Memoryless sliding mode control for nonlinear systems with time delay disturbances using only output information

Xing-Gang Yan * Sarah K. Spurgeon * Christopher Edwards **

* *School of Engineering & Digital Arts, University of Kent, Canterbury, Kent CT2 7NT, United Kingdom (e-mail: X.Yan@kent.ac.uk, S.K.Spurgeon@kent.ac.uk)*
** *Department of Engineering, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom (e-mail: ce14@le.ac.uk)*

Abstract: In this paper, nonlinear systems with time delay disturbances are studied. Both matched and mismatched disturbances are considered. By using an appropriate transformation, the system is transformed to an appropriate regular form, and the nonlinear sliding mode dynamics are derived. A set of sufficient conditions is developed, using a Lyapunov-Razuminkhin approach, such that the sliding motion is uniformly asymptotically stable. An output feedback sliding mode control, independent of the time delay, is proposed to drive the system to the sliding surface in finite time. A simulation example shows the effectiveness of the approach.

1. INTRODUCTION

Nearly all real systems are nonlinear in nature. One of the basic approaches to deal with nonlinear systems is 'linearisation' which can be classified as approximate linearisation (Guardabassi and Savaresi [2001]) and exact/partial linearisation (Marino and Tomei [1995]). However, linearisation techniques are only applicable to a limited class of nonlinear systems. It may be possible to describe the operation of a real system by a linear model in a neighbourhood of a certain point, but it can only describe the 'local' behavior of the system.

Time delay is often encountered in engineering systems (Richard [2003], Gu et al. [2003]), where the delay may appear in the system state, input, output and disturbances. Sometimes even a small delay may affect the performance of the system greatly; a stable system may become unstable, or chaotic behavior may appear if delay arises in the system. This has motivated the study of time delay systems. A variety of control approaches including sliding mode control, H_∞ control and back-stepping techniques have been applied to the control of systems with time delay, and many important results have been achieved (see, e.g. Mazenc and Bliman [2006], Wang et al. [2009]). Much work has focussed on linear systems and/or assumed that all the system state variables are available. In some circumstances it is impossible or prohibitively expensive to measure all of the system state variables. This motivates the need to design control systems using only output information.

It is well known that sliding mode control is completely robust to so-called matched disturbances (Utkin [1992], Edwards and Spurgeon [1998]). This has motivated the application of sliding mode techniques to time delay systems with disturbances. Dynamical output feedback sliding mode control strategies are proposed in Niu et al. [2005], Yan et al. [2010, 2009]; these increase the system dimension and require more hardware for implementation. In comparison with this, static output feedback control is preferable. Note, when compared with state feedback, the static output feedback control problem is much more difficult, even for linear systems without delay (Syrmos et al. [1997]). Much less attention has been paid to systems involving time delay using static output feedback sliding mode control, and only very limited literature is available. A static output sliding mode control scheme for time delay systems is proposed by Janardhanan and Bandyopadhyay [2006] where only a class of linear discrete-time systems is considered. In all the existing results for time-delay systems, it is required that the bounds on the disturbances satisfy a linear growth condition. Recently, the bounds on disturbances/uncertainties have been extended to the nonlinear case for time delay systems (Yan et al. [2010]). However, the designed control explicitly depends on the time delay which requires the time-delay is perfectly known, and requires the nominal system to be largely linear.

As pointed out in Hua et al. [2008], most of the existing sliding mode controllers for nonlinear systems depend on time delay, and thus require that the time delay is known and hence require memory, which is difficult to implement especially for the case of time-varying delay. A memoryless control for a class of linear systems was proposed based on the back-stepping approach in Hua et al. [2008] where the nonlinear disturbances are matched and it is assumed that all the system states are available. Although a memoryless sliding mode control scheme is given in Yan [2003], all the nonlinear terms do not include delay and are assumed matched, which renders the associated sliding mode dynamics to be without delay.

LMI techniques have been widely applied to linear time delay systems (Fridman and Dambrine [2009], Richard [2003], Gu et al. [2003]), and provide a systematic design approach. However, it is impossible to find a systematic design approach for nonlinear systems because nonlinear systems exhibit very rich phenomena. In this paper a robust stabilisation problem is considered for a class of fully nonlinear systems with time-varying delay disturbances. Both the disturbances and the nominal system are nonlinear. It is not required that the nominal systems are linearisable or partially linearisable. The disturbances involved are matched and mismatched, and have nonlinear time delayed bounds. By employing an appropriate coordinate transformation, the system is first transformed to regular form, which provides a good structure for analysis and design. Based on the Lyapunov-Razuminkhin approach, sufficient conditions are derived to guarantee that the sliding motion is uniformly asymptotically stable. A static output feedback sliding mode control law is then proposed to drive the system to the sliding surface in finite time. The developed control is independent of the time delay. Finally, a simulation example is presented to show the effectiveness of the approach.

