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Unitary System - I: Constructing a Unitary Fault Detection Observer

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Abstract: A linear time-invariant system is defined as unitary if all singular values of its transfer matrix are equal. A method of constructing a unitary system in a fault detection observer form is developed in this paper. The singular values of the constructed system can be assigned as an analytical function of frequency with a selectable parameter. As the optimization of singular values related properties is important in balancing the sensitivity and robustness of fault detection observers, the study of unitary system has both theoretical importance and potential applications in model-based fault detection such as the $H_\infty/H\_$ optimization to be discussed in Part II of the paper.

Keywords: singular values, transfer function matrix, unitary system, fault detection observer

1. INTRODUCTION

The singular values of a transfer matrix (Gasparyan, 2008) are non-negative functions of frequency that determine the gains of a multi-input multi-output (MIMO) linear time-invariant (LTI) system for which the matrix represents. Important properties such as H_2 norm, H_∞ norm, and $H\_$ index are defined based on the singular values: over a given frequency range, H_2 norm is the integral of the 2-norm of the singular values vector (Bernstein, 2009); H_∞ is the supreme of the largest singular value (Zhou et al., 1996); $H\_$ is the infimum of the smallest singular value (Wang et al., 2007). In the theories of robust control (Sinha, 2007; Zhou et al., 1996), robust estimation (Simon, 2006), and model-based fault detection (Chen and Patton, 1999), one of the most active research topics is about how to construct a system with those properties being optimized.

As those properties present the features of a system from different aspects, the singular values give a more detailed and accurate description of the system. However, the studies on the singular values of a system and specifically the studies on how to construct a system with pre-defined singular values are still rare. The reason partially lies in the complexity of the singular values of a transfer matrix.

In MacFarlane and Hung (1983), it is shown that the singular values of a transfer matrix – more accurately, the square of the singular values – are roots of a polynomial, whose coefficients are polynomials of complex variable s (usually taken as $s = j\omega$ with ω as frequency) and its conjugate $\bar{s}$. It is thus concluded that these singular values are locally analytical as functions of s. Boyd and Balakrishnan (1990) further proved that the "unordered unsigned" singular values, which belong to a set of real functions, are globally analytical. The analytical forms of these singular values (as functions of s), however, are not available for a generic transfer matrix. Even though, the analytical forms of singular values are available for some specific systems such as the unitary system discussed in this paper.

A unitary system is defined in this paper as an MIMO LTI system with a special property that all singular values of its transfer matrix are equal to each other. We will show that, for a system satisfying certain requirements, a unitary system in a weighted observer form, which is commonly used for the purpose of fault detection, can be constructed with a properly selected gain and weight matrix. Its singular values have the form of $|s+k+1|^{-1}$ with the freedom of selecting k, which is the magnitude of frequency response of a first-order transfer function $1/(s+k+1)$. The singular values of the constructed unitary system and hence all singular values related properties thus can be assigned. This significant characteristic can be applied to solve the problems related to the optimization of H_2, H_∞, and $H\_$ of a system such as the combined $H_\infty/H\_$ optimization shown in Part II of the paper.

The Part I of the paper is organized as follows. Section 2 contains some preliminary information such as the definition of a unitary system. Section 3 presents the method of constructing a unitary system in the fault detection observer form. An example is given in Section 4. The paper ends in Section 5 with conclusions.

2. PRELIMINARY OF UNITARY SYSTEM

In this section, the definition of a unitary system and the problem of constructing a unitary system will be presented. For the simplicity, we will only discuss a square system whose number of inputs and outputs are equal.

2.1 Definition of unitary system

A multi-input multi-output (MIMO) linear time-invariant

(LTI) system can be described by a transfer matrix. The singular values of a transfer matrix are non-negative real functions of frequency. In this paper, a system is defined as unitary if all of its singular values are equal.

Definition: A stable linear time-invariant system of m-inputs and m-outputs is defined as a unitary system if its transfer matrix $G(s)$ satisfies

$$\sigma_1(s)=\sigma_2(s)=\ldots=\sigma_m(s)\,, \tag{1}$$

where $\sigma_i(s)$ is the i^{th} singular value of $G(s)$, whose singular value decomposition (SVD) has the form of

$$G(s)=U(s)\Sigma(s)V^{\sim}(s)\,, \tag{2}$$

where $\Sigma(s)=diag\left(\sigma_1,\ \ldots,\ \sigma_m\right)$ is a diagonal matrix of singular values; $U(s)$ and $V(s)$ are unitary complex matrices such that $U(s)U^{\sim}(s)=V(s)V^{\sim}(s)=I_m$ ("$\sim$" denotes the transpose of the conjugate such that $U^{\sim}(s)=U^T(\overline{s})$).

