ell + F_\ell K_\ell)$  and  $\tilde{A}_\ell = (A_\ell + B_\ell K_\ell)$ . The minimum is attained for  $\sigma(t) = 1, \forall t \geq 0$ . Adopting the Metzler matrix as in (17), Figure 3 shows the objective function of problem (34) and a plane surface representing the value of (36) inside the box  $[0, 60] \times [0, 60]$ , where the minimum value  $J_2^{so} = 2.4660$  occurs at  $(p, q) = (1.0, 0.0)$ . The true value  $J_2(\sigma_{so}) = 1.4223$  represents a significant reduction compared to (36) and the state feedback gain matrices are

$$K_1 = [-0.4698 \ -0.0986], \quad K_2 = [0.0000 \ -0.2361] \quad (37)$$

It is interesting to observe that the state feedback gain  $K_1$  given in (37) does not render the closed-loop system matrix  $A_1 + B_1 K_1$  Hurwitz. Even though, from Theorem 2 the switching strategy  $\sigma_{so}$  is strictly consistent. The next example illustrates another aspect of consistency. Consider the switched linear system (26)-(27) defined by matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 0.5 \end{bmatrix}$$

and the other ones unchanged. The best performance considering each subsystem isolated is

$$J_2(\sigma) = \min_{K_\ell, \ell \in \{1, 2\}} \|\tilde{E}_\ell(sI - \tilde{A}_\ell)^{-1} H\|_2^2 = 11.5388 \quad (38)$$

attained with  $\sigma(t) = 1, \forall t \geq 0$ . Adopting the Metzler matrix as in (17), problem (34) provides  $J_2^{so}$  which together with the plane surface defined by the minimum value (38) are shown in Figure 4. The minimization in



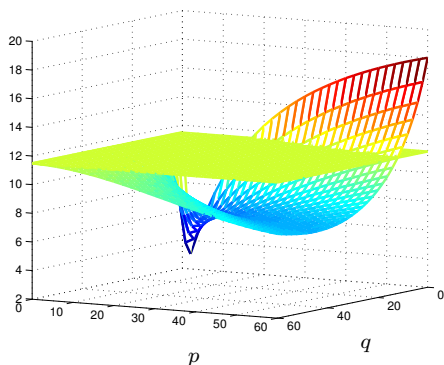


Fig. 3.  $J_2^{so}$  cost determination

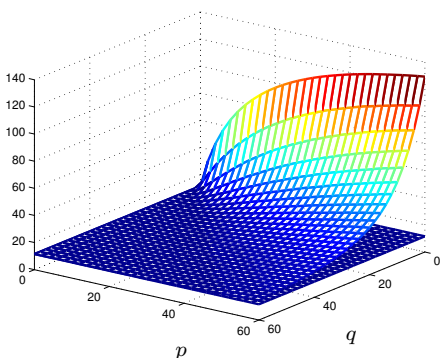


Fig. 4.  $J_2^{so}$  cost determination

the box  $[0, 60] \times [0, 60]$  gives  $(p, q) = (3.0, 11.0)$  and  $J_2^{so} = 11.4662 \approx \min_{\sigma \in C} J_2(\sigma)$ . However, the associated state feedback gains and the switching strategy  $\sigma_{so}(x(t))$  gives the true cost value  $J_2(\sigma_{so}) = 2.9010$  yielding the conclusion that it is strictly consistent. Once again, comparing to (38) the performance improvement is expressive.

## 5. CONCLUSION

In this paper the concept of consistency for switched linear systems has been introduced. It is used to construct a quality certificate for suboptimal solutions of problems involving  $H_2$  and  $H_\infty$  performance indexes as well as state feedback control design. It has been shown how to construct min-type switching strategies that are strictly consistent with respect to the mentioned indexes. The difficulty to calculate a strictly consistent switching strategy stems from the solution of a nonconvex problem with a particular structure. This feature may be relevant for the development of efficient algorithms to cope with this class of optimization problems. Theoretical results have been illustrated by numerical examples.

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