2. PROBLEM FORMULATION

Consider nonlinear systems with time delay disturbances

$$\dot{x} = f(t,x) + g(t,x)\big(u + \Delta g\big(t,x,x_d\big)\big) + \Delta f\big(t,x,x_d\big) \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in \Omega \subset \mathcal{R}^n$ (Ω is a neighbourhood of the origin), $u \in \mathcal{R}^m$ and $y \in \Omega_y \subset \mathcal{R}^p$ are, respectively, the state variables, inputs and outputs with $m \le p < n$. It is assumed the matrix function $g(\cdot) \in \mathcal{R}^{n\times m}$ has full column rank, and the nonlinear vectors $f(\cdot) \in \mathcal{R}^n$ and $h(\cdot) \in \mathcal{R}^p$ with $h(0) = 0$ are known. The terms $\Delta g(\cdot)$ and $\Delta f(\cdot)$ represent the matched and the mismatched disturbances respectively. The notation $x_d := x(t-d)$ and $y_d := y(t-d)$ represents delayed states and delayed outputs respectively, where $d := d(t)$ is the time-varying delay which is assumed to be known, continuous, nonnegative and bounded in $\mathcal{R}^+ := \{t \mid t \ge 0\}$, that is

$$\bar{d} := \sup_{t\in\mathcal{R}^+} \{d(t)\} < \infty$$

The initial condition associated with the time delay is

$$x(t) = \phi(t), \qquad t \in [-\bar{d}, 0] \quad (3)$$

where $\phi(\cdot) \in \Theta$ with Θ the admissible initial value set related to the time delay, which is defined by

$$\Theta := \Big\{\phi(t) \mid \phi(\cdot) \in \mathcal{C}_{[-\bar{d},0]}, \quad \|\phi(t)\| \le q_1\Big\} \quad (4)$$

for some constant $q_1 > 0$. It is assumed that all the nonlinear functions are smooth enough for the subsequent analysis, which guarantees that the unforced system has unique continuous solutions.

Suppose that the Jacobian matrix of the vector-valued function $h(x)$, denoted by J_h, is full rank in the domain Ω. Then, the elements of $h(x)$ are independent of each other, and there exist $n-p$ smooth functions $\delta_i(x)$ for $i = 1, \ldots, n-p$ such that the Jacobian matrix of the vector-valued function

$$T(x) := \begin{bmatrix} \delta_1(x) \\ \vdots \\ \delta_{n-p}(x) \\ h(x) \end{bmatrix} \quad (5)$$

$J_T(\cdot)$ is nonsingular in Ω. This implies that $T(x)$ forms a diffeomorphism in the domain Ω. Let

$$z := [\delta_1(x), \cdots, \delta_{n-p}(x)]^T$$

Clearly the diffeomorphism T in (5) defines a new coordinate system:

$$T: \ x \mapsto \mathrm{col}(z,y) = T(x) \quad (6)$$

Further, it is assumed that the input distribution function matrix $g(t,x)$ satisfies

$$\left[\frac{\partial T(x)}{\partial x} g(t,x)\right] = \begin{bmatrix} 0 \\ G(t,y) \end{bmatrix} \quad (7)$$

where $G(t,y) \in \mathcal{R}^{m\times m}$ is nonsingular in $\mathcal{R}^+ \times \Omega_y$.

Then, in the new coordinate system (z,y) defined by (6), system (1)–(2) can be described by

$$\begin{bmatrix} \dot{z} \\ \dot{y}_1 \end{bmatrix} = F_1(t,z,y_1,y_2) + \Delta F_1(t,z,y,z_d,y_d) \quad (8)$$

$$\dot{y}_2 = F_2(t,z,y_1,y_2) + G(t,y_1,y_2)\big(u + \Delta G(t,z, y,z_d,y_d)\big) + \Delta F_2(t,z,y,z_d,y_d) \quad (9)$$

$$y = \mathrm{col}(y_1,y_2) \quad (10)$$

where $z \in \mathcal{R}^{n-p}$, $y_1 \in \mathcal{R}^{p-m}$ and $y_2 \in \mathcal{R}^m$ form the states in the new coordinate system, and $u \in \mathcal{R}^m$ and $y := \mathrm{col}(y_1,y_2) \in \mathcal{R}^p$ are the system inputs and outputs respectively. In the above, $y_d := y(t-d)$, $z_d := z(t-d)$, and

$$\begin{bmatrix} F_1(\cdot) \\ F_2(\cdot) \end{bmatrix} := \left[\frac{\partial T}{\partial x} f(t,x)\right]_{x=T^{-1}(z,y)}$$

$$\begin{bmatrix} \Delta F_1(\cdot) \\ \Delta F_2(\cdot) \end{bmatrix} := \left[\frac{\partial T}{\partial x} \Delta f(t,x)\right]_{x=T^{-1}(z,y)}$$

$$\Delta G(\cdot) := [\Delta g(t,x,x_d)]_{x=T^{-1}(z,y)}$$

In the new coordinate system (z,y_1,y_2), the domain Ω is transferred to

$$\begin{aligned}\Omega_T &:= \Omega_z \times \Omega_{y_1} \times \Omega_{y_2} \\ &:= \{(z,y_1,y_2) \mid (z,y_1,y_2) = T(x), x \in \Omega\}\end{aligned}$$

where $z \in \Omega_z$, $y_1 \in \Omega_{y_1}$, and $y_2 \in \Omega_{y_2}$. It is clear that system (8)–(9) is in regular form, and that the system outputs are a subset of the state variables, thus facilitating the sliding mode design.

In this paper, the stabilisation problem for system (8)–(10) will be considered. The objective is to design a static output feedback sliding mode control law

$$u = u(t,y) \quad (11)$$

which depends only on time t and the system output y, but is independent of the time delay $d(t)$, such that the closed-loop system formed by applying the control (11) to system (8)–(9) is uniformly asymptotically stable irrespective of the delayed disturbances. Since this control has no delay involved, it is called memoryless control.