2.2 A unitary system in the fault detection observer form

For a transfer matrix $G(s)$ with a minimal realization as

$$G(s)=C\left(sI-A\right)^{-1}F \tag{3}$$

where $A\in R^{n\times n}$, $F\in R^{n\times m}$ and $C\in R^{m\times n}$, the problem addressed in the rest of this paper is to construct a unitary system in the form of

$$G_U(s)=WC(sI-A-LC)^{-1}F \tag{4}$$

with constant matrices L and W. System (4) is a transfer matrix that represents a weighted observer of $G(s)$, which is used for the purpose of fault detection. Consider a system:

$$\begin{aligned}\dot{x}&=Ax+Bu+Ff\\ y&=Cx\end{aligned}\,, \tag{5}$$

where x is the state vector; y is the output vector; u are known inputs; and f are unknown faults to be detected. $G(s)$ is the transfer matrix from f to y. An observer for the purpose of fault detection can be built as

$$\begin{aligned}\dot{\hat{x}}&=A\hat{x}+Bu-L(y-\hat{y})\\ \hat{y}&=C\hat{x}\end{aligned}\,. \tag{6}$$

The estimation errors $\tilde{x}=x-\hat{x}$ and $\tilde{y}=y-\hat{y}$ then satisfy

$$\begin{aligned}\dot{\tilde{x}}&=\left(A+LC\right)\tilde{x}+Ff\\ \tilde{y}&=C\tilde{x}\end{aligned}\,. \tag{7}$$

The weighted estimation errors of outputs are

$$r=W\tilde{y}\,, \tag{8}$$

or equivalently in another form

$$r(s)=G_U(s)f(s)\,, \tag{9}$$

where

$$G_U(s)=WC(sI-A-LC)^{-1}F\,. \tag{10}$$

The problem is thus to select the (observer) gain L and the weight matrix W so that $G_U(s)$ is unitary.

3. CONSTRUCTING A UNITARY SYSTEM

This section presents the method of constructing a unitary system in the form of $G_U(s)$. Section 3.1 will show that $G(s)$ can be transformed to a unitary $G_U(s)$ if the following conditions are satisfied:

1. $rank(CF)=m$ or equivalently CF is non-singular. This is a measurement requirement so that, if satisfied, the states in (5) can be classified into two groups through linear transformation: one group of measured states whose dynamics contains f explicitly and one group of unmeasured states whose dynamics does not contain f;
2. $G(s)$ does not have finite zeros on the imaginary axis. This is required for the purpose of fault detection. If $G(s)$ contains zeros on the imaginary axis, for example $\pm j\omega_o$, then the faults of the frequency ω_o might not be detectable.

If the above conditions cannot be satisfied, then an approximate unitary system can be constructed with the method shown in Section 3.2. In the case of non-square $G(s)$, a solution is given in Section 3.3 so that the resulted $G_U(s)$ has equal singular values.

3.1 Constructing a unitary system

In this section, we will show first that the system in (3) can be transformed to a special form $G_r(s)$ through a first gain L_1 (Lemma 1) and all possible systems in form of (4) can be built from $G_r(s)$ with a second gain L_2 (Lemma 2). In Lemma 3, we show that there exists a companion system $G_2(s)=G_r(s)+CF$ for $G_r(s)$ such that if $G_{2c}(s)$ is unitary, $G_{rc}(s)$ is also unitary, where $G_{2c}(s)$ is an observer-form transfer matrix of $G_2(s)$ and $G_{rc}(s)$ is the observer of $G_2(s)$ with the same observer gain. Lemma 4 demonstrates that there exists a gain such that the singular values of $G_{2c}(s)$ are equal to those of CF. Thus if the singular values of WCF are equal to each other, which can always be satisfied for the non-singular CF through a weight matrix W, then $WG_{2c}(s)$ is unitary. The method of constructing a unitary system is then summarized in Theorem 1 which follows the route of $G(s)$ $\to G_r(s)$ $\to G_2(s)$ $\to G_{2c}(s)$ $\to G_{rc}(s)\to G_U(s)$ with $G_U(s)$ as a unitary system.