The local case will be treated in this paper. In order to avoid unnecessary notation in describing the local region, the domain may not be specifically stated, but each variable's dimension will be clearly shown.

3. PRELIMINARIES

Some definitions are first introduced.

Definition 1. A function $f_0(x_1, x_2, x_3) : \Omega_1 \times \Omega_2 \times \Omega_3 \mapsto \mathcal{R}^n$ is said to satisfy a generalised Lipschitz condition with respect to (w.r.t.) the variables $x_1 \in \Omega_1 \subset \mathcal{R}^{n_1}$ and $x_2 \in \Omega_2 \subset \mathcal{R}^{n_2}$ uniformly for x_3 in $\Omega_3 \subset \mathcal{R}^{n_3}$ if there exist continuous functions $\mathcal{L}_{f_0 1}(\cdot)$ and $\mathcal{L}_{f_0 2}(\cdot)$ defined in Ω_3 such that for any $\hat{x}_1, x_1 \in \Omega_1$ and $\hat{x}_2, x_2 \in \Omega_2$, the inequality

$$\begin{aligned} &\|f_0(x_1, x_2, x_3) - f_0(\hat{x}_1, \hat{x}_2, x_3)\| \\ &\leq \mathcal{L}_{f_0 1}(x_3)\,\|x_1 - \hat{x}_1\| + \mathcal{L}_{f_0 2}(x_3)\,\|x_2 - \hat{x}_2\| \end{aligned}$$

holds for any $x_3 \in \Omega_3$. Further, if $\Omega_1 = \mathcal{R}^{n_1}$ and $\Omega_2 = \mathcal{R}^{n_2}$, then, it is said that $f_0(\cdot)$ satisfies a global generalised Lipschitz condition w.r.t. x_1 and x_2 uniformly for x_3 in Ω_3. Moreover, $f_0(\cdot)$ is called a (global) generalised Lipschitz function, and $\mathcal{L}_{f_0 1}(\cdot)$ and $\mathcal{L}_{f_0 2}(\cdot)$ are called (global) generalised Lipschitz constants.

Remark 1. The generalised Lipschitz condition in Definition 1 has been introduced in Corduneanu [1991]. The generalised Lipschitz constants $\mathcal{L}_{f_0 1}(x_3)$ and $\mathcal{L}_{f_0 2}(x_3)$ are usually functions of x_3 instead of constants. This is different from the normal Lipschitz condition. However, in order to maintain a consistent style, here the continuous functions $\mathcal{L}_{f_0 1}(x_3)$ and $\mathcal{L}_{f_0 2}(x_3)$ are called generalised Lipschitz constants. This generalised Lipschitz condition is more relaxed than the normal Lipschtiz condition.

Definition 2. (see, Khalil [2002], Gu et al. [2003]) A continuous function $\alpha : \ [0, a) \mapsto [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. Further, it is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

In connection with the well known $\mathcal{K}$ function defined above, the following new concept is introduced, which will be termed as a $\mathcal{WS}$ function.

Definition 3. A continuous function $\beta(t, \tau_1, \tau_2)\ :\ \mathcal{R}^+ \times \mathcal{R}^+ \times \mathcal{R}^+ \mapsto \mathcal{R}^+$ with $\beta(t, 0, 0) = 0$ is said to be weak w.r.t the variable τ_1 and strong w.r.t. the variable τ_2 if there exist continuous functions $\chi_1(t, \tau_1, \tau_2)$ and $\chi_2(t, \tau_1, \tau_2)$ such that

$$\beta(t, \tau_1, \tau_2) = \chi_1(t, \tau_1, \tau_2)\tau_1 + \chi_2(t, \tau_1, \tau_2)\tau_2 \qquad (12)$$

where both functions $\chi_1(\cdot, \tau_2)$ and $\chi_2(\cdot, \cdot, \tau_2)$ are nondecreasing w.r.t. the variable τ_2. The function $\beta(t, \tau_1, \tau_2)$ is said to belong to the class $\mathcal{WS}$ function w.r.t. τ_1 and τ_2.

Remark 2. It should be noted that if a continuous function $\alpha(t, \tau_1, \tau_2) : \mathcal{R}^+ \times \mathcal{R}^+ \times \mathcal{R}^+ \mapsto \mathcal{R}^+$ with $\alpha(t, 0, 0) = 0$ is smooth enough, then, the decomposition

$$\alpha(t, \tau_1, \tau_2) = \alpha_1(t, \tau_1, \tau_2)\tau_1 + \alpha_2(t, \tau_1, \tau_2)\tau_2$$

holds. The relevant discussion is available in Banks and Al-jurani [1994]. Further if $\alpha_1(t, \tau_1, \tau_2)$ and $\alpha_2(t, \tau_1, \tau_2)$ are nondecreasing w.r.t. τ_2, then $\alpha(t, \tau_1, \tau_2)$ belongs to a $\mathcal{WS}$ function w.r.t. τ_1 and τ_2.

4. SLIDING MODE DESIGN

In this section, a sliding surface is proposed first, and the stability of the associated sliding mode is analysed. Then a sliding mode control is designed such that the closed-loop system is driven to the sliding surface and maintains a sliding motion on it thereafter.

4.1 Stability of sliding motion

For system (8)–(10), choose the switching function

$$s(x) := y_2 \qquad (13)$$

Then, the output sliding surface is described by

$$\{\mathrm{col}(z, y_1, y_2) \mid y_2 = 0\} \qquad (14)$$

Since system (8)–(9) is in regular form, the sliding mode dynamics associated with the sliding surface (14) can be described in a compact form by

$$\dot{X} = F_{1s}(t, X) + \Delta F_{1s}(t, X, X_d) \qquad (15)$$

where $X := \mathrm{col}(z, y_1) \in \mathcal{R}^{n-m}$. The considered domain is $X \in \Omega_X := \Omega_z \times \Omega_{y_1}$, and

$$\begin{aligned} F_{1s}(t, X) &:= F_1(t, z, y_1, 0), \\ \Delta F_{1s}(t, X, X_d) &:= \Delta F_1(t, z, y_1, 0, z_d, y_{1d}, 0) \end{aligned}$$

The initial value related to the delay for system (15) is obtained from (3) using the coordinate transformation (6).

Assumption 1. There exists a C^1 function $V(\cdot) : \mathcal{R}^+ \times \mathcal{R}^{n-m} \mapsto \mathcal{R}^+$ and positive constants r_i for $i = 1, \ldots, 4$ such that for any $X \in \Omega_X$

i). $r_1\|X\|^2 \leq V(t, X) \leq r_2\|X\|^2$;

ii). $\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial X}\right)^T F_{1s}(t, X) \leq -r_3\|X\|^2$;

iii). $\|\frac{\partial V}{\partial X}\| \leq r_4\|X\|$

where $\frac{\partial V}{\partial X} := \left[\frac{\partial V}{\partial X_1} \ \cdots \ \frac{\partial V}{\partial X_{n-m}}\right]^T$ and $\mathrm{col}(X_1, X_2 \cdots, X_{n-m}) := X$.

Assumption 2. If there exists a continuous nondecreasing function $\rho(\cdot)$ defined in $\mathcal{R}^+$ satisfying $\rho(\tau) > \tau$ for $\tau > 0$, such that for any $\theta \in [-\bar{d},\ 0]$,

$$V(t + \theta, X(t + \theta)) \leq \rho(V(t, X(t))) \qquad (16)$$

then, there exists a constant $c_0 > 1$ such that

$$\|X(t + \theta)\| \leq c_0\|X(t)\| \qquad (17)$$

for any $\theta \in [-\bar{d},\ 0]$ where $V(\cdot)$ is given in Assumption 1.

Remark 3. If the nominal system associated with (15) is exponentially stable, then the conditions i)-iii) in Assumption 1 hold (see, e.g. Theorem 4.14 in Khalil [2002]). Assumption 2, which is related to the time delay, is a further limitation on the function $V(\cdot)$ given in Assumption 1. If time delay is not involved, then Assumption 2 will be unnecessary. A class of functions satisfying Assumption 2 is presented in Lemma 1 in the Appendix.

Assumption 3. The disturbance $\Delta F_{1s}(\cdot)$ in (15) satisfies

$$\|\Delta F_{1s}(t, X, X_d)\| \leq \psi\,(t, \|X\|, \|X_d\|) \qquad (18)$$

where $\psi(\cdot, \tau_1, \tau_2)$ is a known class $\mathcal{WS}$ function w.r.t. the variables τ_1 and τ_2.

Since $\psi(\cdot)$ is a class of $\mathcal{WS}$ function, it follows that under Assumption 3, the function $\psi(\cdot)$ has a decomposition as

$$\psi(\cdot) = \psi_1(t, \|X\|, \|X_d\|)\|X\| + \psi_2(t, \|X\|, \|X_d\|)\|X_d\| \quad (19)$$

where the functions $\psi_1(\cdot, \cdot, \tau)$ and $\psi_2(\cdot, \cdot, \tau)$ are nondecreasing w.r.t. the variable $\tau \in \mathcal{R}^+$.

Theorem 1. Under Assumptions 1-3, system (8)–(9) has a uniformly asymptotically stable sliding motion associated with the sliding surface (14) if for $X \in \Omega_X$ and $t \in \mathcal{R}^+$,

$$\psi_1(t, \|X\|, c_0\|X\|) + c_0\psi_2(t, \|X\|, c_0\|X\|) \leq \frac{1}{r_4}(r_3 - \gamma) \tag{20}$$

holds for some $\gamma > 0$ and $c_0 > 1$, where $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are given in (19), and r_3 and r_4 are defined in Assumption 1.

Proof: The sliding mode dynamics of system (8)–(9) in (15) must be proved to be uniformly asymptotically stable. Consider the Lyapunov candidate function $V(\cdot)$ defined in Assumption 1. It follows that the time derivative of $V(\cdot)$ along the trajectory of system (15) is given by

$$\begin{aligned}\dot{V} &= \tfrac{\partial V}{\partial t} + \left(\tfrac{\partial V}{\partial X}\right)^T (F_{1s}(t, X) + \Delta F_{1s}(t, X, X_d)) \\ &\leq \tfrac{\partial V}{\partial t} + \left(\tfrac{\partial V}{\partial X}\right)^T F_{1s}(t, X) + \left\| \left(\tfrac{\partial V}{\partial X}\right)^T \right\| \\ &\quad \cdot\|\Delta F_{1s}(t, X, X_d)\| \\ &\leq -r_3\|X\|^2 + r_4\psi_1(t, \|X\|, \|X_d\|)\|X\|^2 \\ &\quad +r_4\psi_2(t, \|X\|, \|X_d\|)\|X\|\|X_d\|\end{aligned} \tag{21}$$

where equation (19) and Assumptions 1 and 3 are employed. Assume that there exists a function $\rho(\cdot)$ in $\mathcal{R}^+$ which satisfies $\rho(\tau) > \tau$ for $\tau > 0$, such that

$$V(t + \theta, X(t + \theta)) \leq \rho(V(t, X(t))), \quad \theta \in [-\bar{d},\ 0] \tag{22}$$