Lemma 1: A transfer matrix $G(s)=C(sI-A)^{-1}F$ with CF non-singular can be transformed to

$$G_r(s)=C(sI-A-L_1C)^{-1}F=CF\left(s+k\right)^{-1} \tag{11}$$

with a gain L_1, where k is a selectable parameter.

Proof:
Since CF is non-singular, there always exists an invertible matrix

$$T=\begin{bmatrix}(CF)^{-1}C\\ F^{\perp}\end{bmatrix} \tag{12}$$

such that

$$C=\begin{bmatrix}CF & 0\end{bmatrix}T\,,\ TF=\begin{bmatrix}I\\ 0\end{bmatrix} \tag{13}$$

where $F^{\perp}$ is the transform of the basis of the null space of F^T so that $F^{\perp}F=0_{(n-m)\times m}$. With T, the original $G(s)=C(sI-A)^{-1}F$ can be transformed to

$$G(s)=CT^{-1}(sI-\tilde{A})^{-1}TF=\begin{bmatrix}CF & 0\end{bmatrix}(sI-\tilde{A})^{-1}\begin{bmatrix}I\\0\end{bmatrix}, \quad (14)$$

where

$$\tilde{A}=TAT^{-1}=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}. \quad (15)$$

With a selectable k and

$$\tilde{L}_1=\begin{bmatrix}-\left(kI_m+\tilde{A}_{11}\right)(CF)^{-1}\\ -\tilde{A}_{21}(CF)^{-1}\end{bmatrix}, \quad (16)$$

one has

$$\begin{aligned}G_r(s)&=CT^{-1}(sI-\tilde{A}-\tilde{L}_1CT^{-1})^{-1}TF\\ &=CT^{-1}\begin{bmatrix}sI+kI_m & -\tilde{A}_{12}\\ 0 & sI-\tilde{A}_{22}\end{bmatrix}^{-1}TF\\ &=\begin{bmatrix}CF & 0\end{bmatrix}\begin{bmatrix}\left(sI+kI_m\right)^{-1} & \Delta\\ 0 & \left(sI-\tilde{A}_{22}\right)^{-1}\end{bmatrix}\begin{bmatrix}I\\0\end{bmatrix},\\ &=CF(s+k)^{-1}I_m=CF(s+k)^{-1}\end{aligned} \quad (17)$$

where Δ is a matrix calculated from the inverse of the upper triangle block matrix in (17). Therefore,

$$\begin{aligned}G_r(s)&=C(sI-T^{-1}\tilde{A}T-T^{-1}\tilde{L}_1C)^{-1}F\\ &=C(sI-A-T^{-1}\tilde{L}_1C)^{-1}F=CF(s+k)^{-1}\end{aligned} \quad (18)$$

Thus, if the gain L_1 is chosen as

$$L_1=T^{-1}\tilde{L}_1, \quad (19)$$

the original system in (3) is transformed to the system in (11).

Remark 3.1: For matrix $F^T\in R^{m\times n}$ ($m<n$), its null space $N(F^T)$ contains all vectors z that satisfy $F^Tz=0$ so that $N(F^T)=\left\{z\in R^n:\ F^Tz=0\right\}$. $F^{\perp}\in R^{(n-m)\times n}$ is the transpose of the basis of $N(F^T)$ which means the range of $\left(F^{\perp}\right)^T$ is $N(F^T)$ so that $N(F^T)=\left\{\left(F^{\perp}\right)^Tz_o:\ z_o\in R^{n-m}\right\}$. It is thus inferred that for all $z_o\in R^{n-m}$, one has $F^T\left(F^{\perp}\right)^Tz_o=0$, which means $F^T\left(F^{\perp}\right)^T=0_{m\times(n-m)}$ or $F^{\perp}F=0_{(n-m)\times m}$.

From (11), it can be seen that the singular values of $G_r(s)$ are $\left|s+k\right|^{-1}\Sigma_{CF}$, where Σ_{CF} are the singular values of CF. This implies that, if all singular values of CF are equal, the singular values of $G_r(s)$ are also equal to each other. However the stability of $G_r(s)$ cannot be guaranteed. We will show next that all systems in the form of $G_c(s)=C(sI-A-LC)^{-1}F$ can be built based on $G_r(s)$. A unitary system thus can also be built based on $G_r(s)$, if it exists.

Lemma 2: A system $G_c(s)=C(sI-A-LC)^{-1}F$ can always be expressed in the form of

$$G_c=\left[I-C(sI-A-L_1C)^{-1}L_2\right]^{-1}CF\left(s+k\right)^{-1} \quad (20)$$

with

$$L=L_1+L_2, \quad (21)$$

where L_1 is the gain calculated as in Lemma 1.