Then, from Assumption 2, there exists a constant $c_0 > 1$ such that for any $\theta \in [-\bar{d}\ 0]$

$$\|X(t + \theta)\| \leq c_0\|X(t)\| \tag{23}$$

Since $\psi_1(\cdot, \cdot, \tau)$ and $\psi_2(\cdot, \cdot, \tau)$ are nondecreasing *w.r.t.* τ in $\mathcal{R}^+$, it is clear that if (22) holds, then from (23) and (21),

$$\begin{aligned}\dot{V} \leq &-\Big(r_3 - r_4\psi_1(t, \|X\|, c_0\|X\|) \\ &-r_4c_0\psi_2(t, \|X\|, c_0\|X\|)\Big)\|X\|^2 \leq -\gamma\|X\|^2\end{aligned}$$

where inequality (20) is employed above. Hence the conclusion follows from the Razumikhin Theorem (see, e.g. Gu et al. [2003]) . ∇

4.2 Reachability analysis

In order to design a static output feedback sliding mode control law for system (8)–(10), the following assumptions are imposed on the system (9).

Assumption 4. The disturbances $\Delta G(\cdot)$ and $\Delta F_2(\cdot)$ in (9) satisfy

$$\|\Delta G(t, z, y, z_d, y_d)\| \leq \varpi_1(t, z, y, z_d, y_d) \tag{24}$$

$$\|\Delta F_2(t, z, y, z_d, y_d)\| \leq \varpi_2(t, z, y, z_d, y_d) \tag{25}$$

for some known functions $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ which in turn satisfy the generalised Lipschitz condition w.r.t. the variables z, z_d and y_d uniformly for $t \in \mathcal{R}^+$ and $y \in \Omega_y$.

Assumption 5. The nonlinear function $F_2(t, z, y)$ in (9) satisfies the generalised Lipschitz condition w.r.t. the variables z uniformly for $t \in \mathcal{R}^+$ and $y \in \Omega_y$.

Assumptions 4 and 5 imply that the following inequalities

$$\begin{aligned}|\varpi_1(t, z, y, z_d, y_d) - \varpi_1(t, 0, y, 0, 0)| &\leq \mathcal{L}_{\varpi_1 1}(t, y)\|z\| \\ &+ \mathcal{L}_{\varpi_1 2}(t, y)\|z_d\| + \mathcal{L}_{\varpi_1 3}(t, y)\|y_d\|\end{aligned} \tag{26}$$

$$\begin{aligned}|\varpi_2(t, z, y, z_d, y_d) - \varpi_2(t, 0, y, 0, 0)| &\leq \mathcal{L}_{\varpi_2 1}(t, y)\|z\| \\ &+ \mathcal{L}_{\varpi_2 2}(t, y)\|z_d\| + \mathcal{L}_{\varpi_2 3}(t, y)\|y_d\|\end{aligned} \tag{27}$$

$$\|F_2(t, z, y) - F_2(t, 0, y)\| \leq \mathcal{L}_{F_2 1}(t, y)\|z\| \tag{28}$$

hold in the domain Ω_T.

Consider system (8)–(10) in the domain

$$\{(z, y) \mid \|z\| \leq q_2,\ y \in \Omega_y\} \subset \Omega_T \tag{29}$$

where q_2 is a positive constant and $\Omega_y := \Omega_{y_1} \times \Omega_{y_2}$. Let

$$q := \max\{q_1, q_2\} \tag{30}$$

where q_1 and q_2 are defined by (4) and (29) respectively. Construct the control law

$$\begin{aligned}u(t, y) = &-G^{-1}(t, y)F_2(t, 0, y) - G^{-1}(t, y)\Big(\|G(t, y)\| \\ &\cdot\varpi_1(t, 0, y, 0, 0) + \varpi_2(t, 0, y, 0, 0)\Big)\mathrm{sgn}(y_2) \\ &-G^{-1}(t, y)k(t, y)\mathrm{sgn}(y_2)\end{aligned} \tag{31}$$

where $F_2(\cdot)$ is given in (9) and the functions $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ satisfy (24) and (25) respectively. The symbol sgn is the usual signum function, and the function $k(\cdot)$ is the control gain to be determined later. The function $G(t, y)$ is given in (7), which is nonsingular in $\mathcal{R}^+ \times \mathcal{Y}$, and thus the control $u(\cdot)$ in (31) is well defined.

Theorem 2. Consider the nonlinear system (8)–(9). Under Assumptions 4 and 5, system (8)–(9) can be driven to the sliding surface (14) in finite time and maintains a sliding motion on it thereafter by control (31) if the control gain $k(\cdot)$ is chosen as

$$\begin{aligned}k(t, y) := q\Big(&\mathcal{L}_{F_2 1}(t, y) + \|G(t, y)\|\big(\mathcal{L}_{\varpi_1 1}(t, y) + \mathcal{L}_{\varpi_1 2}(t, y) \\ &+\mathcal{L}_{\varpi_1 3}(t, y)\big) + \mathcal{L}_{\varpi_2 1}(\cdot) + \mathcal{L}_{\varpi_2 2}(\cdot) + \mathcal{L}_{\varpi_2 3}(\cdot)\Big) + \eta\end{aligned} \tag{32}$$

for any $\eta > 0$, where the positive constant q is defined in (30), $F_2(\cdot)$ is given in (9) and the functions $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ satisfy (24) and (25) respectively. $\mathcal{L}_\star$ are the associated generalised Lipschitz constants given in (26)–(28).

Proof: Substituting $u(\cdot)$ in (31) into equation (9), it follows from (13) that

$$\begin{aligned}s^T(x)\dot{s}(x) = s^T(x)&\Big(F_2(t, z, y) - F_2(t, 0, y)\Big) + s^T(x) \\ &\cdot G(t, y_1, y_2)\Delta G(t, z, y, z_d, y_d) \\ &-\|G(t, y)\|\varpi_1(t, 0, y, 0, 0)s^T(x)\mathrm{sgn}(y_2) \\ &+s^T(x)\Delta F_2(t, z, y, z_d, y_d) \\ &-\varpi_2(t, 0, y, 0, 0)s^T(x)\mathrm{sgn}(y_2) \\ &-k(t, y)s^T(x)\mathrm{sgn}(y_2)\end{aligned} \tag{33}$$

As $s(x) = y_2$ in (13), it follows that under Assumption 5,

$$\begin{aligned}&s^T(x)\Big(F_2(t, z, y) - F_2(t, 0, y)\Big) \\ &\leq \|s(x)\|\,\|F_2(\cdot) - F_2(t, 0, y)\| \leq q\mathcal{L}_{F_2 1}(t, y)\|s(x)\|\end{aligned} \tag{34}$$

and from the fact that $s^T\text{sgn}(s) \geq \|s\|$ for any vector s, it follows that under Assumption 4,

$$\begin{aligned}
&s^T(x)G(t,y_1,y_2)\Delta G(t,z,y,z_d,y_d) - \|G(t,y)\| \\
&\cdot\varpi_1(t,0,y,0,0)s^T(x)\text{sgn}(y_2) \\
&\leq \|s(x)\|\,\|G(t,y)\|\big(\varpi_1(t,z,y,z_d,y_d) - \varpi_1(t,0,y,0,0)\big) \\
&\leq \|s(x)\|\,\|G(t,y)\|\big(\mathcal{L}_{\varpi_1 1}(t,y)\|z\| + \mathcal{L}_{\varpi_1 2}(t,y)\|z_d\| \\
&\quad+\mathcal{L}_{\varpi_1 3}(t,y)\|y_d\|\big) \\
&\leq q\|s(x)\|\,\|G(t,y)\|\big(\mathcal{L}_{\varpi_1 1}(\cdot) + \mathcal{L}_{\varpi_1 2}(\cdot) + \mathcal{L}_{\varpi_1 3}(\cdot)\big) \quad (35)
\end{aligned}$$

where q is defined by (30), and (26) is employed above. By similar reasoning as in (35),

$$\begin{aligned}
&s^T(x)\Delta F_2(t,z,y,z_d,y_d) - \varpi_2(t,0,y,0,0)s^T(x)\text{sgn}(y_2) \\
&\leq \|s(x)\|\Big(\varpi_2(t,z,y,z_d,y_d) - \varpi_2(t,0,y,0,0)\Big) \\
&\leq q\|s(x)\|\Big(\mathcal{L}_{\varpi_2 1}(t,y) + \mathcal{L}_{\varpi_2 2}(t,y) + \mathcal{L}_{\varpi_2 3}(t,y)\Big) \quad (36)
\end{aligned}$$

and substituting (34)–(36) into (33) yields

$$\begin{aligned}
s^T\dot{s} &\leq q\Big(\mathcal{L}_{F_2 1}(t,y) + \|G(t,y)\|\big(\mathcal{L}_{\varpi_1 1}(t,y) + \mathcal{L}_{\varpi_1 2}(t,y) \\
&\quad+\mathcal{L}_{\varpi_1 3}(t,y)\big) + \mathcal{L}_{\varpi_2 1}(t,y) + \mathcal{L}_{\varpi_2 2}(t,y) \\
&\quad+\mathcal{L}_{\varpi_2 3}(t,y)\Big)\|s\| - k(t,y)\|s\| \\
&= -\eta\|s\|
\end{aligned}$$

where (32) is employed above. This shows that the reachability condition holds. Hence the conclusion follows. ∇

Theorems 1 and 2 together show that the associated closed-loop system is uniformly asymptotically stable.

5. SIMULATION EXAMPLE

Consider a nonlinear system with time delay disturbances

$$\dot{x} = \underbrace{\begin{bmatrix} -3x_1x_2^2 - 3x_1 + x_3^2 \\ 3x_1^2x_2 - 3x_2 - x_3\exp\{-t\}\cos(x_2t) \\ -2x_3 + \frac{1}{2}x_1x_3^2 \end{bmatrix}}_{f(\cdot)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ x_2^2\sin^2 t + 1 \end{bmatrix}}_{g(\cdot)}\big(u + \Delta g(t,x,x_d)\big) + \Delta f(t,x,x_d) \quad (37)$$

$$y =: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \quad (38)$$

where $x = \text{col}(x_1,x_2,x_3) \in \mathcal{R}^3$, $u \in \mathcal{R}$ and $y \in \mathcal{R}^2$ are, respectively, the state variables, input and outputs, and $\Delta g(\cdot)$ and $\Delta f(\cdot)$ are the matched and mismatched disturbances respectively.