Proof:
$G_c(s)$ can be expressed as

$$\begin{aligned}G_c(s)&=C(sI-A-LC)^{-1}F\\ &=\left[I-C(sI-A)^{-1}L\right]^{-1}C(sI-A)^{-1}F\end{aligned}, \quad (22)$$

which also means

$$\begin{aligned}G_c(s)&=C(sI-A-L_1C-L_2C)^{-1}F\\ &=\left[I-C(sI-A-L_1C)^{-1}L_2\right]^{-1}C(sI-A-L_1C)^{-1}F\end{aligned} \quad (23)$$

with $L=L_1+L_2$.
By selecting L_1 as shown in Lemma 1, it is derived that

$$G_c=\left[I-C(sI-A-L_1C)^{-1}L_2\right]^{-1}CF\left(s+k\right)^{-1}. \quad (24)$$

Remark 3.2: From Lemma 2 it can be concluded that if there exists a unitary system for $G(s)$ in the form of $G_U=WC(sI-A-LC)^{-1}F$, it can be constructed from $G_r(s)$ such that $G_U=W\left[I-C(sI-A-L_1C)^{-1}L_2\right]^{-1}G_r(s)$.

Lemma 3: For the following two systems:

$$G_r(s)=C(sI-A)^{-1}F=CF\left(s+k\right)^{-1} \quad (25)$$

$$G_2(s)=G_r(s)+CF=C(sI-A)^{-1}F+CF, \quad (26)$$

their corresponding systems in the observer form:

$$G_{rc}(s)=C(sI-A-LC)^{-1}F \quad (27)$$

$$G_{2c}(s)=C(sI-A-LC)^{-1}(F+LCF)+CF \quad (28)$$

have the following relation: if the singular values of $G_{2c}(s)$ are $\Sigma(s)$, then the singular values of $G_{rc}(s)$ are

$$\Sigma_{rc}(s)=\left|s+k+1\right|^{-1}\Sigma(s). \quad (29)$$

Proof:
The systems in (27) and (28) can also be represented in the following forms:

$$\begin{aligned}G_{rc}(s)&=\left[I-C(sI-A)^{-1}L\right]^{-1}C(sI-A)^{-1}F\\ &=\left[I-C(sI-A)^{-1}L\right]^{-1}CF\left(s+k\right)^{-1}\end{aligned} \quad (30)$$

$$\begin{aligned}G_{2c}(s)&=\left[I-C(sI-A)^{-1}L\right]^{-1}\left[C(sI-A)^{-1}F+CF\right]\\ &=\left[I-C(sI-A)^{-1}L\right]^{-1}CF\left(s+k+1\right)/\left(s+k\right)\\ &=G_{rc}(s)\left(s+k+1\right)\end{aligned}. \quad (31)$$

Since the SVD of $G_{2c}(s)$ is

$$G_{2c}(s)=U(s)\Sigma(s)V^{\sim}(s), \quad (32)$$

it is obvious that

$$G_{rc}(s)=G_{2c}(s)/(s+k+1)=U(s)\Sigma(s)/\left(s+k+1\right)V^{\sim}(s) \quad (33)$$

Thus, the singular values of $G_{rc}(s)$ are

$$\Sigma_{rc}(s)=\left|s+k+1\right|^{-1}\Sigma(s). \quad (34)$$

Lemma 4 (Zhou et al. 1996):

$$G_c(s)=C\left(sI-A-LC\right)^{-1}\left(F+LCF\right)+CF \quad (35)$$

satisfies

$$G_c(s)G_c^{\sim}(s)=\left(CF\right)\left(CF\right)^T \quad (36)$$

if

$$L = -\left[YC^T + F(CF)^T\right]\left[(CF)(CF)^T\right]^{-1}, \tag{37}$$

where Y is a positive-definite solution of the following algebraic Riccati equation:

$$\begin{aligned}&\left[A - F(CF)^{-1}C\right]Y + Y\left[A - F(CF)^{-1}C\right]^T \\ &-YC^T(CF)^{-T}(CF)^{-1}CY = 0\end{aligned}. \tag{38}$$

Remark 3.3: A positive-definite Y exists if: 1) (A, C) is an observable pair, which is satisfied since system (3) is a minimal realization; 2) $G(s) = C(sI - A)^{-1}F$ does not have zeros on the imaginary axis, which is satisfied from the condition 2.