Let $z = x_1$. It is easy to see that the system (37)–(38) is in the form (8)–(10) as follows

$$\begin{bmatrix} \dot{z} \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -3zy_1^2 - 3z + y_2^2 \\ 3z^2y_1 - 3y_1 - y_2\exp\{-t\}\cos(y_1t) \end{bmatrix}}_{F_1(\cdot)} + \Delta F_1(t,z,y,z_d,y_d) \quad (39)$$

$$\dot{y}_2 = \underbrace{-2y_2 + \frac{1}{2}zy_2^2}_{F_2(\cdot)} + \underbrace{(1 + y_1^2\sin^2 t)}_{G(\cdot)}\big(u + \Delta G(t,z,y,z_d,y_d)\big) + \Delta F_2(t,z,y,z_d,y_d) \quad (40)$$

The disturbances $\Delta G(\cdot)$, $\Delta F_1(\cdot)$ and $\Delta F_2(\cdot)$ are assumed to satisfy

$$|\Delta G(\cdot)| \leq \underbrace{|z_d + y_1|\exp\{-t\}}_{\varpi_1(\cdot)} \quad (41)$$

$$\|\Delta F_1(\cdot)\| \leq z_d^2\sin^2(ty_{1d}) + (y_{2d}y_2)^4 \quad (42)$$

$$\|\Delta F_2(\cdot)\| \leq \underbrace{(|z_d| + 1)(y_2^2 + |y_1|)}_{\varpi_2(\cdot)} \quad (43)$$

Choose the switching function

$$s(x) := y_2$$

Then, the sliding mode dynamics are described by

$$\dot{X} = \underbrace{\begin{bmatrix} -3zy_1^2 - 3z \\ 3z^2y_1 - 3y_1 \end{bmatrix}}_{F_{1s}(\cdot)} + \Delta F_{1s}(t,X,X_d) \quad (44)$$

where $X = \text{col}(z,y_1)$, and from (42)

$$\|\Delta F_{1s}(t,X,X_d)\| \leq z_d^2\sin^2(ty_{1d}) \leq \underbrace{z_d^2}_{\psi}$$

and thus equation (19) is satisfied with

$$\psi_1(\cdot) = 0 \qquad \text{and} \qquad \psi_2(\cdot) = \|z_d\|$$

This implies that Assumption 3 is satisfied. Construct a candidate Lyapunov function $V = z^2 + y_1^2$. Assumption 1 holds with $r_1 = r_2 = 1$, $r_3 = 6$, and $r_4 = 2$. From Lemma 1 in the Appendix, Assumption 2 holds with $\rho(\cdot) = 1$ and $c_0 = 1.01$. Let $\gamma = 0.5$, then it follows that all the conditions in Theorem 1 hold in the domain

$$\Omega_X = \{(z,y_1)\big|\ |z| \leq 2.7,\ y_1 \in \mathcal{R}\}$$

which guarantees that the sliding motion is uniformly asymptotically stable.

Then, the control law (31) with the gain $k(\cdot)$ satisfying (32) is well defined. From Theorems 1 and 2, the closed loop system is uniformly asymptotically stable.

For simulation purposes, the delay is chosen as $d(t) = 2 - \sin t$ and the initial condition related to the delay is given by

$$\phi(t) = [\sin t \quad 0 \quad 1 + \cos t]^T$$

and the constants $q = 3$ and $\eta = 1$. The simulation in figure 1 demonstrates the effectiveness of the proposed approach.

6. CONCLUSIONS

A class of fully nonlinear systems with time delay disturbances has been studied in this paper. Based on sliding mode techniques, a static output feedback control law is designed to stabilise the system. The sliding mode dynamics are fully nonlinear and the bounds on the disturbances are nonlinear and time delayed. Both matched and mismatched disturbances are considered. The conservatism is reduced by exploiting the system structure. This work provides a methodology to deal with fully nonlinear systems using static output feedback sliding mode control.

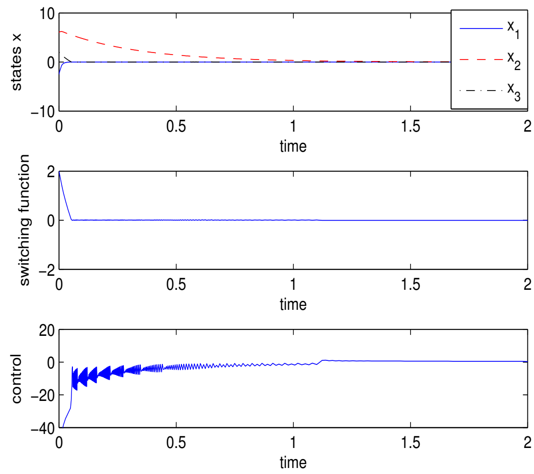

Fig. 1. The time response of the controlled system (37)–(38) (top), the switching function (middle) and the control signal (bottom)

APPENDIX

In this Appendix, a result related to Assumption 2 will be shown.