The procedures of constructing a unitary system in the fault detection observer form are thus summarized in Theorem 1.

Theorem 1: Given $G(s)$ with a minimal realization of $G(s) = C(sI - A)^{-1}F$, where CF is non-singular and $G(s)$ does not have finite zeros on the imaginary axis, a unitary system with singular values of $|s+k+1|^{-1}$ can be constructed as

$$G_U(s) = (CF)^{-1}C(sI - A - LC)^{-1}F, \tag{39}$$

where L is calculated following the procedures of :

1). Calculate:

$$T = \begin{bmatrix}(CF)^{-1}C \\ F^{\perp}\end{bmatrix}, \tag{40}$$

$$\tilde{A} = TAT^{-1} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22}\end{bmatrix}; \tag{41}$$

2). Select k and calculate

$$\tilde{L}_1 = \begin{bmatrix}-\left(kI_m + \tilde{A}_{11}\right)(CF)^{-1} \\ -\tilde{A}_{21}(CF)^{-1}\end{bmatrix}; \tag{42}$$

3). Calculate

$$L_1 = T^{-1}\tilde{L}_1; \tag{43}$$

4). Solve the positive definite Y for the Riccati equation

$$\begin{aligned}&\left[A + L_1C - F(CF)^{-1}C\right]Y + Y\left[A + L_1C - F(CF)^{-1}C\right]^T \\ &-YC^T(CF)^{-T}(CF)^{-1}CY = 0\end{aligned}; \tag{44}$$

5). Solve the gain

$$L_2 = -\left[YC^T + F(CF)^T\right]\left[(CF)(CF)^T\right]^{-1}; \tag{45}$$

6). Calculate the gain

$$L = L_1 + L_2. \tag{46}$$

Proof:
From Lemma 3, the following two systems:

$$\begin{aligned}G_c(s) &= C(sI - A - L_1C - L_2C)^{-1}(F + L_2CF) + CF \\ G_{rc}(s) &= C(sI - A - L_1C - L_2C)^{-1}F = C(sI - A - LC)^{-1}F\end{aligned} \tag{47}$$

satisfy

$$G_{rc}(s) = (s+k+1)^{-1}G_c(s) \tag{48}$$

since

$$G_r(s) = C(sI - A - L_1C)^{-1}F = CF(s+k)^{-1}. \tag{49}$$

With L_2 calculated from steps 2 and 3, it is derived from Lemma 4 that

$$G_c(s)G_c^{\sim}(s) = (CF)(CF)^T, \tag{50}$$

or equivalently

$$(CF)^{-1}G_c(s)G_c^{\sim}(s)(CF)^{-T} = I_m, \tag{51}$$

which means the singular values of $(CF)^{-1}G_c(s)$ are all equal to one. From (33), the following equation is obtained:

$$G_U(s) = (CF)^{-1}G_{rc}(s) = (s+k+1)^{-1}(CF)^{-1}G_c(s). \tag{52}$$

Thus, $G_U(s)$ is a unitary system with singular values of $|s+k+1|^{-1}$. Meanwhile it can be derived that

$$G_U(s) = U(s)|s+k+1|^{-1}V^{\sim}(s) = |s+k+1|^{-1}U_1(s) \tag{53}$$

where $U_1(s) = U(s)V^{\sim}(s)$ is also a unitary matrix.
Theorem 1 is thus proved.

Remark 3.4: $|s+k+1|^{-1}$ is the magnitude of the transfer function $1/(s+k+1)$. The singular values of $G_U(s)$ therefore present a first-order magnitude frequency response characteristic. The singular values of $G_U(s)$ thus can be assigned as $|s+k+1|^{-1}$ or equivalently as a function of frequency $[\omega^2 + (k+1)^2]^{-1/2}$ with a selectable parameter k. $-(k+1)$ is the pole of the transfer function $1/(s+k+1)$. The difference between $G_{rc}(s)$ and $G_U(s)$ is an artificial weight of $(CF)^{-1}$. The singular values of $G_{rc}(s)$ are $|s+k+1|^{-1}\Sigma_{CF}$ which are also magnitudes of first-order transfer functions. These transfer functions have the same pole with different gains.

3.2 Constructing an approximate unitary system

The non-singularity requirement of CF cannot always be satisfied. In such a case, an approximate solution is given in this section.

Lemma 5: For matrices $\Psi, \Phi \in C^{m\times m}$, their singular values satisfy (Bernstein, 2009):

$$\bar{\sigma}(\Psi \pm \Phi) \le \bar{\sigma}(\Psi) + \bar{\sigma}(\Phi) \tag{54}$$

$$\underline{\sigma}(\Psi) + \bar{\sigma}(\Phi) \ge \underline{\sigma}(\Psi \pm \Phi) \ge \underline{\sigma}(\Psi) - \bar{\sigma}(\Phi) \tag{55}$$

$$\bar{\sigma}(\Psi\Phi) \ge \bar{\sigma}(\Psi)\underline{\sigma}(\Phi). \tag{56}$$

Lemma 6: If $\Psi, \Phi, \Gamma \in C^{m\times m}$, $U \in R^{m\times m}$ and $\sigma \in R^1$ satisfy the following 3 conditions:

1. $UU^T = I$;
2. $\Gamma = \Psi(\Phi + \sigma U)$;
3. All singular values of Γ are equal to σ as $\sigma_\Gamma = \sigma$, then the following inequalities hold:

$$\bar{\sigma}(\Psi\Phi) - \sigma \le \sigma^2 / \left[\underline{\sigma}(\Phi) - \sigma\right] \tag{57}$$

$$\bar{\sigma}(\Psi\Phi) - \underline{\sigma}(\Psi\Phi) \le 2\sigma^2 / \left[\underline{\sigma}(\Phi) - \sigma\right] \tag{58}$$

where $\bar{\sigma}$ and $\underline{\sigma}$ denote the largest and smallest singular values of a matrix.

Proof:
With $\Psi\Phi = \Gamma - \sigma\Psi U$ and $\sigma_\Gamma = \sigma$, it is obvious from inequalities (54) and (55) that:

$$\bar{\sigma}(\Psi\Phi) \le \sigma + \bar{\sigma}(\sigma\Psi U) = \sigma + \sigma\bar{\sigma}(\Psi) \quad (59)$$

$$\underline{\sigma}(\Psi\Phi) \ge \sigma - \bar{\sigma}(\sigma\Psi U) = \sigma - \sigma\bar{\sigma}(\Psi) \quad (60)$$

which means $\bar{\sigma}(\Psi B) - \sigma \le \sigma\bar{\sigma}(\Psi)$ and $\bar{\sigma}(\Psi\Phi) - \underline{\sigma}(\Psi\Phi) \le 2\sigma\bar{\sigma}(\Psi)$.

From inequality (56) and conditions 2 and 3, one has $\sigma \ge \bar{\sigma}(\Psi)\underline{\sigma}(\Phi + \sigma U)$ and thus the inequality $\bar{\sigma}(\Psi) \le \sigma / \underline{\sigma}(\Phi + \sigma U)$ holds. From inequality (55), we have $\underline{\sigma}(\Phi + \sigma U) \ge \underline{\sigma}(\Phi) - \sigma$, which means:

$$\bar{\sigma}(\Psi) \le \sigma / \underline{\sigma}(\Phi + \sigma U) \le \sigma / \left[\underline{\sigma}(\Phi) - \sigma\right] \quad (61)$$

It is thus derived that:

$$\bar{\sigma}(\Psi\Phi) - \sigma \le \sigma^2 / \left[\underline{\sigma}(\Phi) - \sigma\right] \quad (62)$$

$$\bar{\sigma}(\Psi\Phi) - \underline{\sigma}(\Psi\Phi) \le 2\sigma^2 / \left[\underline{\sigma}(\Phi) - \sigma\right]. \quad (63)$$

Theorem 2: For a linear transfer matrix $G(s)$ of *m*-inputs and *m*-outputs with a minimum realization of:

$$G(s) = C(sI - A)^{-1} F \quad (64)$$

and its companion system:

$$G_2(s) = G(s) + \sigma U = C(sI - A)^{-1} F + \sigma U \quad (65)$$

where U is any real unitary matrix, a feedback gain will transform the two systems in (64) and (65) to the systems in the following forms:

$$G_c(s) = C(sI - A - LC)^{-1} F \quad (66)$$

$$G_{2c}(s) = C(sI - A - LC)^{-1}(F + \sigma LU) + \sigma U \quad . \quad (67)$$

If the singular values of $G_{2c}(s)$ are all equal and satisfy

$$\sigma\left[G_{2c}(s)\right] = \sigma \quad (68)$$

then the singular values of $G_c(s)$ satisfy the following inequalities:

$$\bar{\sigma}\left[G_c(s)\right] - \sigma \le \sigma / \left\{\underline{\sigma}\left[G(s)\right]/\sigma - 1\right\} \quad (69)$$

$$\bar{\sigma}\left[G_c(s)\right] - \underline{\sigma}\left[G_c(s)\right] \le 2\sigma / \left\{\underline{\sigma}\left[G(s)\right]/\sigma - 1\right\}. \quad (70)$$