Lemma 1. Suppose that $\xi : \mathcal{R}^+ \mapsto \mathcal{R}^+$ is differentiable and strictly increasing with $\xi(0) = 0$, and $P \in \mathcal{R}^{n\times n}$ is symmetric positive definite (*s.p.d.*). Then the function $V(X) := \xi(X^T PX))$ with $X \in \mathcal{R}^n$ is positive definite and satisfies Assumption 2 if there exists $b > 1$ such that $b\xi(r) \le \xi(cr)$ for some constant $c > 0$ and any $r \in \mathcal{R}^+$.

Proof: Since P is s.p.d. and $\xi(\cdot)$ is strictly increasing in $\mathcal{R}^+$ with $\xi(0) = 0$, it follows from $X^T PX \ge 0$ that

$$V(X) = \xi(X^T PX) \ge \xi(0) = 0$$

and $V(0) = 0 \Longleftrightarrow X = 0$. This implies that the function $V(X)$ is positive definite.

The objective now is to prove that $V(\cdot)$ satisfies Assumption 2. Suppose that there exists $b > 1$ such that $b\xi(r) \le \xi(cr)$ for some constant $c > 0$ and any $r \in \mathcal{R}^+$. Then, Let $\rho(r) := br$ $(r \in \mathcal{R}^+)$. Since $\xi : \mathcal{R}^+ \mapsto \mathcal{R}^+$ is strictly increasing in $\mathcal{R}^+$, it follows that when $V(X_d) \le \rho(V(X))$ $(X_d := X(t-d))$,

$$\begin{aligned}
&\xi\left(X_d^T PX_d\right) \le b\xi\left(X^T PX\right) \le \xi(cX^T PX)\\
\Longleftrightarrow\ & X_d^T PX_d \le cX^T PX\\
\Longrightarrow\ & \underline{\lambda}(P)\|X_d\|^2 \le X_d^T PX_d \le cX^T PX \le c\overline{\lambda}(P)\|X\|^2\\
\Longleftrightarrow\ & \|X_d\| \le \sqrt{c\overline{\lambda}(P)/\underline{\lambda}(P)}\|X\|.
\end{aligned}$$

where $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ denote the maximum and minimum eigenvalues of a matrix respectively. Hence the conclusion follows by choosing $c_0 \ge \sqrt{c\overline{\lambda}(P)/\underline{\lambda}(P)}$. #

Lemma 1 shows a class of functions satisfying Assumption 2. For example, the function

$$V(z, y) = (z^T P_1 z + y^T P_2 y)^\alpha$$

belongs to such a class of functions if the matrices P_1 and P_2 are *s.p.d.* with appropriate dimensions, and $\alpha \ge 1$ is a constant.

REFERENCES

S. P. Banks and S. K. Al-jurani. Lie algebra and the stability of nonlinear systems. *Int. J. Control*, 60:315–329, 1994.

C. Corduneanu. *Integral equations and applications*. New York: Cambridge University Press, 1991.

C. Edwards and S. K. Spurgeon. *Sliding mode control: theory and applications*. London: Taylor and Francis Ltd, 1998.

E. Fridman and M. Dambrine. Control under quantization, saturation and delay: An LMI approach. *Automatica*, 45 (10):2258–64, 2009.

K. Gu, V. L. Kharitonov, and J. Chen. *Stability of time-delay systems*. Boston: Birkhäuser, 2003.

G. O. Guardabassi and S. M. Savaresi. Approximate linearization via feedback – an overview. *Automatica*, 37(1):1–15, 2001.

C. Hua, Q. Wang, and X. Guan. Exponential stabilization controller design for interconnected time delay systems. *Automatica*, 44(10):2600–06, 2008.

S. Janardhanan and B. Bandyopadhyay. Output feedback discrete-time sliding mode control for time delay systems. *IEE Proc. Part D: Control Theory Appl.*, 153(4): 387–396, 2006.

H. K. Khalil. *Nonlinear Systems* (Third Edition). New Jersey: Prentice Hall, Inc., 2002.

R. Marino and P. Tomei. *Nonlinear Control Design*. Englewood Cliff: Prentice Hall International, 1995.

F Mazenc and P. A. Bliman. Backstepping design for time-delay nonlinear systems. *IEEE Trans. on Automat. Control*, 51(1):149–154, 2006.

Y. Niu, D.W.C. Ho, and J. Lam. Robust integral sliding mode control for uncertain stochastic systems with time-varying delay. *Automatica*, 41(5):873–880, 2005.

J. P. Richard. Time-delay systems: An overview of some recent advances and open problems. *Automatica*, 39(10): 1667–1694, 2003.

V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis. Static output feedback—a survey. *Automatica*, 33 (2):125–137, 1997.

V. I. Utkin. *Sliding modes in control optimization*. Berlin: Springer-Verlag, 1992.

Z. Wang, D. W. C. Ho, Y. Liu, and X. Liu. Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements. *Automatica*, 45(3): 684–91, 2009.

J. J. Yan. Memoryless adaptive decentralized sliding mode control for uncertain large-scale systems with time-varying delays. *ASME Journal Dynamics Systems, Measurement and Control*, 125(2):172–76, 2003.

X. G. Yan, S. K. Spurgeon, and C. Edwards. Global stabilisation for a class of nonlinear time-delay systems based on dynamical output feedback sliding mode control. *Int. J. Control*, 82(12):2293–303, 2009.

X. G. Yan, S. K. Spurgeon, and C. Edwards. Sliding mode control for time-varying delayed systems based on a reduced-order observer. *Automatica*, 46(8):1354–62, 2010.