Proof:

The transfer matrices in (66) and (67) are the same as:

$$G_c(s) = \left[I - C(sI - A)^{-1} L\right]^{-1} C(sI - A)^{-1} F \quad (71)$$

$$G_{2c}(s) = \left[I - C(sI - A)^{-1} L\right]^{-1} \left[C(sI - A)^{-1} F + \sigma U\right], \quad (72)$$

which also means that, with $G_L(s) = \left[I - C(sI - A)^{-1} L\right]^{-1}$:

$$G_c(s) = G_L(s) G(s) \quad (73)$$

$$G_{2c}(s) = G_L(s)\left[G(s) + \sigma U\right]. \quad (74)$$

If the singular values of $G_{2c}(s)$ are equal to σ, all 3 conditions of Lemma 6 are satisfied with $\Psi = G_L(s)$, $\Phi = G(s)$, and $\Gamma = G_{2c}(s)$. Thus (64) and (65) hold.

Remark 3.5: According to Lemma 4, a gain L can be calculated so that the singular values of $G_{2c}(s)$ are equal to those of σU. Since U is a selectable matrix, $G_{2c}(s)$ satisfies inequality (74). With the same *L,* $G_c(s)$ will satisfy inequalities (69) and (70). As σ and U can be chosen as any values, an approximate unitary system can be built by selecting $\sigma << \underline{\sigma}\left[G(s)\right]$ for all $s = j\omega$ in the frequency range of interest. The inaccuracy of the approximation can be calculated with inequalities (69) and (70).

Remark 3.6: For $G(s)$ with a non-singular *CF*, only an approximate solution is given in this paper. A part of our future work is to find an exact solution, where a dynamic gain and weight might be required in the form of $L(s)$ and $W(s)$.

3.3 Solutions to a non-square system

In this section, a non-square system in the following form is considered:

$$G(s) = C(sI - A)^{-1} F \quad (75)$$

where, $A \in R^{n\times n}$, $F \in R^{n\times r}$ and $C \in R^{m\times n}$ with the assumptions of: 1) $m > r$; 2) $rank(CF) = r$ so that *CF* has full column rank. There are two ways of transforming (75) into a unitary system.

1). Reduce the dimension of *C*
For such a system, a corresponding square system can be constructed as:

$$G_a(s) = W_1 C(sI - A)^{-1} F \quad (76)$$

where, $W_1 \in R^{r\times m}$ is a real constant matrix so that $rank(W_1 CF) = r.$ If $G_a(s)$ does not have zeros on the imaginary axis, a unitary system can be constructed as

$$G_u(s) = \left(W_1 CB\right)^{-1} W_1 C(sI - A - LC)^{-1} F. \quad (77)$$

2). Increase the dimension of *F*
Another method is to construct a square transfer matrix as:

$$G_b(s) = C(sI - A)^{-1}\left[F \quad F_o\right] \quad (78)$$

where F_o is selected so that $rank\left(C\left[F \quad F_o\right]\right) = m.$ If $G_b(s)$ does not have zeros on the imaginary axis, a unitary system thus can be constructed for $G_b(s)$ in the form of

$$G_u(s) = \left(C\left[F \quad F_o\right]\right)^{-1} C(sI - A - LC)^{-1}\left[F \quad F_o\right]. \quad (79)$$

Thus, all non-zero singular values of

$$G_{u1}(s) = \left(C\left[F \quad F_o\right]\right)^{-1} C(sI - A - LC)^{-1} F \quad (80)$$

are equal.

4. AN EXAMPLE OF UNITARY SYSTEM

In this example, a unitary system

$$G_U(s) = (CF)^{-1} C(sI - A - LC)^{-1} F$$

is constructed for a system

$$G(s) = C(sI - A)^{-1} F,$$

which is randomly generated as:

$$A = \begin{bmatrix} 0.5046 & 0.0070 & 0.2952 & 1.2408 \\ 1.6265 & 0.4572 & 0.5554 & 1.8668 \\ 0.1535 & 0.0027 & 0.3623 & 1.9462 \\ 0.8160 & 0.3919 & 0.2873 & 0.4583 \end{bmatrix},$$

$$F=\begin{bmatrix} 0.5529 & 0.6416 & 0.6332 \\ 0.7702 & 0.2557 & 1.4140 \\ 0.3895 & 0.0384 & 1.1211 \\ 0.1005 & 1.4039 & 0.0108 \end{bmatrix},$$

$$C=\begin{bmatrix} 0.7756 & 0.2396 & 0.4075 & 0.4421 \\ 0.7088 & 0.8486 & 0.9788 & 0.3494 \\ 0.6068 & 0.2505 & 0.9409 & 0.6687 \end{bmatrix}.$$

L is calculated following the procedures in Theorem 1:

1). Calculate (the choice of $F^{\perp}$ may be varied)

$$T=\begin{bmatrix} (CF)^{-1}C \\ F^{\perp} \end{bmatrix}=\begin{bmatrix} 2.4183 & 1.1270 & -2.7754 & -1.2346 \\ 0.2034 & -0.4835 & 0.4882 & 0.6940 \\ -1.1166 & -0.0796 & 1.6183 & 0.4805 \\ -0.5818 & 0.6379 & -0.4776 & 0.1628 \end{bmatrix}$$

and calculate $\tilde{A}=TAT^{-1}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$;

2). Select k=10 and calculate

$$\tilde{L}_1=\begin{bmatrix} -\left(kI_2+\tilde{A}_{11}\right)(CF)^{-1} \\ -\tilde{A}_{21}(CF)^{-1} \end{bmatrix};$$

3). Calculate the feedback gain $L_1=T^{-1}\tilde{L}_1$;

4). Solve the Algebraic Riccati equation for a positive-definite Y:

$$\left[A+L_1C-F(CF)^{-1}C\right]Y+Y\left[A+L_1C-F(CF)^{-1}C\right]^T \\ -YC^T(CF)^{-T}(CF)^{-1}CY=0 ;$$

5). Calculate $L_2=-\left[YC^T+F(CF)^T\right]\left[(CF)(CF)^T\right]^{-1}$;

6) Calculate

$$L=L_1+L_2=\begin{bmatrix} -19.9902 & -0.3428 & 10.6200 \\ -5.0800 & -8.5893 & 6.8353 \\ 16.8744 & -4.1290 & -12.9409 \\ -8.6851 & 14.2991 & -16.4507 \end{bmatrix}.$$

The singular values plots of $G(s)$ and $G_U(s)$ are shown in Fig. 1, from which it can be seen that $G(s)$ with three different singular values is transformed to $G_U(s)$ which has three equal singular values as $1/|s+11|$ or as a function of frequency $1/(\omega^2+11^2)^{1/2}$.

5. CONCLUSIONS

In this paper, a unitary system is defined as a multi-input multi-output linear time-invariant system whose singular values of transfer matrix are equal. A non-unitary system can be transformed to a unitary system in the fault detection observer form as long as certain requirements are met. The singular values of the resulted unitary system are $|s+k+1|^{-1}$, which is the magnitude frequency response of the transfer function $1/(s+k+1)$. With the method presented in this paper, the singular values of a unitary system thus can be assigned as a function of s in the form of $|s+k+1|^{-1}$ or equivalently, as a function of frequency of $[\omega^2+(k+1)^2]^{-1/2}$. Singular values related properties, such as H_2 norm, H_∞ norm, and $H\_$ index, can be determined based on this function. The study on unitary system therefore not only helps the investigation of linear systems, but also has potential applications in the areas of robust control, robust estimation and robust fault diagnosis, and fault-tolerant control where the optimization of singular values related properties are of key importance. In the Part II of the paper, we will study the application of a unitary system in the combined H_∞/ $H\_$ optimization for the purpose of fault detection observer design.

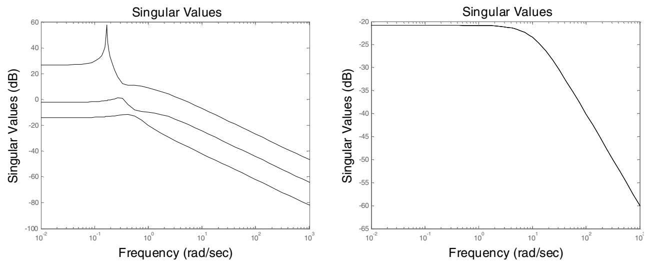

Fig. 1: Singular values: non-unitary $G(s)$ vs. unitary $G_U(s)$

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